Electronic Journal of Differential Equations, Vol. 2009(2009), No. 71, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF POSITIVE SOLUTIONS FOR QUASILINEAR ELLIPTIC SYSTEMS INVOLVING THE p-LAPLACIAN

XUDONG SHANG, JIHUI ZHANG

Abstract. In this article, we study the existence of positive solutions for the quasilinear elliptic system

$$\begin{gathered}-\Delta_p u = f(x,u,v) \quad x \in \Omega,\\ -\Delta_p v = g(x,u,v) \quad x \in \Omega,\\ u = v = 0 \quad x \in \partial\Omega.\end{gathered}$$

Using degree theoretic arguments based on the degree map for operators of type $(S)_+$, under suitable assumptions on the nonlinearities, we prove the existence of positive weak solutions.

1. Introduction and main result

In this paper we study the existence of positive solution for the nonlinear elliptic system

$$\begin{gathered}-\Delta_p u = f(x,u,v) \quad x \in \Omega,\\ -\Delta_p v = g(x,u,v) \quad x \in \Omega,\\ u = v = 0 \quad x \in \partial\Omega\end{gathered} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, and Ω is a smooth bounded region in $\mathbb{R}^N$ for $N \geq 1$, $p > 1$.

Elliptic systems have several practical applications. For example they can describe the multiplicative chemical reaction catalyzed by grains under constant or variant temperature, a correspondence of the stable station of dynamical system determined by the reaction-diffusion system. In recent years, many publications have appeared concerning quasilinear elliptic systems which have been used in a great variety of applications, we refer the readers to [1, 2, 3, 4, 5, 6] and the references therein. Existence and multiplity results for quasilinear elliptic systems with variational have been broadly investigate. Djellit and Tas [2] studied a class of quasilinear elliptic systems involving the p-Laplacian operator, the right hand sides of systems being closed related to the critical Sobolev exponents. Under some additional assumptions on the nonlinearities, they proved the existence of at least one nontrivial solution. In [4], the authors studied the existence and multiplicity of non-negative solutions for the quasilinear elliptic system in both bounded and

2000 *Mathematics Subject Classification.* 34A34, 34B18.
Key words and phrases. p-Laplacian system; positive solutions; operator of type $(S)_+$.

Submitted March 6, 2009. Published June 1, 2009.

unbounded domain in $\mathbb{R}^N$. Zhang [1] used the Leray-Schauder degree to obtain a positive solution of the nonlinear elliptic system.

In this work, we show the existence of positive solutions for system (1.1). Using the degree theory for $(S)_+$ operator initiated by Browder [7]. Our main goal in this article is to extend the main result of [1] to the quasilinear case. The relevant studies about $(S)_+$ operators can be found in [8, 9, 10].

Through this paper, $(u, v) \in \mathbb{R}^2$. As to the nonlinearities f, g, we assume that they are Caratheodory functions satisfying the following growth conditions:
(i) There exist $a_i \geq 0, c_i \geq 0 (i = 1, 2)$ such that

$$0 \leq f(x, u, v) \leq a_1 |(u, v)|^{q-1} + c_1,$$
$$0 \leq g(x, u, v) \leq a_2 |(u, v)|^{q-1} + c_2$$

where $1 < p < q < p^* = \frac{Np}{N-p}$ if $p < N$, or $p < q < +\infty$ if $p \geq N$.
(ii) There existence an $\epsilon^{'} > 0$, $c_3 > 0$, $1 < p < \theta < p^*$ such that

$$f(x, u, v)u + g(x, u, v)v \leq (\lambda_1 - \epsilon')(|u|^p + |v|^p) + c_3(|u|^\theta + |v|^\theta)$$

where λ_1 stands for the first eigenvalue of the operator $-\Delta_p$ in $W_0^{1,p}(\Omega)$.
(iii) $f(x, u, v)$ and $g(x, u, v)$ also satisfies

$$\liminf_{|(u,v)|\to\infty} \frac{f(x, u, v)}{|(u, v)|^{p-1}} = +\infty, \quad \liminf_{|(u,v)|\to\infty} \frac{g(x, u, v)}{|(u, v)|^{p-1}} = +\infty.$$

The main result of this paper is as follows.

Theorem 1.1. *Suppose that* (i)–(iii) *hold. Then* (1.1) *has a positive weak solution.*

The plan of this paper is as follows. In section 2, we shall present some lemmas in order to prove our main results. The main results is proved in section 3.

2. Preliminaries

We start this section by recalling the definition for operator of type $(S)_+$.

Definition 2.1 ([10])**.** Let X be a reflexive Banach space and X^* its topological dual. A mapping $A : X \to X^*$ is of type $(S)_+$, if for each sequence u_n in X satisfying $u_n \rightharpoonup u_0$ in X and

$$\limsup_{n\to+\infty} \langle A(u_n), u_n - u_0 \rangle \leq 0,$$

we have $u_n \to u_0$.

If the operator A satisfies the above condition, then it is possible to define its degree. Now we consider triples (A, Ω, x_0) such that Ω is a nonempty, bounded, open set in X, $A : \overline{\Omega} \to X^*$ is a demicontinuous mapping of type $(S)_+$ and $x_0 \notin A(\partial\Omega)$. On such triples Browder [7] defined a degree denoted by $\deg(A, \Omega, x_0)$, which has the following three basic properties:

(i) (Normality) If $x_0 \in A(\Omega)$ then $\deg(A, \Omega, x_0) = 1$;
(ii) (Domain additivity) If Ω_1, Ω_2 are disjoint open subsets of Ω and $x_0 \notin A(\overline{\Omega} \backslash (\Omega_1 \cup \Omega_2))$ then $\deg(A, \Omega, x_0) = \deg(A, \Omega_1, x_0) + \deg(A, \Omega_2, x_0)$;
(iii) (Homotopy invariance) If $\{A_t\}_{t\in[0,1]}$ is a homotopy of type $(S)_+$ such that A_t is bounded for every $t \in [0, 1]$ and $x_0 : [0, 1] \to X^*$ is a continuous map such that $x_0(t) \notin A_t(\partial\Omega)$ for all $t \in [0, 1]$, then $\deg(A_t, \Omega, x_0(t))$ is independent of $t \in [0, 1]$.

Remark 2.2. If the operator A is of type $(S)_+$, and K is compact, then $A+K$ is of type $(S)_+$.

Lemma 2.3. *Assume A is of type $(S)_+$. Suppose that for $u \in X$ and $\|u\|_X = r$. $\langle Au, u\rangle > 0$ is satisfied. Then*
$$\deg(A, B_r(0), 0) = 1.$$

In this paper, we denote by Z the product space $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$. The space Z will be endowed with the norm
$$\|z\|_Z^p = \|u\|_{W_0^{1,p}(\Omega)}^p + \|v\|_{W_0^{1,p}(\Omega)}^p, \qquad z = (u, v) \in Z,$$
where $\|u\|_{W_0^{1,p}(\Omega)} = (\int_\Omega |\nabla u|^p dx)^{1/p}$.

As usual, a weak solution of system (1.1) is any $(u, v) \in Z$ such that
$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\xi dx + \int_\Omega |\nabla v|^{p-2}\nabla v\nabla\zeta dx - \int_\Omega f(x,u,v)\xi dx - \int_\Omega g(x,u,v)\zeta dx = 0$$
for every $(\xi, \zeta) \in Z$.

Next let us introduce the functionals $I_i, F_i : Z \to Z^*$ $(i = 1, 2)$ as follows:
$$\begin{gathered}
\langle I_1(u,v), (\xi,\zeta)\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\xi dx,\\
\langle I_2(u,v), (\xi,\zeta)\rangle = \int_\Omega |\nabla v|^{p-2}\nabla v\nabla\zeta dx,\\
\langle F_1(u,v), (\xi,\zeta)\rangle = \int_\Omega f(x,u,v)\xi dx,\\
\langle F_2(u,v), (\xi,\zeta)\rangle = \int_\Omega g(x,u,v)\zeta dx.
\end{gathered}$$
Define the operator
$$A = I_1 + I_2 - F_1 - F_2.$$

Lemma 2.4. *The mapping $B = I_1 + I_2$ is of type $(S)_+$.*

Proof. Assume that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in Z and
$$\limsup_{n\to+\infty} \langle B(u_n, v_n), (u_n - u_0, v_n - v_0)\rangle \le 0.$$
From the weak convergence we have that
$$\limsup_{n\to+\infty} \langle B(u_0, v_0), (u_n - u_0, v_n - v_0)\rangle = 0.$$
Then we get
$$\limsup_{n\to+\infty} \langle B(u_n, v_n) - B(u_0, v_0), (u_n - u_0, v_n - v_0)\rangle \le 0.$$
By the monotonicity property of B, we obtain
$$\limsup_{n\to+\infty} \langle B(u_n, v_n) - B(u_0, v_0), (u_n - u_0, v_n - v_0)\rangle = 0.$$
This implies
$$\begin{aligned}
&\lim_{n\to+\infty} \int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_0|^{p-2}\nabla u_0)(\nabla u_n - \nabla u_0)dx\\
&+ \int_\Omega (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v_0|^{p-2}\nabla v_0)(\nabla v_n - \nabla v_0)dx = 0.
\end{aligned}$$

Observe that for all $x, y \in \mathbb{R}^N$,

$$|x-y|^p \le \begin{cases} (|x|^{p-2}x - |y|^{p-2}y)(x-y) & \text{if } p \ge 2, \\ [(|x|^{p-2}x - |y|^{p-2}y)(x-y)]^{p/2}(|x|+|y|)^{(2-p)p/2} & \text{if } 1 < p < 2. \end{cases}$$

So, we obtain

$$\lim_{n\to+\infty} \int_\Omega (|\nabla u_n - \nabla u_0|^p + |\nabla v_n - \nabla v_0|^p)dx = 0.$$

Thus, we have $(\nabla u_n, \nabla v_n) \to (\nabla u_0, \nabla v_0)$ in $L^p(\Omega)\times L^p(\Omega)$. Also $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in Z, which implies that $(u_n, v_n) \to (u_0, v_0)$ in $L^p(\Omega) \times L^p(\Omega)$. Hence

$$(u_n, v_n) \to (u_0, v_0) \quad \text{in } Z.$$

The proof is complete. □

Lemma 2.5. *The mapping $F = F_1 + F_2$ is compact.*

The proof of the above lemma follows easily from Hypotheses (i). By Remark 2.2 we have the following result.

Lemma 2.6. *The operator A is type of $(S)_+$.*

3. Proof of main theorem

Define $B_R^K = \{(u,v) \in K : \|(u,v)\|_Z < R\}$, $K = \{(u,v) \in K : u \ge 0, v \ge 0, \text{a.e.} x \in \Omega\}$. Now, we give the proofs of the main results.

Proof of Theorem 1.1. There exists $R_0 > 0$ such that

$$\deg(A, B_R^K, 0) = 0 \quad \text{for all } R \ge R_0. \tag{3.1}$$

Let

$$\langle L(u,v), (\xi,\zeta)\rangle = \int_\Omega ((k+\epsilon)u^{p-1}\xi + (k+\epsilon)v^{p-1}\zeta)dx, \quad \forall (\xi,\zeta) \in Z \tag{3.2}$$

where k is a real number, $0 < \epsilon < \epsilon'$. Since L is a completely continuous operator, the homotopy $H_t : [0,1] \times K \to Z^*$ defined by

$$\begin{aligned} \langle H_t(u,v), (\xi,\zeta)\rangle &= \int_\Omega (|\nabla u|^{p-2}\nabla u \nabla \xi + |\nabla v|^{p-2}\nabla v \nabla \zeta)dx \\ &\quad - (1-t)\int_\Omega (f(x,u,v)\xi + g(x,u,v)\zeta)dx \\ &\quad - t\int_\Omega ((k+\epsilon)u^{p-1}\xi + (k+\epsilon)v^{p-1}\zeta)dx, \end{aligned}$$

where the value of k will be fixed later. Clearly H_t is of type $(S)_+$. We claim that there exists $R_0 > 0$ such that

$$H_t(u,v) \ne 0 \quad \text{for all } t \in [0,1],\ (u,v) \in \partial B_R^K\ R \ge R_0.$$

Suppose that is not true. Then we can find sequences $\{t_n\} \subset [0,1]$ and $\{(u_n, v_n)\} \subset Z$ such that $t_n \to t \in [0,1]$, $\|(u_n, v_n)\|_Z \to \infty$ and

$$\begin{aligned}
&\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n \nabla\xi + |\nabla v_n|^{p-2}\nabla v_n\nabla\zeta)dx \\
&= t_n \int_\Omega ((k+\epsilon)u_n^{p-1}\xi + (k+\epsilon)v_n^{p-1}\zeta)dx \\
&\quad + (1-t_n)\int_\Omega (f(x,u_n,v_n)\xi + g(x,u_n,v_n)\zeta)dx
\end{aligned} \tag{3.3}$$

for every $(\xi,\zeta) \in Z$. Put $(\omega_n, \psi_n) = \frac{(u_n,v_n)}{\|(u_n,v_n)\|_Z}$, we may assume that there exists $(\omega_0, \psi_0) \in Z$ satisfying

$$\begin{gathered}
(\omega_n, \psi_n) \rightharpoonup (\omega_0, \psi_0) \quad \text{in } Z, \\
(\omega_n, \psi_n) \to (\omega_0, \psi_0) \quad \text{in } L^p(\Omega) \times L^p(\Omega), \\
(\omega_n, \psi_n) \to (\omega_0, \psi_0) \quad \text{a.e. } x \in \Omega.
\end{gathered}$$

Applying the test function $(\omega_n - \omega_0, \psi_n - \psi_0) \in Z$ in (3.3), we find

$$\begin{aligned}
&\int_\Omega (|\nabla\omega_n|^{p-2}\nabla\omega_n\nabla(\omega_n-\omega_0) + |\nabla\psi_n|^{p-2}\nabla\psi_n\nabla(\psi_n-\psi_0))dx \\
&= (1-t_n)\int_\Omega (\frac{f(x,u_n,v_n)}{\|(u_n,v_n)\|_Z^{p-1}}(\omega_n-\omega_0) + \frac{g(x,u_n,v_n)}{\|(u_n,v_n)\|_Z^{p-1}}(\psi_n-\psi_0))dx \\
&\quad + t_n\int_\Omega ((k+\epsilon)\omega_n^{p-1}(\omega_n-\omega_0) + (k+\epsilon)\psi_n^{p-1}(\psi_n-\psi_0))dx\,.
\end{aligned} \tag{3.4}$$

We know that

$$\begin{gathered}
(1-t_n)\int_\Omega (\frac{f(x,u_n,v_n)}{\|(u_n,v_n)\|_Z^{p-1}}(\omega_n-\omega_0) + \frac{g(x,u_n,v_n)}{\|(u_n,v_n)\|_Z^{p-1}}(\psi_n-\psi_0))dx \to 0 \quad n\to\infty, \\
t_n\int_\Omega ((k+\epsilon)\omega_n^{p-1}(\omega_n-\omega_0) + (k+\epsilon)\psi_n^{p-1}(\psi_n-\psi_0))dx \to 0 \quad n\to\infty.
\end{gathered}$$

By (3.4), we have

$$\lim_{n\to+\infty}\int_\Omega (|\nabla\omega_n|^{p-2}\nabla\omega_n\nabla(\omega_n-\omega_0) + |\nabla\psi_n|^{p-2}\nabla\psi_n\nabla(\psi_n-\psi_0))dx = 0;$$

i.e.,

$$\lim_{n\to+\infty}\langle B(\omega_n,\psi_n), (\omega_n,\psi_n) - (\omega_0,\psi_0)\rangle = 0.$$

According to Lemma 2.4 we obtain $(\omega_n, \psi_n) \to (\omega_0, \psi_0)$ in Z as $n \to \infty$ and $\|(\omega_0, \psi_0)\| = 1$. This shows that $(\omega_0, \psi_0) \neq (0,0)$. Let now apply the test function $(\omega_0, \psi_0) \in Z$ in (3.4), we get

$$\begin{aligned}
&\int_\Omega (|\nabla\omega_n|^{p-2}\nabla\omega_n\nabla\omega_0 + |\nabla\psi_n|^{p-2}\nabla\psi_n\nabla\psi_0)dx \\
&= (1-t_n)\int_\Omega (\frac{f(x,u_n,v_n)}{\|(u_n,v_n)\|_Z^{p-1}}\omega_0 + \frac{g(x,u_n,v_n)}{\|(u_n,v_n)\|_Z^{p-1}}\psi_0)dx \\
&\quad + t_n\int_\Omega ((k+\epsilon)\omega_n^{p-1}\omega_0 + (k+\epsilon)\psi_n^{p-1}\psi_0)dx
\end{aligned} \tag{3.5}$$

Passing to the limit as $n \to \infty$, using Fatou's lemma and hypothesis (iii), we obtain

$$\int_\Omega (|\nabla \omega_0|^p + |\nabla \psi_0|^p) dx \geq (k+\epsilon) \int_\Omega (\omega_0^p + \psi_0^p) dx. \tag{3.6}$$

Now we take $k = \frac{\|(\omega_0,\psi_0)\|_Z^p}{\|(\omega_0,\psi_0)\|_{L^p\times L^p}^p}$, by (3.6) we have

$$k+\epsilon \leq \frac{\int_\Omega (|\nabla \omega_0|^p + |\nabla \psi_0|^p) dx}{\int_\Omega (\omega_0^p + \psi_0^p) dx} = k.$$

This contradiction shows the claim. Using the homotopy invariance of the degree map, which through the homotopy H_t yields

$$\deg(A, B_R^K, 0) = \deg(H_1, B_R^K, 0) \quad \text{for all } R \geq R_0.$$

Now we computing $\deg(H_1, B_R^K, 0)$. Let the homotopy $H_t' : [0,1] \times k \to Z^*$ be defined by

$$\begin{aligned} \langle H_t'(u,v), (\xi,\zeta)\rangle &= \int_\Omega (|\nabla u|^{p-2} \nabla u \nabla \xi + |\nabla v|^{p-2} \nabla v \nabla \zeta) dx \\ &\quad - t \int_\Omega (m(x)\xi + m(x)\zeta) dx - \int_\Omega ((k+\epsilon) u^{p-1}\xi + (k+\epsilon) v^{p-1}\zeta) dx \end{aligned}$$

for all $(\xi,\zeta) \in Z$, $t \in [0,1]$, $m(x) \in L_+^\infty(\Omega) = \{u(x) \in L^\infty(\Omega) | u(x) \geq 0, \forall x \in \Omega\}$. Clearly, it is a $(S)_+$ homotopy. So we have

$$\deg(H_1, B_R^K, 0) = \deg(H_t', B_R^K, 0).$$

Similarly, we prove the claim concerning the homotopy H_t'. By the homotopy invariance of the degree map, we have

$$\deg(H_1, B_R^K, 0) = \deg(H_1', B_R^K, 0).$$

Next, we show that $\deg(H_1', B_R^K, 0) = 0$. If $\deg(H_1', B_R^K, 0) \neq 0$, there exits $(u,v) \in B_R^K$ such that

$$\begin{aligned} &\int_\Omega (|\nabla u|^{p-2} \nabla u \nabla \xi + |\nabla v|^{p-2} \nabla v \nabla \zeta) dx \\ &= \int_\Omega (m(x)\xi + m(x)\zeta) dx + \int_\Omega ((k+\epsilon) u^{p-1}\xi + (k+\epsilon) v^{p-1}\zeta) dx. \end{aligned}$$

Clearly $(u,v) \neq (0,0)$, let $(\xi,\zeta) = (u,v)$, then

$$\int_\Omega (|\nabla u|^p dx + |\nabla v|^p dx) \geq (k+\epsilon) \int_\Omega (u^p + v^p) dx,$$

We take $k = \frac{\|(u,v)\|_Z^p}{\|(u,v)\|_{L^p\times L^p}^p}$, which provides a contradiction. Therefore

$$\deg(A, B_R^K, 0) = \deg(H_1', B_R^K, 0) = 0.$$

So (3.1) holds. Then, note that

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx},$$

we have $\|u\|^p_{W_0^{1,p}(\Omega)} \geq \lambda_1 \|u\|^p_{L^p(\Omega)}$. By(ii), we get

$$\begin{aligned}\langle A(u,v),(u,v)\rangle &= \|(u,v)\|_Z^p - \int_\Omega (f(x,u,v)u + g(x,u,v)v)dx \\ &\geq \frac{\epsilon'}{\lambda_1}(\|u\|^p + \|v\|^p) - c_4(\|u\|^\theta + \|v\|^\theta),\end{aligned}$$

where $c_4 > 0$. Since $\theta > p$, there exist $r > 0$ such that

$$\langle A(u,v),(u,v)\rangle > 0,$$

for all $(u,v) \in \partial B_r^K$, where $B_r^K = \{(u,v) \in K | \|(u,v)\|_Z < r\}$. In view of Lemma 2.3, there exists sufficiently small $r > 0$ such that

$$\deg(A, B_r^K, 0) = 1.$$

According to the (3.1), we can take $R > r$ such that

$$\deg(A, B_R^K, 0) = 0.$$

Since the domain additivity of type $(S)_+$, we obtain

$$\deg(A, B_R^K \backslash B_r^K, 0) = -1.$$

So we are led to the existence of $(u,v) \in B_{r,R}^K = \{(u,v) \in K | r < \|(u,v)\|_Z < R\}$ such that $A(u,v) = 0$. Hence, system(1.1) has a positive solution. □

Acknowledgments. The authors would like to thank the anonymous referee for his/her valuable suggestions.

References

[1] J. Zhang; *Existence results for the positive solutions of nonlinear elliptic systems*, Appl. Math. Com. 153 (2004) 833-842.
[2] A. Djellit, S. Tas; *Quasilinear elliptic systems with critical Sobolev exponents in $\mathbb{R}^N$*, Nonlinear Anal. 66 (2007) 1485-1497.
[3] A. Djellit, S. Tas; *On some nonlinear elliptic systems*, Nonlinear Anal. 59 (2004) 695-706.
[4] P. Drabek, N. M. Stavrakakis, N. B. Zographopoulos; *Multiple nonsemitrivial solutions for quasilinear systems*, Differential Integral Equations 16 (12) (2003) 1519-1531.
[5] D. D. Hai, H. wang; *Nontrivial solutions for p-Laplacian systems*, J. Math. Anal. Appl 330 (2007) 186-194.
[6] A. Djellit, S. Tas; *Existence of solutions for a class of elliptic systems in $\mathbb{R}^N$ involving the p-Laplacian*, Electronic J. Diff. Eqns. Vol. 2003 (2003), No. 56, 1-8.
[7] F. Browder; *Fixed point theory and nonlinear problems*, Bull. Amer. Math. Soc. (N.S.) 9 (1983) 1-39.
[8] H. Lu, D.O. Regna, R.P. Agrarwa; *On the existence of multiple periodic solutions for the vector p-Laplacian via critical point theory*, Applications of Math. 6 (2005) 555-568.
[9] H. Aman; *A note on degree theory for gradient mappings*, Proc. Amer. Math. Soc 85(1982) 591-595.
[10] E. Zeidler; *Nonlinear Function Analysis and its applications II B*, Springer-Verlag, New York, 1990.

Xudong Shang
Department of Mathematics, Nanjing Normal University Taizhou College, Taizhou 225300, Jiangsu, China
E-mail address: xudong-shang@163.com

Jihui Zhang
Institute of Mathematics, School of Mathematics and Computer Sciences, Nanjing Normal University, Nanjing 210097, Jiangsu, China
E-mail address: jihuiz@jlonline.com