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EXISTENCE OF SOLUTIONS FOR SECOND-ORDER NONLINEAR IMPULSIVE BOUNDARY-VALUE PROBLEMS

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Abstract. We prove the existence of solutions for a second-order nonlinear impulsive boundary-value problem by applying Schaefer's fixed point theorem. Results for periodic and anti-periodic impulsive boundary-value problems can be obtained as special cases of the results in this article.

1. Introduction

Impulsive boundary-value problems have been extensively studied in recent years. The study of impulsive differential equations provide a natural description of observed evolution processes of several real world problems in biology, physics, engineering, etc. For the general theory of impulsive differential equations, we refer the reader to [6, 11, 12, 16]. Some recent results for periodic and anti-periodic nonlinear impulsive boundary-value problems can be found in [1, 2, 3, 4, 5, 8, 9, 10, 13, 14, 15]. Bai and Yang [2] applied Schaefer's fixed point theorem to establish the existence of solutions for second-order nonlinear impulsive differential equations with periodic boundary conditions. Motivated by the studies in [2], we study the existence of solutions for the impulsive nonlinear boundary-value problem

$$\begin{gathered} u''(t) = f(t, u(t), u'(t)), \quad t \in [0, T],\ t \neq t_1, \\ u(t_1^+) - u(t_1^-) = I(u(t_1)), \quad u'(t_1^+) - u'(t_1^-) = J(u(t_1)), \\ u(0) = \mu u(T), \quad u'(0) = \mu u'(T), \end{gathered} \tag{1.1}$$

where $f : [0, T]\backslash\{t_1\} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $I, J : \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions defining the impulse at $t_1 \in (0, T)$ and μ is a fixed real number with $|\mu| \geq 1$. We assume that $f(t_1^+, x, y) = \lim_{t\to t_1^+} f(t, x, y)$ and $f(t_1^-, x, y) = \lim_{t\to t_1^-} f(t, x, y)$ both exist with $f(t_1^-, x, y) = f(t_1, x, y)$. For the sake of simplicity (as in [4]), we consider only one impulse at $t = t_1 \in (0, T)$. An arbitrary finite number of impulses can be addressed similarly.

We remark that the impulsive boundary-value problem (1.1) reduces to a periodic boundary-value problem [2] for $\mu = 1$ and anti-periodic boundary-value problem for $\mu = -1$. Thus, problem (1.1) can be regarded as a generalization of periodic and anti-periodic boundary-value problems.

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Let us define the Banach spaces

$$PC([0,T],\mathbb{R}^n)=\big\{u\in C([0,T]\backslash\{t_1\}\times\mathbb{R}^n),\ u \text{ is left continuous at } t=t_1, \\ \text{and the right hand limit } u(t_1^+) \text{ exists}\big\},$$

$$PC^1([0,T],\mathbb{R}^n)=\big\{u\in PC([0,T],\mathbb{R}^n),\ u' \text{ is left continuous at } t=t_1, \\ \text{and the right hand limit } u'(t_1^+) \text{ exists}\big\},$$

with the norms $\|u\|_{PC}=\sup_{t\in[0,T]}|u(t)|$, and $\|u\|_{PC^1}=\max\{\|u\|_{PC},\|u'\|_{PC}\}$, respectively.

A function $u\in PC^1([0,T],\mathbb{R}^n)\cap C^2([0,T]\backslash\{t_1\}\times\mathbb{R}^n)$ is a solution to (1.1) if it satisfies (1.1) for all $t\in[0,T]$.
For $\sigma\in PC([0,T],\mathbb{R}^n)$, $p\geq 0$, $q>0$, consider the linear impulsive problem

$$\begin{gathered} u''(t)-pu'(t)-qu(t)+\sigma(t)=0,\quad t\in[0,t],\ t\neq t_1,\\ u(t_1^+)-u(t_1^-)=I(u(t_1)),\quad u'(t_1^+)-u'(t_1^-)=J(u(t_1)),\\ u(0)=\mu u(T),\quad u'(0)=\mu u'(T),\quad \mu\in\mathbb{R}\quad(\mu\neq 0), \end{gathered} \tag{1.2}$$

whose associated auxiliary equation has the roots

$$r_1=\frac{p+\sqrt{p^2+4q}}{2},\quad r_2=\frac{p-\sqrt{p^2+4q}}{2}.$$

In view of $p\geq 0$, $q>0$, it is clear that r_1 and r_2 are respectively positive and negative real numbers. We need the following lemma for the sequel. The proof of this lemma is omitted as it can be obtained by direct computations.

Lemma 1.1. *$u\in PC^1([0,T],\mathbb{R}^n)\cap C^2([0,T]\backslash\{t_1\}\times\mathbb{R}^n)$ is a solution of (1.2) if and only if it satisfies the following impulsive integral equation*

$$u(t)=\int_0^T G(t,s)\sigma(s)ds-G(t,t_1)J(u(t_1))+W(t,t_1)I(u(t_1)), \tag{1.3}$$

where

$$G(t,s)=\frac{1}{r_1-r_2}\begin{cases}\frac{e^{r_1(t-s)}}{\mu e^{r_1T}-1}+\frac{e^{r_2(t-s)}}{1-\mu e^{r_2T}}, & 0\leq s<t\leq T,\\ \frac{\mu e^{r_1(T+t-s)}}{\mu e^{r_1T}-1}+\frac{\mu e^{r_2(T+t-s)}}{1-\mu e^{r_2T}}, & 0\leq t\leq s\leq T,\end{cases}$$

$$W(t,s)=\frac{1}{r_1-r_2}\begin{cases}\frac{r_2e^{r_1(t-s)}}{\mu e^{r_1T}-1}+\frac{r_1e^{r_2(t-s)}}{1-\mu e^{r_2T}}, & 0\leq s<t\leq T,\\ \frac{\mu r_2e^{r_1(T+t-s)}}{\mu e^{r_1T}-1}+\frac{\mu r_1e^{r_2(T+t-s)}}{1-\mu e^{r_2T}}, & 0\leq t\leq s\leq T,\end{cases}$$

with $(\mu e^{r_1T}-1)\neq 0$ and $(1-\mu e^{r_2T})\neq 0$.

As $r_1\geq -r_2>0$ $(p\geq 0, q>0)$, we find that

$$|G(t,s)|\leq|G_1|,\quad |W(t,s)|\leq r_1|G_1|,\quad |G_t(t,s)|\leq r_1|G_1|,\quad |W_t(t,s)|\leq r_1^2|G_1|, \tag{1.4}$$

where

$$G_1=\frac{\mu(e^{r_1T}-e^{r_2T})}{(r_1-r_2)(\mu e^{r_1T}-1)(1-\mu e^{r_2T})}.$$

Let

$$H=\max\{|G_1|,r_1|G_1|,r_1^2|G_1|\}. \tag{1.5}$$

Define an operator $\Lambda : PC^1([0,T],\mathbb{R}^n) \to PC([0,T],\mathbb{R}^n)$ by

$$\begin{aligned}\Lambda u(t) = \int_0^T & G(t,s)[-f(s,u(s),u'(s)) + pu'(s) + qu(s)]ds \\ & - G(t,t_1)J(u(t_1)) + W(t,t_1)I(u(t_1)), \quad t \in [0,T].\end{aligned} \tag{1.6}$$

It follows by Lemma 1.1 that u is a fixed point of the operator Λ if and only if u is a solution of (1.1).

In view of the continuity of f, I, J, the operators Λ_1, Λ_2 defined by

$$\begin{gathered}\Lambda_1 u(t) = \int_0^T G(t,s)\Big[-f(s,u(s),u'(s)) + pu'(s) + qu(s)\Big]ds, \quad t \in [0,T], \\ \Lambda_2 u(t) = -G(t,t_1)J(u(t_1)) + W(t,t_1)I(u(t_1)), \quad t \in [0,T],\end{gathered}$$

are compact. Thus, $\Lambda = \Lambda_1 + \Lambda_2$ is a compact operator.

2. Existence of solutions

Theorem 2.1. *Let $f : [0,T]\backslash\{t_1\} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $I, J : \mathbb{R}^n \to \mathbb{R}^n$ be continuous functions. If there exist nonnegative constants $\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, M$ such that*

- (A1) *For all $(t,x,y) \in ([0,T]\backslash\{t_1\}) \times \mathbb{R}^n \times \mathbb{R}^n$,*
$$\|f(t,x,y) - py - qx\| \le 2\alpha[\langle x+y, f(t,x,y)\rangle + \|y\|^2] + M,$$
- (A2) $\|I(x)\| \le \beta_1\|x\| + \gamma_1$, $\|J(x)\| \le \beta_2\|x\| + \gamma_2$ *with $r_1\beta_1 + \beta_2 < 1/H$, for all $x \in \mathbb{R}^n$.*

Then problem (1.1) *has at least one solution.*

Proof. From the preceding section, we know that u is a fixed point of the operator Λ if and only if u is a solution of (1.1). Thus we need to show that the operator Λ (indeed compact) has at least one fixed point. For that, we apply Schaefer's theorem to show that all the solutions to the following equation are bounded a priori with the bound independent of λ,

$$u = \Lambda\lambda u, \quad \lambda \in (0,1). \tag{2.1}$$

Letting u to be a solution of (2.1), we have

$$\begin{gathered}u''(t) - pu'(t) - qu(t) = \lambda[f(t,u(t),u'(t)) - pu'(t) - qu(t)], \quad t \in [0,T],\ t \ne t_1, \\ u(t_1^+) - u(t_1^-) = \lambda I(u(t_1)), \quad u'(t_1^+) - u'(t_1^-) = \lambda J(u(t_1)), \\ u(0) = \mu u(T), \quad u'(0) = \mu u'(T), \quad \mu \in \mathbb{R} \quad (|\mu| \ge 1).\end{gathered}$$

Using (A1)-(A2) and (1.4)-(1.5), we have

$$
\begin{aligned}
&\|u(t)\|\\
&= \lambda\|\Lambda u(t)\|\\
&= \| \int_0^T \lambda G(t,s)\Big[f(s,u(s),u'(s)) - pu'(s) - qu(s)\Big]ds\\
&\quad - \lambda G(t,t_1)J(u(t_1)) + \lambda W(t,t_1)I(u(t_1))\|\\
&\le |G_1|\Big[\int_0^T \lambda\|f(s,u(s),u'(s)) - pu'(s) - qu(s)\|ds\\
&\quad + \lambda(\|J(u(t_1))\| + r_1\|I(u(t_1))\|)\Big]\\
&\le |G_1|\Big[\int_0^T (2\alpha(\langle u(s)+u'(s), \lambda f(s,u(s),u'(s))\rangle + \|u'\|^2) + M)ds\\
&\quad + (r_1\beta_1+\beta_2)\|u(t_1)\| + r_1\gamma_1 + \gamma_2\Big]\\
&= |G_1|\Big[\int_0^T (2\alpha(\langle u(s)+u'(s), \lambda f(s,u(s),u'(s)) + (1-\lambda)pu'(s)\\
&\quad + (1-\lambda)qu(s)\rangle + \|u'\|^2) + M)ds - \int_0^T 2\alpha\langle u(s)+u'(s), (1-\lambda)pu'(s)\\
&\quad + (1-\lambda)qu(s)\rangle ds + (r_1\beta_1+\beta_2)\|u(t_1)\| + r_1\gamma_1 + \gamma_2\Big].
\end{aligned}
\tag{2.2}
$$

In view of the fact that $|\mu| \ge 1$, we have

$$
\begin{aligned}
&-2\alpha\int_0^T \langle u(s)+u'(s), (1-\lambda)pu'(s) + (1-\lambda)qu(s)\rangle ds\\
&= -2\alpha(1-\lambda)q\int_0^T \|u(s)\|^2ds - 2\alpha(1-\lambda)p\int_0^T \|u'(s)\|^2ds\\
&\quad + 2\alpha(1-\lambda)(p+q)\int_0^T \langle u(s), u'(s)\rangle ds\\
&\le 2\alpha(1-\lambda)(p+q)\int_0^T \langle u(s), u'(s)\rangle ds\\
&= \alpha(1-\lambda)(p+q)\int_0^T \frac{d}{ds}(\|u(s)\|^2)ds\\
&= \alpha(1-\lambda)(p+q)(\|u(T)\|^2 - \|u(0)\|^2)\\
&\le \alpha(1-\lambda)(p+q)(1-\mu^2)\|u(T)\|^2 \le 0.
\end{aligned}
\tag{2.3}
$$

Using (2.3) in (2.2), we obtain

$$
\begin{aligned}
&\|u(t)\|\\
&= \lambda\|\Lambda u(t)\|\\
&\le |G_1|\Big[\int_0^T (2\alpha(\langle u(s)+u'(s), \lambda f(s,u(s),u'(s)) + (1-\lambda)pu'(s)\\
&\quad + (1-\lambda)qu(s)\rangle + \|u'(s)\|^2) + M)ds + (r_1\beta_1+\beta_2)\|u(t_1)\| + r_1\gamma_1 + \gamma_2\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= |G_1|\Big[\int_0^T (2\alpha(\langle u(s)+u'(s),u''(s)\rangle + \langle u(s)+u'(s),u'(s)\rangle \\
&\quad - \langle u(s),u'(s)\rangle) + M)ds + (r_1\beta_1+\beta_2)\|u(t_1)\| + r_1\gamma_1+\gamma_2\Big] \\
&\le |G_1|\Big[\int_0^T (2\alpha(\langle u(s)+u'(s),u''(s)+u'(s)\rangle + M)ds \\
&\quad + (r_1\beta_1+\beta_2)\|u(t_1)\| + r_1\gamma_1+\gamma_2\Big] \\
&= |G_1|\Big[\int_0^T (\alpha\frac{d}{ds}(\|u(s)+u'(s)\|^2) + M)ds + (r_1\beta_1+\beta_2)\|u(t_1)\| + r_1\gamma_1+\gamma_2\Big] \\
&= |G_1|\Big[\alpha(\|u(T)+u'(T)\|^2 - \|u(0)+u'(0)\|^2) + MT \\
&\quad + (r_1\beta_1+\beta_2)\|u(t_1)\| + r_1\gamma_1+\gamma_2\Big] \\
&= |G_1|[\alpha(1-\mu^2)\|u(T)+u'(T)\|^2 + MT + (r_1\beta_1+\beta_2)\|u(t_1)\| + \gamma_1+\gamma_2\Big] \\
&\le |G_1|\Big[MT + (r_1\beta_1+\beta_2)\|u(t_1)\| + r_1\gamma_1+\gamma_2\Big],
\end{aligned}
$$

where we have used the fact that $\alpha(1-\mu^2)\|u(T)+u'(T)\|^2 \le 0$ (by the assumption $|\mu|\ge 1$). Taking supremum on $[0,T]$, we obtain

$$
\sup_{t\in[0,T]}\|u(t)\| \le \frac{|G_1|[MT+r_1\gamma_1+\gamma_2]}{1-|G_1|(r_1\beta_1+\beta_2)}.
$$

Similarly, it can be shown that

$$
\sup_{t\in[0,T]}\|u'(t)\| \le \frac{H[MT+r_1\gamma_1+\gamma_2]}{1-H(r_1\beta_1+\beta_2)}.
$$

Thus, we have

$$
\begin{aligned}
\|u\|_{PC^1} &= \max\{\frac{|G_1|[MT+r_1\gamma_1+\gamma_2]}{1-|G_1|(r_1\beta_1+\beta_2)}, \frac{H[MT+r_1\gamma_1+\gamma_2]}{1-H(r_1\beta_1+\beta_2)}\} \\
&= \frac{H[MT+r_1\gamma_1+\gamma_2]}{1-H(r_1\beta_1+\beta_2)},
\end{aligned}
$$

which is the desired bound independent of λ. Hence, by Schaefer's fixed point theorem [7], the operator Λ has at least one fixed point which implies that the problem (1.1) has at least one solution. This completes the proof. □

Example. Consider the scalar nonlinear impulsive problem

$$
\begin{gathered}
u''(t) = (u(t)+u'(t))^3 + u'(t) + 2u(t) + 2t, \quad t\in[0,1],\ t\ne t_1, \\
u(t_1^+) - u(t_1^-) = \frac{1}{6}u(t_1), \quad u'(t_1^+) - u'(t_1^-) = \frac{1}{8}u(t_1), \\
u(0) = \mu u(T), \quad u'(0) = \mu u'(T), \quad \mu\in\mathbb{R} \quad (|\mu|\ge 1).
\end{gathered}
\tag{2.4}
$$

Here, $T=1$, $f(t,x,y) = (x+y)^3 + y + 2x + 2t$, $p=1$, $q=2$, $r_1=2$, $r_2=-1$, $\beta_1 = 1/6$, $\beta_2 = 1/8$, $\gamma_1=\gamma_2=0$, $1/H = 0.3$. Moreover, for $\alpha = 2/3$, $M = 8/3$, we find that

$$
\begin{aligned}
&2\alpha[(x+y)f(t,x,y) + y^2] + M \\
&= \frac{4}{3}[(x+y)^4 + (x+y)^2 + x(x+y) + 2t(x+y) + y^2] + \frac{8}{3}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{4}{3}[(x+y)^4 + (x+y)^2] + \frac{4}{3}(x + \frac{1}{2}y)^2 + \frac{8}{3}t(x+y) + y^2 + \frac{8}{3} \\
&\geq \frac{4}{3}[(x+y)^4 + (x+y)^2] + \frac{4}{3}(x + \frac{1}{2}y)^2 - \frac{8}{3}|x+y| + y^2 + \frac{8}{3} \\
&= \frac{4}{3}[(x+y)^4 + (|x+y|-1)^2 + (x + \frac{1}{2}y)^2] + y^2 + \frac{4}{3} \\
&\geq |x+y|^3 + y^2 + 1, \quad \forall (t,x,y) \in ([0,1]\backslash\{t_1\}) \times \mathbb{R} \times \mathbb{R}.
\end{aligned}
$$

Thus, for all $(t,x,y) \in ([0,1]\backslash\{t_1\}) \times \mathbb{R} \times \mathbb{R}$,

$$
|f(t,x,y) - 2x - y| \leq 2\alpha[(x+y)f(t,x,y) + y^2] + M.
$$

Hence, the assumptions (A1)-(A2) are satisfied. Therefore, by Theorem 2.1, problem (2.4) has at least one solution.

Remarks. (1) If the function f does not depend on $u'(t)$, then the assumption (A1) takes the form

$$
\|f(t,x) - qx\| \leq 2\alpha\langle x, f(t,x)\rangle + M, \quad (t,x) \in ([0,T]\backslash\{t_1\}) \times \mathbb{R}^n.
$$

For example, consider a scalar function

$$
f(t,x) = x^5 + x + 2t, \quad (t,x) \in ([0,1]\backslash\{t_1\}) \times \mathbb{R}.
$$

For $\alpha = 1/2$, $M = 2$, we obtain

$$
\begin{aligned}
2\alpha\langle x, f(t,x)\rangle + M &= x^6 + x^2 + 2tx + 2 \\
&\geq x^6 + x^2 - 2|x| + 2 \\
&= x^6 + (|x|-1)^2 + 1 \\
&\geq |x|^5 + 1, \quad \forall (t,x) \in ([0,1]\backslash\{t_1\}) \times \mathbb{R}.
\end{aligned}
$$

Thus, $|f(t,x) - x| \leq 2\alpha x f(t,x) + M$, for all $(t,x) \in [0,1] \times \mathbb{R}$.

(2) A similar proof follows for a modified form of Theorem 2.1 obtained by replacing the assumption (A1) by the condition

$$
\|f(t,x,y) - py - qx\| \leq 2\alpha\langle y, f(t,x,y)\rangle + M, \quad (t,x,y) \in ([0,T]\backslash\{t_1\}) \times \mathbb{R}^n \times \mathbb{R}^n.
$$

(3) The results presented in this paper are new and a variety of special cases can be recorded by fixing the value of μ. For instance, if we take $\mu = 1$ in the problem (1.1), the results for impulsive periodic boundary-value problems [2] appear as a special case while $\mu = -1$ in (1.1) yields the existence results for anti-periodic second order boundary-value problems.

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