Electronic Journal of Differential Equations, Vol. 2009(2009), No. 66, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

MULTIPLE NONNEGATIVE SOLUTIONS FOR SECOND-ORDER BOUNDARY-VALUE PROBLEMS WITH SIGN-CHANGING NONLINEARITIES

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Abstract. In this paper, we study the existence of multiple nonnegative solutions for second-order boundary-value problems of differential equations with sign-changing nonlinearities. Our main tools are the fixed-point theorem in double cones and the Leggett-Williams fixed point theorem. We present also the integral kernel associated with the boundary-value problem.

1. Introduction

Boundary-value problems with nonnegative solutions describe many phenomena in the applied science, and they are widely used in fields, such as chemistry, biological, etc.; see for example [2, 4, 5, 6, 7, 8]. Problems with integral boundary conditions have been applied in heat conduction, chemical engineering, underground water flow-elasticity, etc. The existence of nonnegative solutions to these problems have received a lot of attention; see [3, 8, 9, 10, 11, 12] and reference therein.

Recently, by constructing a special cone and using the fixed point index theory, Liu and Yan [9] proved the existence of multiple solutions to the singular boundary-value problem

$$
\begin{gathered}
(p(t)x'(t))' + \lambda f(t, x(t), y(t)) = 0 \\
(p(t)y'(t))' + \lambda g(t, x(t), y(t)) = 0 \\
\alpha x(0) - \beta x'(0) = \gamma x(1) + \delta x'(1) = 0 \\
\alpha y(0) - \beta y'(0) = \gamma y(1) + \delta y'(1) = 0,
\end{gathered}
$$

where the parameter λ in $\mathbb{R}^+$, $p \in C([0,1], \mathbb{R}^+)$, $\alpha, \beta, \gamma, \delta \geq 0$, $\beta\gamma + \alpha\delta + \alpha\gamma > 0$, $f \in C((0,1) \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+)$, $g \in C((0,1) \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$, but g must be controlled by f.

2000 *Mathematics Subject Classification.* 34B10, 34B18.
Key words and phrases. Nonnegative solutions; fixed-point theorem in double cones; integral kernel; integral boundary conditions.

Submitted December 8, 2008. Published May 14, 2009.
Supported by grant 05EZ53 from the Foundation of Educational Commission of Shanghai.

By using fixed point index theory in a cone, Yang [10] studied the existence of positive solutions to a system of second-order nonlocal boundary value problems

$$\begin{gathered} -u'' = f(t,u,v) \\ -v'' = g(t,u,v) \\ u(0) = v(0) = 0 \\ u(1) = H_1\Big(\int_0^1 u(\tau)d\alpha(\tau)\Big) \\ v(1) = H_2\Big(\int_0^1 v(\tau)d\beta(\tau)\Big), \end{gathered}$$

where α and β are increasing nonconstant functions defined on $[0,1]$ with $\alpha(0) = 0 = \beta(0)$ and $f, g \in C((0,1)\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$, $H_i \in C(\mathbb{R}^+,\mathbb{R}^+)$.

By using fixed point theory in a cone, Feng [12] studied positive solutions for the boundary-value problem, with integral boundary conditions in Banach spaces,

$$x'' + f(t,x) = 0$$

with

$$x(0) = \int_0^1 g(t)x(t)dt, \quad x(1) = 0$$

or

$$x(0) = 0, x(1) = \int_0^1 g(t)x(t)dt,$$

where $f \in C([0,1]\times P, P), g \in L^1[0,1]$, and P is a cone of E. All of these, we can find the nonlinear term f is nonnegative.

In this paper, by using the fixed point theorem in double cones and the Leggett-Williams fixed point theorem, we study the existence of multiple nonnegative solutions to the boundary value problem

$$\begin{gathered} u_1''(t) + f_1(t,u_1(t),u_2(t)) = 0 \\ u_2''(t) + f_2(t,u_1(t),u_2(t)) = 0 \\ u_1(0) = u_2(0) = 0 \\ u_1(1) = \int_0^1 g_1(s)u_1(s)ds, u_2(1) = \int_0^1 g_2(s)u_2(s)ds, \end{gathered} \tag{1.1}$$

where $f_1, f_2 \in C((0,1)\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R})$, and g_1, g_2 are nonnegative functions in $L^1[0,1]$.

In this paper we assume that the following conditions:

- (H1) $f_i \in C((0,1)\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R})$, $g_i \in L^1[0,1]$ is nonnegative, $i = 1,2$;
- (H2) $1 - \int_0^1 sg_i(s)ds > 0$;
- (H3) $f_1(t,0,u_2(t)) \geq 0 (\not\equiv 0)$, $f_2(t,u_1(t),0) \geq 0(\not\equiv 0)$, $t \in [0,1]$.

2. Preliminaries

Let X be a Banach space with norm $\|\cdot\|$ and $K \subset X$ be a cone. For a constant $r > 0$, denote $K_r = \{x \in K : \|x\| < r\}$, $\partial K_r = \{x \in K : \|x\| = r\}$. Suppose $\alpha : K \to \mathbb{R}^+$ is a continuously increasing functional; i.e. α is continuous and $\alpha(\lambda x) \leq \alpha(x)$ for $\lambda \in (0,1)$. Let

$$K(b) = \{x \in K : \alpha(x) < b\}, \partial K(b) = \{x \in K : \alpha(x) = b\}.$$

and $K_a(b) = \{x \in K : \|x\| > a, \alpha(x) < b\}$. $\phi : K \to \mathbb{R}^+$ is a continuously concave functional. Denote

$$K(\phi, a, b) = \{x \in K : \phi(x) \geq a, \|x\| \leq b\}.$$

We will use the following two theorem.

Theorem 2.1 ([1]). *Let X be a real Banach space with norm $\|\cdot\|$ and $K, K' \subset X$ two cones with $K' \subset K$. Suppose $T : K \to K$ and $T^* : K' \to K'$ are two completely continuous operators and $\alpha : K' \to \mathbb{R}^+$ is a continuously increasing functional satisfying $\alpha(x) \leq \|x\| \leq M\alpha(x)$ for all $x \in K'$, where $M \geq 1$ is a constant. If there are constants $b > a > 0$ such that*

- (C1) *$\|Tx\| < a$, for $x \in \partial K_a$;*
- (C2) *$\|T^*x\| < a$, for $x \in \partial K'_a$ and $\alpha(T^*x) > b$ for $x \in \partial K'(b)$;*
- (C3) *$Tx = T^*x$, for $x \in K'_a(b) \cap \{u : T^*u = u\}$.*

Then T has at least two fixed points y_1 and y_2 in K, such that

$$0 \leq \|y_1\| < a < \|y_2\|, \quad \alpha(y_2) < b.$$

Theorem 2.2 (Leggett-Williams fixed point theorem [13]). *Let $A : \overline{K_c} \to \overline{K_c}$ be completely continuous and ϕ be a nonnegative continuous concave functional on K such that $\phi(x) \leq \|x\|$ for all $x \in \overline{K_c}$. Suppose that there exist $0 < d < a < b \leq c$ such that*

- (C4) *$\{x \in K(\phi, a, b) : \phi(x) > a\} \neq \emptyset$ and $\phi(Ax) > a$ for $x \in K(\phi, a, b)$;*
- (C5) *$\|Ax\| < d$ for $\|x\| \leq d$;*
- (C6) *$\phi(Ax) > a$ for $x \in K(\phi, a, c)$ with $\|Ax\| > b$.*

Then A has at least three fixed points x_1, x_2, x_3 in $\overline{K_c}$ satisfying

$$\|x_1\| < d, \quad a < \phi(x_2), \quad \|x_3\| > d, \quad \phi(x_3) < a.$$

Lemma 2.3. *Assume that* (H2) *holds. Then for any $y_i \in C[0,1]$, the boundary value problem*

$$u_i''(t) + y_i(t) = 0 \tag{2.1}$$

$$u_i(0) = 0, u_i(1) = \int_0^1 g_i(s)u_i(s)ds, \tag{2.2}$$

has a unique solution

$$u_i(t) = \int_0^1 H_i(t,s)y_i(s)ds, \quad i = 1, 2, \tag{2.3}$$

where

$$H_i(t,s) = G(t,s) + \frac{t\int_0^1 g_i(r)G(r,s)dr}{1 - \int_0^1 sg_i(s)ds}, \quad i = 1, 2,$$

$$G(t,s) = \begin{cases} t(1-s), & \text{if } 0 \leq t \leq s \leq 1, \\ s(1-t), & \text{if } 0 \leq s \leq t \leq 1, \end{cases}$$

The proof is similar to [12, Lemma 2.1], and is omitted.

Lemma 2.4. *Assume that* (H2) *holds. Let $\delta \in (0, \frac{1}{2})$, then for all $t \in [\delta, 1-\delta], \sigma, s \in [0,1]$, we have*

$$H_i(\sigma, s) \geq 0, \quad H_i(t,s) \geq \delta H_i(\sigma, s). \tag{2.4}$$

Proof. It is clear that $H_i(\sigma, s) \geq 0$, From the properties of $G(t, s)$, we obtain

$$G(t, s) \geq \delta G(\sigma, s), \quad t \in [\delta, 1-\delta],\ \sigma, s \in [0, 1],$$

then

$$\begin{aligned} H_i(t, s) &= G(t, s) + \frac{t\int_0^1 g_i(r)G(r, s)dr}{1-\int_0^1 sg_i(s)ds} \\ &\geq \delta G(\sigma, s) + \frac{\delta\int_0^1 g_i(r)G(r, s)dr}{1-\int_0^1 sg_i(s)ds} \\ &\geq \delta G(\sigma, s) + \frac{\delta\sigma\int_0^1 g_i(r)G(r, s)dr}{1-\int_0^1 sg_i(s)ds} = \delta H_i(\sigma, s) \end{aligned}$$

The proof is complete. □

Lemma 2.5. *Assume that* (H2) *holds. If $y_i \in C[0, 1], y_i \geq 0$, then the unique solution $u_i(t)$ of the boundary-value problem* (2.1)-(2.2) *satisfies $u_i(t) \geq 0$ and $\min_{t\in[\delta,1-\delta]} u_i(t) \geq \delta\|u_i\|, i = 1, 2$.*

Proof. It is clear that $u_i(t) \geq 0$, for all $t \in [0, 1]$, $i = 1, 2$. In fact, from (2.3) and (2.4), for any $t \in [\delta, 1-\delta], s, \sigma \in [0, 1], i = 1, 2$, we have

$$u_i(t) = \int_0^1 H_i(t, s)y_i(s)ds \geq \int_0^1 \delta H_i(\sigma, s)y_i(s)ds = \delta u_i(\sigma).$$

Hence,

$$u_i(t) \geq \delta \max_{0\leq\sigma\leq1} |u_i(\sigma)| = \delta\|u_i\|,$$

and $\min_{\delta\leq t\leq1-\delta} u_i(t) \geq \delta\|u_i\|$. The proof is complete. □

Let $X = C[0, 1] \times C[0, 1]$ with the norm $\|(u_1, u_2)\| := \|u_1\| + \|u_2\|$, $K = \{(u_1, u_2) \in X : u_i \geq 0, i = 1, 2\}$ and $K' = \{(u_1, u_2) \in K\colon\ u_i(t)$ is concave in $[0, 1], \min_{t\in[\delta,1-\delta]} u_i(t) \geq \delta\|u_i\|, i = 1, 2\}$.

Clearly, $K, K' \subset X$ are cones with $K' \subset K$. Let $T_i : K \to C[0, 1], i = 1, 2$ be defined by

$$T_1(u_1, u_2)(t) = \Big(\int_0^1 H_1(t, s)f_1(s, u_1(s), u_2(s))ds\Big)^+, \quad t \in [0, 1],$$
$$T_2(u_1, u_2)(t) = \Big(\int_0^1 H_2(t, s)f_2(s, u_1(s), u_2(s))ds\Big)^+, \quad t \in [0, 1],$$

where $(B)^+ = \max\{B, 0\}$. Let

$$T(u_1, u_2)(t) = (T_1(u_1, u_2)(t), T_2(u_1, u_2)(t)),$$
$$A_1(u_1, u_2)(t) = \int_0^1 H_1(t, s)f_1(s, u_1(s), u_2(s))ds, \quad t \in [0, 1],$$
$$A_2(u_1, u_2)(t) = \int_0^1 H_2(t, s)f_2(s, u_1(s), u_2(s))ds, \quad t \in [0, 1],$$
$$A(u_1, u_2)(t) = (A_1(u_1, u_2)(t), A_2(u_1, u_2)(t)).$$

For $(u_1, u_2) \in X$, define $\theta : X \to K$ by

$$(\theta(u_1, u_2))(t) = (\max\{u_1(t), 0\}, \max\{u_2(t), 0\}),$$

then $T = \theta \circ A$.

Let $T_i^*: K' \to C[0,1], i = 1,2$ be defined by

$$\begin{aligned} T_1^*(u_1,u_2)(t) &= \int_0^1 H_1(t,s) f_1^+(s,u_1(s),u_2(s))ds, \quad t \in [0,1], \\ T_2^*(u_1,u_2)(t) &= \int_0^1 H_2(t,s) f_2^+(s,u_1(s),u_2(s))ds, \quad t \in [0,1], \end{aligned} \tag{2.5}$$

and

$$T^*(u_1,u_2)(t) = (T_1^*(u_1,u_2)(t), T_2^*(u_1,u_2)(t)).$$

Define $\alpha : K' \to R^+$ by

$$\alpha(u_1,u_2) = \min_{\delta \le t \le 1-\delta} u_1(t) + \min_{\delta \le t \le 1-\delta} u_2(t).$$

It is clear that α is a continuous increasing functional and $\alpha(u_1,u_2) \le \|(u_1,u_2)\|$. For $u \in K'$, we have

$$\alpha(u_1,u_2) = \min_{\delta \le t \le 1-\delta} u_1(t) + \min_{\delta \le t \le 1-\delta} u_2(t) \ge \delta\|u_1\| + \delta\|u_2\| = \delta\|(u_1,u_2)\|.$$

Therefore,

$$\alpha(u_1,u_2) \le \|(u_1,u_2)\| \le \frac{1}{\delta}\alpha(u_1,u_2).$$

Lemma 2.6. *Suppose $A : K \to X$ is completely continuous. Then $\theta \circ A : K \to K$ is also a completely continuous operator.*

Proof. The complete continuity of A implies that A is continuous and maps each bounded subset of K to a relatively compact set of X. Let $D \subset K$ be a bounded set, for any $\epsilon > 0$, there exist $P_i(x_i,y_i) \in X, i = 1,2,\dots,m$, such that

$$AD \subset \cup_{i=1}^m B(P_i,\epsilon),$$

where $B(P_i,\epsilon) := \{(u_1,u_2) \in K : \|u_1 - x_i\| + \|u_2 - y_i\| < \epsilon\}$. Then for any $Q^*(x_Q^*,y_Q^*) \in (\theta \circ A)(D)$, there exists $Q(x_Q,y_Q) \in AD$, such that

$$(x_Q^*,y_Q^*) = (\max\{x_Q,0\}, \max\{y_Q,0\}).$$

We choose a $P_i \in \{P_1,P_2,\dots,P_m\}$, such that

$$\|x_Q - x_i\| + \|y_Q - y_i\| < \epsilon.$$

Since

$$\|x_Q^* - x_i^*\| + \|y_Q^* - y_i^*\| \le \|x_Q - x_i\| + \|y_Q - y_i\| < \epsilon,$$

we have $Q^*(x_Q^*,y_Q^*) \in B(P_i^*,\epsilon)$, and so $(\theta \circ A)(D)$ is relatively compact.

For each $\epsilon > 0$, there exists $\eta > 0$, such that $\|A(x_1,y_1) - A(x_2,y_2)\| < \epsilon$, for $\|x_1 - x_2\| + \|y_1 - y_2\| < \eta$. Since

$$\begin{aligned} &\|(\theta \circ A)(x_1,y_1) - (\theta \circ A)(x_2,y_2)\| \\ &= \|\Big(\max\{A_1(x_1,y_1),0\} - \max\{A_1(x_2,y_2),0\}, \\ &\quad \max\{A_2(x_1,y_1),0\} - \max\{A_2(x_2,y_2),0\}\Big)\| \\ &\le \|A(x_1,y_1) - A(x_2,y_2)\| < \epsilon. \end{aligned}$$

We have $\|(\theta \circ A)(x_1,y_1) - (\theta \circ A)(x_2,y_2)\| < \epsilon$, for $\|x_1 - x_2\| + \|y_1 - y_2\| < \eta$.

Hence, $\theta \circ A$ is continuous in K and $\theta \circ A$ is completely continuous. The proof is complete. □

Since f_i is continuous, it is clear that $A : K \to X$ and $T^* : K' \to X$ are completely continuous. From Lemmas 2.6 and 2.5, we have $T : K \to K$ and $T^* : K' \to K'$ are completely continuous.

Lemma 2.7. *If (u_1, u_2) is a fixed point of T, then (u_1, u_2) is a fixed point of A.*

Proof. Suppose (u_1, u_2) is a fixed point of T, obviously, we just need to prove that $A_i(u_1, u_2)(t) \geq 0$, $i = 1, 2$, for $t \in [0, 1]$.

If there exist $t_0 \in (0, 1)$ and an i ($i = 1, 2$) such that $u_i(t_0) = T_i(u_1, u_2)(t_0) = 0$ but $A_i(u_1, u_2)(t_0) < 0$. Without loss of generalization, let $i = 1$ and (t_1, t_2) be the maximal interval and contains t_0 such that $A_1(u_1, u_2)(t) < 0$ for all $t \in (t_1, t_2)$. Obviously, $(t_1, t_2) \neq (0, 1)$. Or else, $T_1(u_1, u_2)(t) = u_1(t) = 0$, for all $t \in [0, 1]$. This is in contradiction with (H3).

Case i: If $t_2 < 1$, then $A_1(u_1, u_2)(t_2) = 0$. Thus, $A_1'(u_1, u_2)(t_2) \geq 0$, We obtain

$$A_1''(u_1, u_2)(t) = -f_1(t, 0, u_2) \leq 0, \quad \text{for } t \in (t_1, t_2).$$

So

$$A_1'(u_1, u_2)(t) \geq 0, \quad \text{for } t \in [t_1, t_2]$$

We obtain $t_1 = 0$, and $A_1'(u_1, u_2)(0) \geq 0$, $A_1(u_1, u_2)(0) < 0$. This is in contradiction with $A_1(u_1, u_2)(0) = 0$.

Case ii: If $t_1 > 0$, we have $A_1(u_1, u_2)(t_1) = 0$. Thus $A_1'(u_1, u_2)(t_1) \leq 0$. We obtain

$$A_1''(u_1, u_2)(t) = -f_1(t, 0, u_2) \leq 0, \quad \text{for } t \in (t_1, t_2).$$

So

$$A_1'(u_1, u_2)(t) < 0, \quad \text{for } t \in [t_1, t_2].$$

We obtain $t_2 = 1, A_1'(u_1, u_2)(1) \leq 0$.

On the other hand, $A_1(u_1, u_2)(t) < 0$, for $t \in (t_1, t_2), A_1'(u_1, u_2)(1) \leq 0$ imply $A_1(u_1, u_2)(1) < 0$. By (H1), $A_1(u_1, u_2)(1) = \int_0^1 g_1(s)u_1(s)ds \geq 0$. This is a contradiction. The proof is complete. □

3. Main result

Denote

$$M_i = \max_{t\in[0,1]} \int_0^1 H_i(t, s)ds, \quad m_i = \min_{t\in[\delta,1-\delta]} \int_\delta^{1-\delta} H_i(t, s)ds, i = 1, 2$$

Theorem 3.1. *Suppose that condition* (H1)–(H3) *hold. Assume that there exist positive numbers $\delta, a, b, \lambda_i, \mu_i$, $i = 1, 2$, such that $\delta \in (0, \frac{1}{2})$, $0 < a < \delta b < b$, $\lambda_1 + \lambda_2 \leq 1$, $\mu_1 + \mu_2 > 1$, and satisfy*

- (H4) *$f_i(t, u_1, u_2) \geq 0$, for $t \in [0, 1], u_1 + u_2 \in [0, b]$;*
- (H5) *$f_i(t, u_1, u_2) < \frac{\lambda_i a}{M_i}$, for $t \in [0, 1], u_1 + u_2 \in [0, a]$;*
- (H6) *$f_i(t, u_1, u_2) \geq \frac{\mu_i \delta b}{m_i}$, for $t \in [\delta, 1 - \delta], u_1 + u_2 \in [\delta b, b]$.*

Then, (1.1) *has at least two nonnegative solutions (u_1, u_2) and (u_1^*, u_2^*) such that $0 \leq \|(u_1, u_2)\| < a < \|(u_1^*, u_2^*)\|$, $\alpha(u_1^*, u_2^*) < \delta b$.*

Proof. For all $(u_1, u_2) \in \partial K_a$, from (H5) we have

$$\begin{aligned}
\|T_i(u_1,u_2)\| &= \max_{t\in[0,1]} (\int_0^1 H_i(t,s) f_i(s,u_1(s),u_2(s))ds)^+ \\
&= \max_{t\in[0,1]} \max\{\int_0^1 H_i(t,s) f_i(s,u_1(s),u_2(s))ds, 0\} \\
&< \frac{\lambda_i a}{M_i} \max_{t\in[0,1]} \int_0^1 H_i(t,s)ds = \lambda_i a.
\end{aligned}$$

Therefore,

$$\|T(u_1,u_2)\| = \|T_1(u_1,u_2)\| + \|T_2(u_1,u_2)\| < \lambda_1 a + \lambda_2 a \le a.$$

So (C1) of Theorem 2.1 is satisfied.

For $(u_1, u_2) \in \partial K'_a$, from (H5), we have

$$\begin{aligned}
\|T_i^*(u_1,u_2)\| &= \max_{t\in[0,1]} \int_0^1 H_i(t,s) f_i^+(s,u_1(s),u_2(s))ds \\
&< \frac{\lambda_i a}{M_i} \max_{t\in[0,1]} \int_0^1 H_i(t,s)ds = \lambda_i a.
\end{aligned}$$

We also obtain

$$\|T_i^*(u_1,u_2)\| = \|T_1^*(u_1,u_2)\| + \|T_2^*(u_1,u_2)\| < \lambda_1 a + \lambda_2 a \le a.$$

For $(u_1, u_2) \in \partial K'(\delta b)$, i.e., $\alpha(u_1, u_2) = \delta b$, For $t \in [\delta, 1-\delta]$, by Lemma 2.5, we have $\delta b \le u_1(t) + u_2(t) \le b$. From (H6), we obtain

$$\begin{aligned}
\alpha(T^*(u_1,u_2)) &= \min_{t\in[\delta,1-\delta]} \int_0^1 H_1(t,s) f_1^+(s,u_1(s),u_2(s))ds \\
&\quad + \min_{t\in[\delta,1-\delta]} \int_0^1 H_2(t,s) f_2^+(s,u_1(s),u_2(s))ds \\
&\ge \min_{t\in[\delta,1-\delta]} \int_\delta^{1-\delta} H_1(t,s) f_1^+(s,u_1(s),u_2(s))ds \\
&\quad + \min_{t\in[\delta,1-\delta]} \int_\delta^{1-\delta} H_2(t,s) f_2^+(s,u_1(s),u_2(s))ds \\
&\ge \frac{\mu_1 \delta b}{m_1} \min_{t\in[\delta,1-\delta]} \int_\delta^{1-\delta} H_1(t,s)ds + \frac{\mu_2 \delta b}{m_2} \min_{t\in[\delta,1-\delta]} \int_\delta^{1-\delta} H_2(t,s)ds \\
&= \mu_1 \delta b + \mu_2 \delta b > \delta b.
\end{aligned}$$

Therefore (C2) of Theorem 2.1 is satisfied.

Finally, we show that (C3) of Theorem 2.1 is satisfied. Let $(u_1, u_2) \in K'_a(\delta b) \cap \{(u_1, u_2) : T^*(u_1, u_2) = (u_1, u_2)\}$, we have

$$\alpha(u_1,u_2) < \delta b, \|(u_1,u_2)\| > a.$$

From Lemma 2.5, we know that

$$\begin{gathered}
\|(u_1,u_2)\| \le \frac{1}{\delta}\alpha(u_1,u_2) < b, \\
0 \le u_1(t) + u_2(t) < b.
\end{gathered}$$

From (H4), we obtain

$$f_i^+(s, u_1(s), u_2(s)) = f_i(s, u_1(s), u_2(s)).$$

This implies that $T(u_1, u_2) = T^*(u_1, u_2)$ for

$$(u_1, u_2) \in K'_a(\delta b) \cap \{(u_1, u_2) : T^*(u_1, u_2) = (u_1, u_2)\}.$$

By Theorem 2.1 and Lemma 2.7, we know that (1.1) has at least two nonnegative solutions (u_1, u_2) and (u_1^*, u_2^*) such that

$$0 \le \|(u_1, u_2)\| < a < \|(u_1^*, u_2^*)\|, \alpha(u_1^*, u_2^*) < b.$$

The proof is complete. □

Define $\phi : K \to R^+$ by

$$\phi(u_1, u_2) = \min_{\delta \le t \le 1-\delta} u_1(t) + \min_{\delta \le t \le 1-\delta} u_2(t)$$

Theorem 3.2. *Suppose that condition* (H1)–(H3) *hold. There exist* $\delta \in (0, \frac{1}{2})$, $a, b, \lambda_i, \mu_i > 0$, $i = 1, 2$, *such that* $0 < a < \delta b < b$, $\lambda_1 + \lambda_2 \le 1, \mu_1 + \mu_2 > 1$, *and* (H5), (H6) *hold, and satisfy*

- (H7) $f_i(t, u_1, u_2) \ge 0$, *for* $t \in [0, 1]$, $u_1 + u_2 \in [\delta b, b]$.
- (H8) $f_i(t, u_1, u_2) \le \frac{\lambda_i b}{M_i}$, *for* $t \in [0, 1], u_1 + u_2 \in [0, b]$.

Then, (1.1) *has at least three nonnegative solutions* (u_1, u_2), (u_1^*, u_2^*), (u_1^{**}, u_2^{**}), *such that* $0 \le \|(u_1, u_2)\| < a < \|(u_1^*, u_2^*)\|, \phi(u_1^*, u_2^*) < b, \phi(u_1^{**}, u_2^{**}) \ge b$.

Proof. Firstly, we prove $T : \overline{K_b} \to \overline{K_b}$ is a completely continuous operator. From (H8), for $i = 1, 2$, we obtain

$$\begin{aligned} \|T_i(u_1, u_2)\| &= \max_{t \in [0,1]} \Big(\int_0^1 H_i(t, s) f_i(s, u_1(s), u_2(s)) ds \Big)^+ \\ &= \max_{t \in [0,1]} \max \Big\{ \int_0^1 H_i(t, s) f_i(s, u_1(s), u_2(s)) ds, 0 \Big\} \\ &< \frac{\lambda_i b}{M_i} \max_{t \in [0,1]} \int_0^1 H_i(t, s) ds = \lambda_i b. \end{aligned}$$

Therefore,

$$\|T(u_1, u_2)\| = \|T_1(u_1, u_2)\| + \|T_2(u_1, u_2)\| < \lambda_1 b + \lambda_2 b \le b.$$

From Lemma 2.6, we know $T : \overline{K_b} \to \overline{K_b}$ is a completely continuous operator. For the operator T and any $u_1 + u_2 \in [0, a]$, from (H5) and Theorem 3.1, we know (C5) of Theorem 2.2 is satisfied.

Next, we show that (C4) of Theorem 2.2 is satisfied. Clearly,

$$\{(u_1, u_2) \in K(\phi, \delta b, b) : \phi(u_1, u_2) > \delta b\} \ne \emptyset.$$

Assume $(u_1,u_2)\in K(\phi,\delta b,b)$, for any $t\in[\delta,1-\delta]$, we have $\delta b\le u_1+u_2\le b$. From (H6) and (H7) we obtain

$$\begin{aligned}
\phi(T(u_1,u_2)) &= \min_{t\in[\delta,1-\delta]}\Big(\int_0^1 H_1(t,s)f_1(s,u_1(s),u_2(s))ds\Big)^+\\
&\quad+\min_{t\in[\delta,1-\delta]}\Big(\int_0^1 H_2(t,s)f_2(s,u_1(s),u_2(s))ds\Big)^+\\
&\ge \min_{t\in[\delta,1-\delta]}\int_0^1 H_1(t,s)f_1(s,u_1(s),u_2(s))ds\\
&\quad+\min_{t\in[\delta,1-\delta]}\int_0^1 H_2(t,s)f_2(s,u_1(s),u_2(s))ds\\
&\ge \min_{t\in[\delta,1-\delta]}\int_\delta^{1-\delta} H_1(t,s)f_1(s,u_1(s),u_2(s))ds\\
&\quad+\min_{t\in[\delta,1-\delta]}\int_\delta^{1-\delta} H_2(t,s)f_2(s,u_1(s),u_2(s))ds\\
&\ge \frac{\mu_1\delta b}{m_1}\min_{t\in[\delta,1-\delta]}\int_\delta^{1-\delta} H_1(t,s)ds+\frac{\mu_2\delta b}{m_2}\min_{t\in[\delta,1-\delta]}\int_\delta^{1-\delta} H_2(t,s)ds\\
&= \mu_1\delta b+\mu_2\delta b>\delta b.
\end{aligned}$$

Finally, for $(u_1,u_2)\in K(\phi,\delta b,b)$ and $\|T(u_1,u_2)\|>b$, it is easy to prove that

$$\phi(T(u_1,u_2))\ge\delta\|T(u_1,u_2)\|>\delta b.$$

Then (C6) of Theorem 2.2 is satisfied. Therefore from Theorem 2.2 and Lemma 2.7 we know that (1.1) has at least three nonnegative solutions (u_1,u_2), (u_1^*,u_2^*), (u_1^{**},u_2^{**}), such that

$$0\le\|(u_1,u_2)\|<a<\|(u_1^*,u_2^*)\|,\quad \alpha(u_1^*,u_2^*)<b,\quad \alpha(u_1^{**},u_2^{**})\ge b.$$

The proof is complete. □

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