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NECESSARY AND SUFFICIENT CONDITIONS FOR THE OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO NEUTRAL DELAY DYNAMIC EQUATIONS

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Abstract. This article concerns the asymptotic behaviour of solutions to nonlinear first-order neutral delay dynamic equations involving coefficients with opposite signs. We present necessary and sufficient conditions for the solutions to oscillate or to converge to zero. The coefficient associated with the neutral part is considered in three distinct ranges, in one of which the coefficient is allowed to oscillate. Illustrative examples show that the existing results do not apply to these examples and hence they show the significance of our results. The results of this article are also new for the particular choices of the time scale $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

1. Introduction

In this paper, we study the asymptotic and oscillatory behaviour of solutions to the equation

$$\big[x(t) + A(t)x(\alpha(t))\big]^{\Delta} + B(t)F(x(\beta(t))) - C(t)F(x(\gamma(t))) = \varphi(t) \tag{1.1}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, where $t_0 \in \mathbb{T}$, $\sup\{\mathbb{T}\} = \infty$, $A \in \mathrm{C}_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $B, C \in \mathrm{C}_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$, $F \in \mathrm{C}_{\mathrm{rd}}(\mathbb{R}, \mathbb{R})$, $\alpha, \beta, \gamma \in \mathrm{C}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ are strictly increasing and unbounded functions.

For the sake of completeness in the paper, we find it useful to recall the following basic concepts related to the notion of time scale calculus. A *time scale* is a nonempty closed subset of real numbers, and denoted by the notation $\mathbb{T}$. On a time scale $\mathbb{T}$, the *forward jump operator*, the *backward jump operator* and the *graininess function* are defined respectively by

$$\sigma(t) := \inf(t, \infty)_{\mathbb{T}}, \quad \rho(t) := \sup(-\infty, t)_{\mathbb{T}} \quad \text{and} \quad \mu(t) := \sigma(t) - t,$$

for $t \in \mathbb{T}$. For convenience, the interval with a $\mathbb{T}$ index below is used to denote the intersection of the usual interval with $\mathbb{T}$. The delta derivative (or derivative in

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short) of a function $f : \mathbb{T} \to \mathbb{R}$ is defined by

$$f^{\Delta}(t) := \begin{cases} \dfrac{f(\sigma(t)) - f(t)}{\mu(t)}, & \mu(t) > 0 \\ \lim\limits_{s \to t} \dfrac{f(t) - f(s)}{t - s}, & \mu(t) = 0, \end{cases}$$

where $t \in \mathbb{T}^{\kappa}$ (provided that limit exists), and $\mathbb{T}^{\kappa} := \mathbb{T}\backslash\{\sup \mathbb{T}\}$ if $\sup \mathbb{T} = \max \mathbb{T}$ and $\rho(\max \mathbb{T}) \neq \max \mathbb{T}$; otherwise, $\mathbb{T}^{\kappa} := \mathbb{T}$. A function f is called *right-dense continuous* (or *rd-continuous* in short) provided that f is continuous at every right-dense points in $\mathbb{T}$, and has a finite limit at every left-dense point in $\mathbb{T}$. The set of rd-continuous functions are denoted by $\mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$, and $\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ denotes the set of functions of which derivative is also in $\mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. For $s, t \in \mathbb{T}$ and a differentiable function $f \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$, the Cauchy integral of f^{Δ} is defined by

$$\int_s^t f^{\Delta}(\eta)\Delta\eta = f(t) - f(s).$$

Table 1 displays the explicit forms of forward jump, delta derivative and delta integral on the well-known time scales. For further details in the time scales, we refer the readers to the books [6, 7] which summarize and organize most of the time scale theory.

Table 1. Examples of some time scales

$\mathbb{T}$	$\sigma(t)$	$f^{\Delta}(t)$	$\int_s^t f(\eta)\Delta\eta$
$\mathbb{R}$	t	$f'(t)$	$\int_a^b f(\eta)\mathrm{d}\eta$
$\mathbb{Z}$	$t+1$	$\Delta f(t)$	$\sum_{\eta=s}^{t-1} f(\eta)$
$\overline{q^{\mathbb{Z}}}$, $(q > 1)$	qt	$\dfrac{f(qt) - f(t)}{(q-1)t}$	$(q-1)\sum_{\eta=\log_q(s)}^{\log_q(t)-1} f(q^{\eta})q^{\eta}$
$\mathbb{N}_0^q$, $(q > 0)$	$\big(t^{1/q}+1\big)^q$	$\dfrac{f((t^{1/q}+1)^q) - f(t)}{(t^{1/q}+1)^q - t}$	$\sum_{\eta=s^{1/q}}^{t^{1/q}-1} f(\eta^q)\big((\eta+1)^q - \eta^q\big)$

In [4, 8, 9, 25, 29, 32, 33], the authors study the dynamic equation

$$x^{\Delta}(t) + A(t)x(\alpha(t)) = 0 \tag{1.2}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, where $A \in \mathrm{C}_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$ and $\alpha \in \mathrm{C}_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ satisfy $\lim_{t\to\infty} \alpha(t) = \infty$ and $\alpha(t) \leq t$ for all sufficiently large t, and they present oscillation and stability criteria for this equation.

Then later, in [14], the authors extend some of the results stated for (1.2) to the equation

$$x^{\Delta}(t) + A(t)x(\alpha(t)) - B(t)x(\beta(t)) = 0$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, where $A, B \in \mathrm{C}_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$ and $\alpha, \beta \in \mathrm{C}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ satisfy $\lim_{t\to\infty} \alpha(t) = \lim_{t\to\infty} \beta(t) = \infty$ and $\alpha(t) \leq \beta(t) \leq t$ for all sufficiently large t. The authors ([14]) unified some of the well-known results stated for the corresponding difference and/or differential equations.

In a very recent paper [16, 17], the authors study

$$\big[x(t)+A(t)x(\alpha(t))\big]^{\Delta}+B(t)F(x(\beta(t)))-C(t)G(x(\gamma(t)))=\varphi(t)$$

for $t\in[t_0,\infty)_{\mathbb{T}}$, where $A\in\mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $B,C\in\mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$, $F,G\in\mathrm{C}_{\mathrm{rd}}(\mathbb{R},\mathbb{R})$, $\alpha,\beta,\gamma\in\mathrm{C}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ are unbounded strictly increasing functions such that $\alpha(t),\beta(t),\gamma(t)\le t$ holds for all sufficiently large t. In this paper, we weaken the assumptions on the coefficients that are assumed to hold for [10, 17, 24, 26, 27, 28], and improve their results by providing necessary and sufficient conditions.

We go on with the following example, which shows the significance and applicability of our results.

Example 1.1. Consider the neutral delay differential equation

$$\big[x(t)+2x(t-\pi)\big]'+\Big(\frac{1}{t^p}+1\Big)x(t-7\pi/2)-\frac{1}{t^p}x(t-3\pi/2)=0 \tag{1.3}$$

for $t\in[1,\infty)_{\mathbb{R}}$, where $p\in(0,\infty)_{\mathbb{R}}$ is a constant. In view of (1.1), we have $A(t)\equiv 2$, $\alpha(t)=t-\pi$, $B(t)=1/t^p+1$, $\beta(t)=t-7\pi/2$, $F(\lambda)=\lambda$, $C(t)=1/t^p$, $\gamma(t)=t-3\pi/2$, $G(\lambda)=\lambda$ and $\varphi(t)\equiv 0$ for $t\in[1,\infty)_{\mathbb{R}}$ and $\lambda\in\mathbb{R}$. To the best of our knowledge, none of the existing results in the literature can be applied to this equation. For instance, [17, Theorem 1], [21, Theorem 1] and [30, Theorem 1] can not be applied since $A(t)\equiv 2\not\le 0$, and [16, Theorem 1], [24, Theorem 4], [26, Theorem 2.2] and [28, Theorem 2.1] can not be applied to this equation when $p\in(0,1]_{\mathbb{R}}$ holds since the improper integral of $C(t)=1/t^p$ is divergent, but our results (see Theorem 2.1) do not fail revealing asymptotic properties of the solutions of this equation. It is easy to see that $x(t)=\sin(t)$ and $x(t)=\cos(t)$ for $t\in[1,\infty)_{\mathbb{R}}$ are oscillating solutions of (1.3).

As is seen from the example given above, our results can be employed in some cases when the results in the literature fail to apply. Roughly speaking about the technique of this paper, the work depends on revealing asymptotic behaviour of nonoscillatory bounded solutions to (1.1), and then we introduce the conditions that ensure nonexistence of unbounded nonoscillatory solutions to deal with unbounded solutions. Therefore, the method applied here is a little bit different than the ones employed in the literature.

Set $t_{-1}:=\min\{\alpha(t_0),\beta(t_0),\gamma(t_0)\}$. By a *solution* of (1.1), we mean a function $x:[t_{-1},\infty)_{\mathbb{T}}\to\mathbb{R}$ with $x+A(t)x\circ\alpha\in\mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ satisfying (1.1) identically on $[t_0,\infty)_{\mathbb{T}}$. A solution of (1.1) is called *nonoscillatory* if it is eventually of constant sign; otherwise, it is called *oscillatory*.

2. Main results

For an arbitrary function $f:\mathbb{T}\to\mathbb{R}$, we define $f^{\pm}(t):=\max\{\pm f(t),0\}$ for $t\in\mathbb{T}$. It is easy to see that $f^+\equiv f$ provided that f is nonnegative while $f^-\equiv -f$ provided that f is nonpositive, and note that we have $f^+,f^-\ge 0$, $f\equiv f^+-f^-$ and $f^+\ge f\ge -f^-$. Moreover, we have $\lim_{t\to\infty}f^+(t)=\lim_{t\to\infty}f^-(t)=0$ if $\lim_{t\to\infty}f(t)=0$ is true.

We list our assumptions on the coefficient A as follows:

(R1) $\limsup_{t\to\infty}A^+(t)+\limsup_{t\to\infty}A^-(t)<1$.
(R2) $\limsup_{t\to\infty}A(t)<\infty$ and $\liminf_{t\to\infty}A(t)>1$.
(R3) $\liminf_{t\to\infty}A(t)>-1$.
(R4) $\limsup_{t\to\infty}A(t)<-1$ and $\liminf_{t\to\infty}A(t)>-\infty$.

Next, we list assumptions on the nonlinear function F and the forcing term φ:

- (H1) $F \in \mathrm{C}_{\mathrm{rd}}(\mathbb{R},\mathbb{R})$ satisfies $F(\lambda)/\lambda > 0$ for all $\lambda \in \mathbb{R}\backslash\{0\}$.
- (H2) there exists a function $\Phi \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $\Phi^{\Delta} = \varphi$ and that $\lim_{t\to\infty}\Phi(t) = 0$ hold.

Set $\upsilon := \gamma^{-1}\circ\beta$ and suppose that $\upsilon \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ satisfies $\upsilon([s,\infty)_{\mathbb{T}}) = [\upsilon(s),\infty)_{\mathbb{T}}$ for some $s \in [t_0,\infty)_{\mathbb{T}}$, and it is trivial that υ is strictly increasing because of the increasing nature of β and γ. From now on, we suppose that $D \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ defined by

$$D(t) := \begin{cases} B(t) - \upsilon^{\Delta}(t)C(\upsilon(t)), & t \in [\upsilon^{-1}(t_0),\infty)_{\mathbb{T}} \\ B(t) - \upsilon^{\Delta}(t_0)C(\upsilon(t_0)), & t \in [t_0,\upsilon^{-1}(t_0))_{\mathbb{T}} \end{cases}$$

is eventually nonnegative.

- (H3) $\alpha(t) \le t$ for all sufficiently large t.
- (H4) $\limsup_{\lambda\to\pm\infty}\big[F(\lambda)/\lambda\big] < \infty$.
- (H5) there exists a bounded function $\Phi \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $\Phi^{\Delta} = \varphi$.

We again list our additional assumptions on the coefficients B and C as follows:

- (A1) $\int_{t_0}^{\infty} D(\eta)\Delta\eta = \infty$.
- (A2) $\lim_{t\to\infty}\int_{\upsilon(t)}^{t} C(\eta)\Delta\eta = 0$.
- (A3) $\lim_{t\to\infty}\big[\int_{\upsilon(t)}^{t} C(\eta)\Delta\eta\big]^{+} = 0$.

Our first result studies the asymptotic behaviour of bounded solutions of (1.1) when A satisfies the condition (R1).

Theorem 2.1. *Assume that* (H1)*,* (H2)*,* (A1)*,* (A2) *hold. If A satisfies* (R1)*, then every nonoscillatory bounded solution of* (1.1) *tends to zero at infinity.*

Proof. Let x be a nonoscillatiory bounded solution of (1.1). We may assume without loss of generality that x is eventually positive, this is possible because of (H1) and (H2). Say $\upsilon([t_1,\infty)_{\mathbb{T}}) = [\upsilon(t_1),\infty)_{\mathbb{T}}$ and $x(t), x(\alpha(t)), x(\beta(t)), x(\gamma(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$ for some $t_1 \in [t_0,\infty)_{\mathbb{T}}$. Now, for $t \in [t_1,\infty)_{\mathbb{T}}$, set

$$y_x(t) := x(t) + A(t)x(\alpha(t)) \tag{2.1}$$

and

$$z_x(t) := y_x(t) - \int_{\upsilon(t)}^{t} C(\eta)F(x(\gamma(\eta)))\Delta\eta - \Phi(t). \tag{2.2}$$

Obviously, y_x and z_x are bounded because of (A2), boundedness of x and A. Now, considering [6, Theorem 1.98], for all $t \in [t_1,\infty)_{\mathbb{T}}$, we may rewrite z_x in the following form

$$\begin{aligned} z_x(t) =& y_x(t) - \int_{\upsilon(t_1)}^{t} C(\eta)F(x(\gamma(\eta)))\Delta\eta - \int_{\upsilon(t)}^{\upsilon(t_1)} C(\eta)F(x(\gamma(\eta)))\Delta\eta - \Phi(t) \\ =& y_x(t) - \int_{\upsilon(t_1)}^{t} C(\eta)F(x(\gamma(\eta)))\Delta\eta - \int_{t}^{t_1} \upsilon^{\Delta}(\eta)C(\upsilon(\eta))F(x(\beta(\eta)))\Delta\eta - \Phi(t). \end{aligned}$$

Then, applying [6, Theorem 1.117] to the resulting and considering (1.1), we get

$$\begin{aligned} z_x^{\Delta}(t) =& y_x^{\Delta}(t) - C(t)F(x(\gamma(t))) + \upsilon^{\Delta}(t)C(\upsilon(t))F(x(\beta(t))) - \varphi(t) \\ =& -D(t)F(x(\beta(t))) \le 0 \end{aligned} \tag{2.3}$$

for all $t \in [t_2, \infty)_{\mathbb{T}}$. Therefore, z_x is nonincreasing over $[t_2, \infty)_{\mathbb{T}}$; i.e., $\lim_{t\to\infty} z_x(t)$ exists and is finite. This implies that $\lim_{t\to\infty} y_x(t)$ exists and moreover satisfies $\lim_{t\to\infty} y_x(t) = \lim_{t\to\infty} z_x(t)$ by (H2) and (A2), boundedness of x and A. Integrating (2.3) over $[t_2, \infty)_{\mathbb{T}}$, we get

$$\infty > z_x(t_2) - \lim_{t\to\infty} z_x(t) = \int_{t_2}^{\infty} D(\eta)F(x(\beta(\eta)))\Delta\eta,$$

which implies

$$\liminf_{t\to\infty} x(t) = 0 \tag{2.4}$$

by (H1), (A1) and boundedness of x. Set

$$M_x := \limsup_{t\to\infty} x(t). \tag{2.5}$$

Let $\{\varsigma_k\}_{k\in\mathbb{N}}, \{\zeta_k\}_{k\in\mathbb{N}} \in [t_1, \infty)_{\mathbb{T}}$ be two increasing divergent sequences such that as k tends to infinity, $x(\varsigma_k)$ tends to the inferior limit 0, while $x(\zeta_k)$ tends to the superior limit M_x. Because of (R1), we may pick $L, l \in [0, 1)_{\mathbb{R}}$ with $L + l < 1$ such that $L \geq A^+(t)$ and $l \geq A^-(t)$ for all sufficiently large t. Since x is bounded, we may suppose that $x(\alpha(\varsigma_k))$ and $x(\alpha(\zeta_k))$ converge to a limit which can not exceed M_x (due to Bolzano-Weierstrass theorem, such subsequences of $\{x(\alpha(\varsigma_k))\}_{k\in\mathbb{N}}$ and $\{x(\alpha(\zeta_k))\}_{k\in\mathbb{N}}$ always exit). Now, we prove $M_x = 0$, but first, recall that y_x has a finite limit. Indeed, for all $k \in \mathbb{N}$, we have

$$\begin{aligned} y_x(\varsigma_k) - y_x(\zeta_k) &\leq x(\varsigma_k) + A^+(\varsigma_k)x(\alpha(\varsigma_k)) - x(\zeta_k) + A^-(\zeta_k)x(\alpha(\zeta_k)) \\ &\leq x(\varsigma_k) + Lx(\alpha(\varsigma_k)) - x(\zeta_k) + lx(\alpha(\zeta_k)), \end{aligned}$$

which yields $0 \leq (L + l - 1)M_x$ by letting k tend to infinity, and this shows that $M_x = 0$ since $L + l < 1$. The proof is hence completed. □

The following two examples illustrate the significance of Theorem 2.1.

Example 2.2. Let $\mathbb{T} = \mathbb{Z}$. Consider the neutral difference equation

$$\Big[x(t) + \frac{(-1)^t}{3}x(t+3)\Big]^{\Delta} + \Big(2 + \frac{2}{t}\Big)\sqrt[3]{x(t)} - \frac{2}{t}\sqrt[3]{x(t+2)} = 0 \tag{2.6}$$

for $t \in [1, \infty)_{\mathbb{Z}}$. Here, we have $A(t) = (-1)^t/3$, $\alpha(t) = t + 3$, $F(\lambda) = \sqrt[3]{\lambda}$, $B(t) = 2 + 2/t$, $\beta(t) = t$, $C(t) = 2/t$, $\gamma(t) = t + 2$ and $\varphi(t) \equiv 0$ for $t \in [1, \infty)_{\mathbb{Z}}$ and $\lambda \in \mathbb{R}$. In this case, we have $\nu(t) = t - 2$, $D(t) = 2 + 2/t - 2/(t - 2)$ and $\varphi(t) \equiv 0$ for $t \in [1, \infty)_{\mathbb{Z}}$. In the literature, none of the existing results can be applied to this equation since $A(t) = (-1)^t/3$ for $[1, \infty)_{\mathbb{Z}}$ is oscillatory but not tending to zero at infinity and/or the infinite series of $C(t) = 2/t$ is not convergent on $[1, \infty)_{\mathbb{Z}}$. Clearly, A is in (R1) since

$$\limsup_{t\to\infty} \Big[\frac{(-1)^t}{3}\Big]^+ + \limsup_{t\to\infty} \Big[\frac{(-1)^t}{3}\Big]^- = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} < 1.$$

The forcing term $\varphi = 0$ satisfies (H2) with $\Phi = 0$, and nonlinear term F satisfies both (H1) and (H4). On the other hand, we have

$$\sum_{\eta=3}^{\infty} \Big(2 + \frac{2}{\eta} - \frac{2}{\eta - 2}\Big) = \infty \quad \text{and} \quad \lim_{t\to\infty} \sum_{\eta=t-2}^{t-1} \frac{2}{\eta} = 0.$$

Due to Theorem 2.1, we know that every bounded solution of (2.6) is oscillatory or asymptotically convergent to zero, and $x(t) = (-1)^t$ for $t \in [1, \infty)_{\mathbb{Z}}$ is an oscillating bounded solution of (2.6).

Example 2.3. Let $\mathbb{T}=\mathbb{R}$. Consider the following neutral differential equation:

$$\Big[x(t)+\frac{\sin(t)}{4}x(t+1/t^2)\Big]^{\Delta}+2x(t-1/t)-x(t)=0 \tag{2.7}$$

for $t\in[1,\infty)_{\mathbb{R}}$. Here, we have $A(t)=\sin(t)/4$, $\alpha(t)=t+1/t^2$, $F(\lambda)=\lambda$, $B(t)\equiv 2$, $\beta(t)=t-1/t$, $C(t)\equiv 1$, $\gamma(t)=t$ and $\varphi(t)\equiv 0$ for $t\in[1,\infty)_{\mathbb{R}}$ and $\lambda\in\mathbb{R}$. Obviously, $\upsilon(t)=t-1/t$, $D(t)=1-1/t^2$ and $\Phi(t)\equiv 0$ for $t\in[1,\infty)_{\mathbb{R}}$. So that, the arguments of this equation satisfy all the assumptions of Theorem 2.1, and hence every bounded solution of (2.7) is oscillatory or asymptotically convergent to zero.

Next, we state Theorem 2.1 for (R2).

Theorem 2.4. *Assume that* (H1)*,* (H2)*,* (A1)*,* (A2) *hold. If A satisfies* (R2)*, then every nonoscillatory bounded solution of* (1.1) *tends to zero at infinity.*

Proof. Without loss of generality suppose that x is an eventually positive solution. Say $x(t),x(\alpha(t)),x(\beta(t)),x(\gamma(t))>0$ for all $t\in[t_1,\infty)_{\mathbb{T}}$ for some $t_1\in[t_0,\infty)_{\mathbb{T}}$. For $t\in[t_1,\infty)_{\mathbb{T}}$, define y_x and z_x as in (2.1) and (2.2), respectively. Then, following similar arguments to that in the proof of Theorem 2.1, we get (2.4). Considering (R2), we may pick $L,l\in(1,\infty)_{\mathbb{R}}$ satisfying $L\geq A(\alpha^{-1}(t))\geq l$ for all sufficiently large t. We may suppose that $x(\alpha^{-1}(\varsigma_k))$ tends to limits which is not greater than M_x defined by (2.5). Then, for all $k\in\mathbb{N}$, we get

$$\begin{aligned} y_x(\alpha^{-1}(\varsigma_k))-y_x(\alpha^{-1}(\zeta_k)) &\leq x(\alpha^{-1}(\varsigma_k))+A(\alpha^{-1}(\varsigma_k))x(\varsigma_k)-A(\alpha^{-1}(\zeta_k))x(\zeta_k)\\ &\leq x(\alpha^{-1}(\varsigma_k))+Lx(\varsigma_k)-lx(\zeta_k), \end{aligned}$$

which says that $0\leq(1-l)M_x$ holds by letting k tend to infinity. Thus, we have $M_x=0$ because of $l>1$, and this completes the proof. □

The following result is inferred from Theorem 2.1 and Theorem 2.4.

Corollary 2.5. *Assume that* (H1)*,* (H2)*,* (A1)*,* (A2) *hold, A satisfies* (R1) *or* (R2)*. Then, every bounded solution oscillates or converges to zero asymptotically.*

With the following example, we show applicability of Theorem 2.4 on the non-standard time scale quantum set.

Example 2.6. Let $\mathbb{T}=\overline{2^{\mathbb{Z}}}$. For $t\in[1,\infty)_{\overline{2^{\mathbb{Z}}}}$, consider the neutral dynamic equation

$$\big[x(t)+2x(t/2)\big]^{\Delta}+\frac{9}{4t}x(t/2)-\frac{1}{t^2}x(t)=-\frac{1}{t^3}. \tag{2.8}$$

Here, we have $A(t)\equiv 2$, $\alpha(t)=t/2$, $F(\lambda)=\lambda$, $B(t)=9/(4t)$, $\beta(t)=t/2$, $C(t)=1/t^2$, $\gamma(t)=t$ and $\varphi(t)=-1/t^3$ for $t\in[1,\infty)_{\overline{2^{\mathbb{Z}}}}$ and $\lambda\in\mathbb{R}$. For $t\in[1,\infty)_{\overline{2^{\mathbb{Z}}}}$, we have $\nu(t)=t/2$, $D(t)=9/(4t)-2/t^2$ and $\Phi(t)=2/(3t^2)$. Moreover, we calculate

$$\sum_{\eta=0}^{\infty}\Big(\frac{9}{4}-\frac{2}{2^{\eta}}\Big)=\infty \quad\text{and}\quad \lim_{t\to\infty}\frac{2}{t}=0,$$

which show (A1) and (A2) hold. By Theorem 2.4 since A is in (R2), every bounded solution of (2.8) oscillates or asymptotically converges to zero. Clearly, $x(t)=1/t$ for $t\in[1,\infty)_{\overline{2^{\mathbb{Z}}}}$ is a solution, which tends to zero at infinity. With the initial conditions $x(1/2)=x(1)=1$, we get the graphics shown in Figure 1 for the solution with 50 iterates.

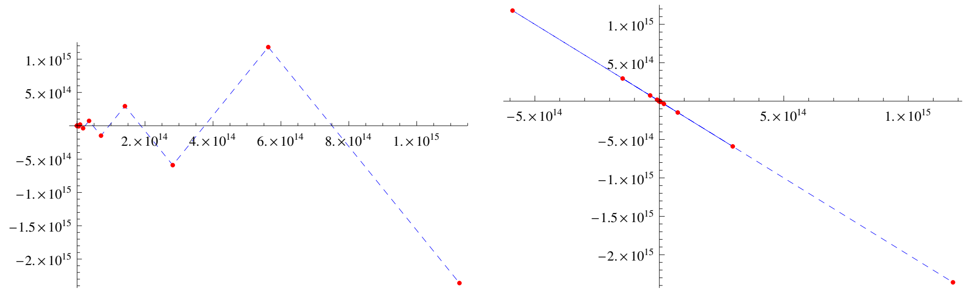

FIGURE 1. Graphs of $(t, x(t))$ (left), and of $(x(t), x(2t))$ (right)

We may guess that this solution oscillates unboundedly. However, Theorem 2.4 is not yet stated for unbounded solutions.

The following result ensures nonexistence of unbounded nonoscillatory solutions when A satisfies (R3).

Theorem 2.7. *Let A satisfy* (R3)*. If* (H1), (H3)–(H5), (A3) *hold. Then* (1.1) *has no unbounded nonoscillatory solutions.*

Proof. For the sake of contradiction, suppose that x is a nonoscillatiory unbounded solution of (1.1), which can be assumed to be eventually positive. Clearly, following the steps in Theorem 2.1, we obtain (2.3) on $[t_1, \infty)_{\mathbb{T}}$. Thus, z_x is eventually nonincreasing; i.e., $\lim_{t\to\infty} z_x(t) < \infty$. Since A is in (R3), we may pick $l \in [0, 1)_{\mathbb{R}}$ such that $A(t) \geq -l$ for all $t \in [t_2, \infty)_{\mathbb{T}}$ for some $t_2 \in [t_1, \infty)_{\mathbb{T}}$. Now, define the sets $\mathcal{I}(t) := \{\eta \in [\upsilon(t), t)_{\mathbb{T}} : x(\gamma(\eta)) \leq 1\}$ and $\mathcal{J}(t) := \{\eta \in [\upsilon(t), t)_{\mathbb{T}} : x(\gamma(\eta)) > 1\}$ for $t \in [t_2, \infty)_{\mathbb{T}}$ (see [5, § 6]). Note that $\mathcal{I}(t) \cap \mathcal{J}(t) = \emptyset$ and $\mathcal{I}(t) \cup \mathcal{J}(t) = [\upsilon(t), t)_{\mathbb{T}}$ for all $t \in [t_2, \infty)_{\mathbb{T}}$. Let $\{\xi_k\}_{k\in\mathbb{N}} \subset [t_0, \infty)_{\mathbb{T}}$ be an increasing divergent sequence such that $\{x(\xi_k)\}_{k\in\mathbb{N}}$ is increasing and divergent and $x(\xi_k) \geq \sup\{x(\eta) : \eta \in [t_0, \xi_k)_{\mathbb{T}}\} \geq 1$ is true for all $k \in \mathbb{N}$. On one hand, for all sufficiently large k, we have

$$\int_{\mathcal{I}(\xi_k)} C(\eta) F(x(\gamma(\eta)))\Delta\eta \leq \frac{1}{3}(1-l) \leq \frac{1}{3}(1-l)x(\xi_k) \tag{2.9}$$

since $x \circ \gamma$ is bounded above by 1 on $I(t)$ for all $t \in [t_2, \infty)_{\mathbb{T}}$ and (A3) is true. On the other hand, since for $\lambda \in (1, \infty)_{\mathbb{R}}$, $F(\lambda)/\lambda$ has no discontinuities, we learn that $\sup_{\lambda\in(1,\infty)_{\mathbb{R}}}\{F(\lambda)/\lambda\}$ is a finite constant by (H4). Thus, by this reasoning and (A3), for all sufficiently large k, we have

$$\sup_{\lambda\in(1,\infty)_{\mathbb{R}}} \Big\{\frac{F(\lambda)}{\lambda}\Big\}\Big[\int_{\upsilon(\xi_k)}^{\xi_k} C(\eta)\Delta\eta\Big]^+ \leq \frac{1}{3}(1-l). \tag{2.10}$$

Therefore, using (2.10), for all k sufficiently large, we deduce

$$\begin{aligned}
\int_{\mathcal{J}(\xi_k)} C(\eta)F(x(\gamma(\eta)))\Delta\eta \leq & \Big[\int_{\upsilon(\xi_k)}^{\xi_k} C(\eta)F(x(\gamma(\eta)))\Delta\eta\Big]^+ \\
= & \Big[\int_{\upsilon(\xi_k)}^{\xi_k} C(\eta)\frac{F(x(\gamma(\eta)))}{x(\gamma(\eta))}x(\gamma(\eta))\Delta\eta\Big]^+ \\
\leq & \sup_{\lambda\in(1,\infty)_{\mathbb{R}}}\Big\{\frac{F(\lambda)}{\lambda}\Big\}\Big[\int_{\upsilon(\xi_k)}^{\xi_k} C(\eta)x(\gamma(\eta))\Delta\eta\Big]^+ \\
\leq & \sup_{\lambda\in(1,\infty)_{\mathbb{R}}}\Big\{\frac{F(\lambda)}{\lambda}\Big\}\Big[\int_{\upsilon(\xi_k)}^{\xi_k} C(\eta)\Delta\eta\Big]^+ x(\xi_k) \\
\leq & \frac{1}{3}(1-l)x(\xi_k).
\end{aligned} \tag{2.11}$$

Summing (2.9) and (2.11), for all sufficiently large k, we get

$$\int_{\upsilon(\xi_k)}^{\xi_k} C(\eta)F(x(\gamma(\eta)))\Delta\eta \leq \frac{2}{3}(1-l)x(\xi_k). \tag{2.12}$$

Then, taking (H3), (H5), (2.1), (2.2) and (2.12) into account, as $k\to\infty$, we obtain

$$\begin{aligned}
z_x(\xi_k) \geq & y_x(\xi_k) + \frac{2}{3}(1-l)x(\xi_k) - \Phi(\xi_k) \\
\geq & (1-l)x(\xi_k) + \frac{2}{3}(1-l)x(\xi_k) - \Phi(\xi_k) \\
= & \frac{1}{3}(1-l)x(\xi_k) - \Phi(\xi_k) \to \infty,
\end{aligned}$$

which contradicts to $\lim_{t\to\infty} z_x(t) < \infty$. Hence, every nonoscillatory solution of (1.1) is bounded. □

Remark 2.8. Under the assumptions of Theorem 2.7, every unbounded solution of (1.1) is oscillatory.

By Theorem 2.1, Theorem 2.4 and Theorem 2.7, we have the following result.

Corollary 2.9. *Assume that* (H1)–(H4), (A1), (A2) *hold. If A satisfies either* (R1) *or* (R2), *then every solution oscillates or converges to zero asymptotically.*

We give the following example, which is an application of Theorem 2.7.

Example 2.10. Let $\mathbb{T} = \sqrt{\mathbb{N}_0}$, and for $t \in [2,\infty)_{\sqrt{\mathbb{N}_0}}$, consider the dynamic equation

$$\begin{aligned}
&\Big[x(t) + \frac{t\big[(-1)^{t^2}\big]^+ - \big[(-1)^{t^2}\big]^-}{2}x(\sqrt{t^2-2})\Big]^\Delta \\
&+ \frac{1}{t^2(\sqrt{t^2+1}-t)}x(\sqrt{t^2-1}) - \frac{1}{t^4(\sqrt{t^2+1}-t)}x(t) = 0.
\end{aligned} \tag{2.13}$$

For this equation, we see that $A(t) = (t[(-1)^{t^2}]^+ - [(-1)^{t^2}]^-)/2$, $\alpha(t) = \sqrt{t^2-2}$, $B(t) = 1/(t^2(\sqrt{t^2+1}-t))$, $\beta(t) = \sqrt{t^2-1}$, $C(t) = 1/(t^4(\sqrt{t^2+1}-t))$ and $\gamma(t) = t$ for $t \in [2,\infty)_{\sqrt{\mathbb{N}_0}}$. Thus, we obtain $\upsilon(t) = \sqrt{t^2-1}$ and

$$D(t) = \frac{1}{t^2(\sqrt{t^2+1}-t)} - \frac{t-\sqrt{t^2-1}}{(\sqrt{t^2+1}-t)(t^4-2t^2+1)(t-\sqrt{t^2-1})}$$

for $t \in [2,\infty)_{\sqrt{\mathbb{N}_0}}$. Note here that $\upsilon([2,\infty)_{\sqrt{\mathbb{N}_0}}) = [\sqrt{3},\infty)_{\sqrt{\mathbb{N}_0}} = [\upsilon(2),\infty)_{\sqrt{\mathbb{N}_0}}$. One can show that all the conditions of Theorem 2.7 hold, and thus every unbounded solution of (2.13) is oscillatory. The following graphics belong to a solution with the initial conditions $x(\sqrt{2}) = x(\sqrt{3}) = x(2) = 1$ and 40 iterates are shown in Figure 2 below.

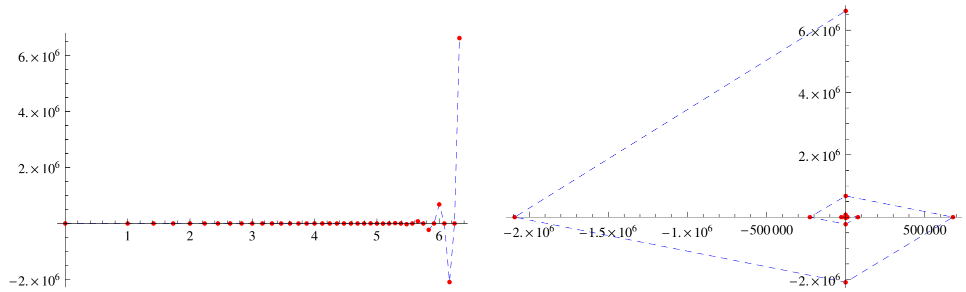

FIGURE 2. Graphs of $(t, x(t))$ (left), and of $(x(t), x(\sqrt{t^2+1}))$ (right).

We may guess from this graphic that this solution is unboundedly oscillating. Since the solution grows very rapidly, in the first graphic, the preceding the points seem very closer to the horizontal axis.

The following result states the asymptotic behaviour for nonoscillatory bounded solutions of (1.1) when A satisfies (R4).

Theorem 2.11. *Assume that* (H1)*,* (H2)*,* (A1)*,* (A2) *hold, A satisfies of* (R4)*. Then, every nonoscillatory bounded solution of* (1.1) *tends to zero at infinity.*

Proof. Suppose without loss of generality that x is an eventually positive solution. There exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t), x(\alpha(t)), x(\beta(t)), x(\gamma(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$. For $t \in [t_1,\infty)_{\mathbb{T}}$, define y_x and z_x as in (2.1) and (2.2), respectively. As in the proof of Theorem 2.1, we obtain (2.4). Set M_x as in (2.5). Following similar arguments to those in the proofs of Theorem 2.1 and/or Theorem 2.4, we get $0 \geq (1-l)M_x$, where $L, l \in (1,\infty)_{\mathbb{R}}$ and $-l \geq A(t) \geq -L$ for all sufficiently large t by (R4). In this present case, we again have $M_x = 0$ since $l > 1$. The proof is completed. $\square$

Example 2.12. For $\mathbb{T} = \mathbb{R}$, consider the dynamic equation

$$\Big[x(t) - \frac{3}{2}x(t+\cos(t)/2)\Big]^{\Delta} + \frac{2}{t}\arctan(x(t+\sin(t)/2)) - \frac{1}{t}\arctan(x(t)) = 0 \quad (2.14)$$

for $t \in [1,\infty)_{\mathbb{R}}$. For this equation, the parameters are $A(t) \equiv -3/2$, $\alpha(t) = t + \cos(t)/2$, $F(\lambda) = \arctan(\lambda)$, $B(t) = 2/t$, $\beta(t) = t + \sin(t)/2$, $C(t) = 1/t$ and $\gamma(t) = t$ for $t \in [1,\infty)_{\mathbb{R}}$ and $\lambda \in \mathbb{R}$. Hence, for $t \in [1,\infty)_{\mathbb{R}}$, we obtain $\upsilon(t) = t + \sin(t)/2$ (strictly increasing) and $D(t) = 2/t - (2+\cos(t))/(2t+\sin(t))$. It is easy to verify that

$$\int_1^{\infty}\Big(\frac{2}{\eta} - \frac{2+\cos(\eta)}{2\eta+\sin(\eta)}\Big)\mathrm{d}\eta = \infty \quad \text{and} \quad \lim_{\lambda\to\pm\infty}\frac{\arctan(\lambda)}{\lambda} = 0.$$

Moreover, we have

$$\lim_{t\to\infty}\Big[\ln\Big(\frac{2t}{2t+\sin(t)}\Big)\Big]^{+} = 0.$$

Since all the conditions of Theorem 2.11 are satisfied, every bounded solution of (2.14) oscillates or tends to zero at infinity.

Theorem 2.11 cannot be stated for unbounded solutions in its present form, this fact is shown with the following example which possesses an unbounded nonoscillatory solution and satisfies all the assumptions of Theorem 2.11 .

Example 2.13. Let $\mathbb{T}=\mathbb{P}_{1,1}$, where $\mathbb{P}_{a,b}:=\cup_{\ell\in\mathbb{Z}}[(a+b)\ell,(a+b)\ell+a]_{\mathbb{R}}$ for $a,b>0$. And consider the following dynamic equation

$$\big[x(t)-2x(t-6)\big]^{\Delta}+\frac{2}{t-4}x(t-4)-\frac{1}{t-2}x(t-2)=0 \tag{2.15}$$

for $t\in[6,\infty)_{\mathbb{P}_{1,1}}$. For this equation, we see that $A(t)\equiv-2$, $\alpha(t)=t-6$, $B(t)=2/(t-4)$, $\beta(t)=t-4$, $C(t)=1/(t-2)$ and $\gamma(t)=t-2$ for $t\in[6,\infty)_{\mathbb{P}_{1,1}}$. Thus, we deduce that $\upsilon(t)=t-2$ and $D(t)=1/(t-4)$ for $t\in[6,\infty)_{\mathbb{P}_{1,1}}$. One can check that all the conditions of Theorem 2.11 are satisfied but (2.15) admits a nonoscillatory unbounded solution $x(t)=t$ for $t\in[6,\infty)_{\mathbb{P}_{1,1}}$.

Remark 2.14. Under the assumptions of Theorem 2.11, the statement in Corollary 2.5 is still valid.

With the following theorem, we are able to study existence of nonoscillatory solutions, which does not asymptotically tend to zero. Clearly, we have to prove existence of a solution of which superior (inferior) limit is a positive (negative) finite, and as we infer from the proofs of Theorem 2.1, Theorem 2.4 and Theorem 2.11, we have to prove that inferior (superior) limit of the solution must be positive (negative). Otherwise, since Φ may have a finite limit at infinity, we may proceed as in the proofs of the mentioned theorems and obtain that the solution is asymptotically tending to zero.

Theorem 2.15. *Suppose that* (H5), (A2) *hold, and that A satisfies* (R1). *If* (A1) *does not hold, then* (1.1) *has a bounded nonoscillatory solution, which does not tend to zero asymptotically.*

Proof. To prove existence of such nonoscillatory solution, we apply Krasnoselkii's fixed point theorem (see [34, Lemma 5]). Let $K\in(0,\infty)_{\mathbb{R}}$ and $t_1\in[t_0,\infty)_{\mathbb{T}}$ satisfy $|\Phi(t)|\le K$ for all $t\in[t_1,\infty)_{\mathbb{T}}$. Since A satisfies (R1), then we can pick $L,l\in[0,1)_{\mathbb{R}}$ with $L+l<1$ and $M,m\in(0,\infty)_{\mathbb{R}}$ with $M>m$ such that $L\ge A^{+}(t)$, $l\ge A^{-}(t)$, $K=[(1-l-L)M-m]/6$ and

$$\max_{\lambda\in[m,M]_{\mathbb{R}}}\{F(\lambda)\}\Big|\int_{\upsilon(t)}^{t}C(\eta)\Delta\eta\Big|\le K \tag{2.16}$$

for all $t\in[t_2,\infty)_{\mathbb{T}}$ for a sufficiently large $t_2\in[t_1,\infty)_{\mathbb{T}}$. There exists $t_3\in[t_2,\infty)_{\mathbb{T}}$ satisfying

$$\max_{\lambda\in[m,M]_{\mathbb{R}}}\{F(\lambda)\}\int_{t}^{\infty}D(\eta)\Delta\eta\le K \tag{2.17}$$

for all $t\in[t_3,\infty)_{\mathbb{T}}$. Let $\mathrm{BC_{rd}}([t_3,\infty)_{\mathbb{T}},\mathbb{R})$ be the Banach space of all bounded rd-continuous functions on $[t_3,\infty)_{\mathbb{T}}$ equipped with the supremum norm

$$\|x\|:=\sup\{|x(\eta)|:\eta\in[t_3,\infty)_{\mathbb{T}}\},$$

and set

$$\Omega:=\big\{x\in\mathrm{BC_{rd}}([t_3,\infty)_{\mathbb{T}},\mathbb{R}):m\le x(\eta)\le M\quad\text{for }\eta\in[t_3,\infty)_{\mathbb{T}}\big\}. \tag{2.18}$$

Pick $t_4 \in [t_3, \infty)_{\mathbb{T}}$ satisfying $\delta(t_4) \geq t_3$ and set $N := [(1 - l + L)M + m]/2$. Define now two mappings $\Gamma, \Psi : \Omega \to \Omega$ as follows:

$$\Gamma x(t) := \begin{cases} \Gamma x(t_4), & t \in [t_3, t_4)_{\mathbb{T}} \\ N - A(t)x(\alpha(t)) + \Phi(t), & t \in [t_4, \infty)_{\mathbb{T}} \end{cases}$$

and

$$\Psi x(t) := \begin{cases} \Psi x(t_4), & t \in [t_3, t_4)_{\mathbb{T}} \\ \displaystyle\int_{\upsilon(t)}^{t} C(\eta)F(x(\gamma(\eta)))\Delta\eta + \int_{t}^{\infty} D(\eta)F(x(\gamma(\upsilon(\eta))))\Delta\eta, & t \in [t_4, \infty)_{\mathbb{T}}. \end{cases}$$

We assert that $\Gamma x + \Psi x = x$ has a fixed point in Ω by the means of Krasnoselkii's fixed point theorem. First, we show $\Gamma x + \Psi y \in \Omega$ for all $x, y \in \Omega$. Clearly, from (2.16) and (2.17), for any $x, y \in \Omega$, we obtain

$$\Gamma x(t) + \Psi y(t) \leq N + lM + 3K = M$$

and

$$\Gamma x(t) + \Psi y(t) \geq N - LM - 3K = m$$

for all $t \in [t_3, \infty)_{\mathbb{T}}$, which proves that the claim is true. Γ is a contraction mapping since $\max\{l, L\} < 1$ and $\|\Gamma x - \Gamma y\| \leq \max\{l, L\}\|x - y\|$ on $[t_3, \infty)_{\mathbb{T}}$. Next, we show that Ψ is a completely continuous mapping; i.e., Ψ is continuous and maps bounded sets into relatively compact sets. Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence in Ω, which converges to $x \in \Omega$. For all $t \in [t_4, \infty)_{\mathbb{T}}$ and $k \in \mathbb{N}$, we have

$$\begin{aligned} \big|\Psi x_k(t) - \Psi x(t)\big| = \Big| & \int_{\upsilon(t)}^{t} C(\eta)\big[F(x_k(\gamma(\eta))) - F(x(\gamma(\eta)))\big]\Delta\eta \\ & + \int_{t}^{\infty} D(\eta)\big[F(x_k(\gamma(\upsilon(\eta)))) - F(x(\gamma(\upsilon(\eta))))\big]\Delta\eta\Big|. \end{aligned}$$

Since Lebesgue's dominated convergence theorem (see [6, § 5]) holds for delta integrals, for all $t \in [t_4, \infty)_{\mathbb{T}}$, we have

$$\lim_{k\to\infty} \big|\Psi x_k(t) - \Psi x(t)\big| = 0,$$

which proves continuity of Ψ on Ω. To show relatively compactness of $\Psi\Omega$, we shall verify the assumptions of Arzelá-Ascoli theorem (see [2, Lemma 2.6]). Obviously, Ω is uniformly bounded. For every $\varepsilon > 0$, there exists $t_5 \in [t_4, \infty)_{\mathbb{T}}$ such that

$$\max_{\lambda\in[m,M]_{\mathbb{R}}}\{F(\lambda)\}\Big|\int_{\upsilon(t)}^{t} C(\eta)\Delta\eta\Big| \leq \frac{\varepsilon}{4}, \quad \max_{\lambda\in[m,M]_{\mathbb{R}}}\{F(\lambda)\}\Big|\int_{t}^{\infty} D(\eta)\Delta\eta\Big| \leq \frac{\varepsilon}{4}.$$

for all $t \in [t_5, \infty)_{\mathbb{T}}$. Therefore, $\Psi\Omega$ is uniformly Cauchy since for every $s, t \in [t_5, \infty)_{\mathbb{T}}$, we have $|\Psi x(t) - \Psi x(s)| \leq \varepsilon$. On the other hand, for every ε, there exists $\delta > 0$ such that

$$\max_{\lambda\in[m,M]_{\mathbb{R}}}\{F(\lambda)\}\Big|\int_{\upsilon(s)}^{\upsilon(t)} C(\eta)\Delta\eta\Big| \leq \frac{\varepsilon}{3}, \quad \max_{\lambda\in[m,M]_{\mathbb{R}}}\{F(\lambda)\}\Big|\int_{s}^{t} C(\eta)\Delta\eta\Big| \leq \frac{\varepsilon}{3},$$

$$\max_{\lambda\in[m,M]_{\mathbb{R}}}\{F(\lambda)\}\Big|\int_{s}^{t} D(\eta)\Delta\eta\Big| \leq \frac{\varepsilon}{3}$$

for every $s, t \in [t_4, t_5]_{\mathbb{T}}$ with $|t - s| \leq \delta$. The above arguments imply $|\Psi x(t) - \Psi x(s)| \leq \varepsilon$ whenever $|t - s| \leq \delta$ for $s, t \in [t_4, t_5]_{\mathbb{T}}$; i.e., $\Psi\Omega$ are locally equicontiuous. Therefore, by Arzelá-Ascoli theorem, $\Psi\Omega$ is relatively compact in $\mathrm{BC}_{\mathrm{rd}}([t_3, \infty)_{\mathbb{T}})$, and thus we conclude that Ψ is completely continuous. It follows from Krasnoselkii's fixed point theorem that there exists $x \in \Omega$ for which $\Gamma x + \Psi x = x$ holds. Therefore the proof is completed. □

Theorem 2.16. *Suppose that* (H5), (A2) *hold, and that A satisfies* (R2). *If* (A1) *does not hold, then* (1.1) *has a bounded nonoscillatory solution, which does not tend to zero asymptotically.*

Proof. To prove existence of such nonoscillatory solution, we apply Krasnoselkii's fixed point theorem. Let $K \in (0, \infty)_{\mathbb{R}}$ and $t_1 \in [t_0, \infty)_{\mathbb{T}}$ satisfy $|\Phi(\alpha^{-1}(t))| \leq K$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Since A satisfies (R2), we may pick $L, l \in (1, \infty)_{\mathbb{R}}$ with $L > l$, and $M, m \in (0, \infty)_{\mathbb{R}}$ with $M > m$ such that $L \geq A(\alpha^{-1}(t)) \geq l$, $K = [(1-l)M - Lm]/6$ and (2.16) for all $t \in [t_2, \infty)_{\mathbb{T}}$ for a sufficiently large $t_2 \in [t_1, \infty)_{\mathbb{T}}$. There exists $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that (2.17) holds. Let Ω defined in (2.18) be the subset of $\mathrm{BC}_{\mathrm{rd}}([t_3, \infty)_{\mathbb{T}}, \mathbb{R})$, and set $\Gamma, \Psi : \Omega \to \Omega$ as follows:

$$\Gamma x(t) := \begin{cases} \Gamma x(t_4), & t \in [t_3, t_4)_{\mathbb{T}} \\ \dfrac{1}{A(\alpha^{-1}(t))}\big[N - x(\alpha^{-1}(t)) + \Phi(\alpha^{-1}(t))\big], & t \in [t_4, \infty)_{\mathbb{T}} \end{cases}$$

and

$$\Psi x(t) := \begin{cases} \Psi x(t_4), & t \in [t_3, t_4)_{\mathbb{T}} \\ \dfrac{1}{A(\alpha^{-1}(t))}\Big(\displaystyle\int_{\upsilon(\alpha^{-1}(t))}^{\alpha^{-1}(t)} C(\eta)F(x(\gamma(\eta)))\Delta\eta & \\ \quad + \displaystyle\int_{\alpha^{-1}(t)}^{\infty} D(\eta)F(x(\gamma(\upsilon(\eta))))\Delta\eta\Big), & t \in [t_4, \infty)_{\mathbb{T}}, \end{cases}$$

where $t_4 \in [t_3, \infty)_{\mathbb{T}}$ satisfies $\delta(t_4) \geq t_3$ and $N := [(1 + l)M + Lm]/2$. Then, it is not hard to show that $\Gamma x + \Psi y \in \Omega$ holds for all $x, y \in \Omega$ holds. Moreover, Γ is a contraction mapping since $\|\Gamma x - \Gamma y\| < (1/l)\|x - y\|$ and Ψ is completely continuous. By Krasnoselkii's fixed point theorem, $\Gamma x + \Psi x = x$ has a solution in $x \in \Omega$. The proof for this case is hence completed. □

Corollary 2.17. *Assume that* (H1), (H2), (H4), (A2) *hold and A satisfies either* (R1) *or* (R2). *Every solution of* (1.1) *oscillates or converges to zero at infinity if and only if* (A1) *holds.*

Corollary 2.18. *Assume that* (H1), (H3), (H4), (A2) *hold and A satisfies either* (R1) *or* (R2). *Every unbounded solution of* (1.1) *oscillates if and only if* (A1) *holds.*

Theorem 2.19. *Suppose that* (H5), (A2) *hold, and that A satisfies* (R4). *If* (A1) *does not hold, then* (1.1) *has a bounded nonoscillatory solution, which does not tend to zero asymptotically.*

Proof. For this case the proof is very similar to that in the proof of Theorem 2.16 by letting $M, m \in (0, \infty)_{\mathbb{R}}$ with $M > m$ satisfy $K = [(1 - l)M - Lm]/6$ and $N := [(1 - l)M + Lm]/2$, where $L, l \in (1, \infty)_{\mathbb{R}}$ with $L > l$ satisfies $-l \geq A(t) \geq -L$ for all sufficiently large t by (R4). Finally, we find that the fixed point of $\Gamma x + \Psi x = x \in \Omega$ is the desired solution of (1.1). Therefore the proof is completed. □

Corollary 2.20. *Assume that* (H1), (H2), (H4), (A2) *hold and A satisfies* (R4). *Every bounded solution of* (1.1) *oscillates or converges to zero at infinity if and only if* (A1) *holds.*

The following example is an application for Theorem 2.15, Theorem 2.16 and Theorem 2.19.

Example 2.21. Let $\mathbb{T}$ be any of the sets $\mathbb{R}$, $\mathbb{Z}$ or $\mathbb{P}_{1/2,1/2}$. For $\lambda \neq \pm 1$, consider the following dynamic equation

$$\big[x(t) + \lambda x(t-1)\big]^{\Delta} + \frac{2}{t^2}(x(t-3))^2 - \frac{1}{t^2}(x(t-1))^2 = \frac{1}{t^2} \tag{2.19}$$

for $t \in [1,\infty)_{\mathbb{T}}$, where $A(t) \equiv \lambda$, $\alpha(t) = t-1$, $F(\lambda) = \lambda^2$, $B(t) = 2/t^2$, $\beta(t) = t-3$, $C(t) = 1/t^2$, $\gamma(t) = t-1$ and $\varphi(t) = 1/t^2$ for $t \in [1,\infty)_{\mathbb{T}}$ and $\lambda \in \mathbb{R}$. This equation satisfies all the assumptions of Theorem 2.15 for $\lambda \in (-1,1)_{\mathbb{R}}$, Theorem 2.16 for $\lambda > 1$ and Theorem 2.19 for $\lambda < -1$. Thus, (2.19) admits a nonoscillatory bounded solutions which does not asymptotically tend to zero, and $x(t) \equiv 1$ for $t \in [1,\infty)_{\mathbb{T}}$ is such a solution.

3. Final comments

Our results proved in the pervious section are still true for bounded solutions when D is eventually nonpositive. Also, Theorem 2.1, Theorem 2.4, Theorem 2.7 and Theorem 2.11 apply for the following type of equations:

$$\big[x(t) + A(t)x(\alpha(t))\big]^{\Delta} + B(t)H(x(\beta(t))) - C(t)F(x(\gamma(t))) = \varphi(t) \tag{3.1}$$

for $t \in [t_0,\infty)_{\mathbb{T}}$, where $A, \alpha, B, C, F, \beta, \gamma, \varphi$ are as mentioned before and $H \in \mathrm{C}_{\mathrm{rd}}(\mathbb{R},\mathbb{R})$ satisfies $H(\lambda)/F(\lambda) \geq 1$ for all $\lambda \in \mathbb{R}\backslash\{0\}$.

It would be a significant interest to study the asymptotic properties of unbounded solutions when F in (3.1) provides superlinear a growth when $\upsilon(t) \leq t (\not\equiv t)$ holds for all sufficiently large t; i.e., (H4) and (A3) do not hold simultaneously. The most important improvement of this paper is that the nonlinear term F needs neither to be nondecreasing as in [26, 27, 28] nor needs to satisfy $\liminf_{\lambda\to\infty}\big[F(\lambda)/\lambda\big] > 0$ as in [28]. Moreover, unlike to all of the results in the papers [12, 14, 15, 16, 18, 20, 21, 22, 23, 24, 26, 27, 30, 31], we do not need υ to be a delay function; i.e, $\upsilon(t) < t$ for all sufficiently large t; i.e., $t - \upsilon(t)$ is allowed to alternate in sign infinitely many times (see Example 2.12).

Now, consider the neutral dynamic equation

$$\big[x(t) + A(t)x(\alpha(t))\big]^{\Delta} + B(t)F(x(\beta(t))) = \varphi(t) \tag{3.2}$$

for $t \in [t_0,\infty)_{\mathbb{T}}$, where $A, \alpha, F, \beta, \varphi$ are as stated previously and $B \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is allowed to oscillate, then (3.2) can be rewritten in the form

$$\big[x(t) + A(t)x(\alpha(t))\big]^{\Delta} + B^{+}(t)F(x(\beta(t))) - B^{-}(t)F(x(\beta(t))) = \varphi(t)$$

for $t \in [t_0,\infty)_{\mathbb{T}}$, which has the same form with (1.1). Hence, our results can be applied to (3.2), and thus, we not only extend the results of [10] but also improve the results of [10, 16].

We finalize the work with the following example, which illustrates the importance of the assumption (A2).

Example 3.1. Let $\mathbb{T} = [0,\infty)_{\mathbb{R}}$ and $a \in (-1,1)_{\mathbb{R}} \cup (1,9)_{\mathbb{R}}$. Consider the following linear homogeneous differential equation:

$$\big[x(t) + ax(t/9)\big]' + \frac{4}{t}x(t/4) - \Big(\frac{5}{2} + \frac{a}{6}\Big)\frac{1}{t}x(t) = 0 \tag{3.3}$$

for $t \in [1,\infty)_{\mathbb{R}}$. For this equation, we have $A(t) \equiv a$, $\alpha(t) = t/9$, $B(t) = 4/t$, $\beta(t) = t/4$, $F(\lambda) = \lambda$, $C(t) = (5/2 + a/6)/t$, $\gamma(t) = t$ and $\varphi(t) = \Phi(t) \equiv 0$ for $t \in [0,\infty)_{\mathbb{R}}$ and $\lambda \in \mathbb{R}$. One can check that all the assumptions of Theorem 2.1 for $a \in (-1,1)_{\mathbb{R}}$ and Theorem 2.4 for $a \in (1,9)_{\mathbb{R}}$ are satisfied except (A2) since

$$\lim_{t\to\infty} \int_{t/4}^{t} \Big(\frac{5}{2} + \frac{a}{6}\Big)\frac{1}{\eta}\mathrm{d}\eta = \Big(5 + \frac{a}{3}\Big)\ln(2) \neq 0.$$

And (3.3) admits a nonoscillatory unbounded solution $x(t) = \sqrt{t}$ for $t \in [1,\infty)_{\mathbb{R}}$, which asymptotically tends to infinity.

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