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EXISTENCE AND UNIQUENESS FOR MAGNETOHYDRODYNAMIC FLOWS IN PIPES WITH VISCOSITY DEPENDENT ON THE TEMPERATURE

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Abstract. The steady motion of a viscous fluid in pipes of arbitrary cross-sections under a transverse magnetic field is studied, assuming that the viscosity and the electric and thermal conductivity are given functions of the temperature. Theorems of existence and uniqueness for the nonlinear elliptic system governing the problem are presented.

1. Introduction

In this paper we study a class of steady, incompressible rectilinear flows of viscous electrically and thermally conducting fluids along cylindrical channels of arbitrary cross-section in the framework of the equations of magnetohydrodynamics. The open and bounded subset Ω of $\mathbb{R}^2$ representing the cross-section of the pipe is referred to the orthogonal frame Oxy with unit vectors $\mathbf{i}$ and $\mathbf{j}$. Oz is the axis of the channel with $\mathbf{k}$ as unit vector. The magnetic field $\mathbf{H}$ is assumed of the form

$$\mathbf{H} = \mathbf{M} + h(x, y)\mathbf{k},$$

where $\mathbf{M}$ is a vector constant and parallel to Ω. Rotating the Oxy frame we can write

$$\mathbf{H} = M\mathbf{i} + h(x, y)\mathbf{k}. \tag{1.1}$$

Since the flow is laminar and rectilinear,

$$\mathbf{v} = v(x, y)\mathbf{k} \tag{1.2}$$

is the velocity of the fluid and $p = p(z)$ the pressure. In the steady state, Maxwell's equations reads

$$\nabla \times \mathbf{E} = 0, \tag{1.3}$$

$$\nabla \times \mathbf{H} = \mathbf{J}, \tag{1.4}$$

where $\mathbf{E}$ is the electric field and $\mathbf{J}$ the current density. Taking the curl of Ohm's law

$$\rho\mathbf{J} = \mathbf{E} + \mathbf{v} \times \mathbf{H}$$

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where ρ is the resistivity and recalling (1.1), (1.2), (1.3) and (1.4), we have

$$\nabla \cdot (\rho \nabla h) + M\frac{\partial v}{\partial x} = 0. \tag{1.5}$$

Moreover, in view of (1.1) and (1.2) the equation of motion reduces to

$$\nabla \cdot (\eta \nabla v) + M\frac{\partial h}{\partial x} = -k \tag{1.6}$$

where k is the constant pressure gradient and η the viscosity. The system of partial differential equations (1.5), (1.6) has been studied in [3] and [4] for its relevance in applications, as e.g. in electromagnetic flow-measurements [5]. Crucial in this treatment is the hypothesis of a constant viscosity and resistivity. In this paper we study a non-linear version of the system (1.5), (1.6) in which viscosity and resistivity are given functions of the temperature θ. In practical cases this dependence can be quite strong. Thus we need to add the energy equation

$$-\nabla \cdot (\kappa \nabla \theta) = \eta |\nabla v|^2 + \rho |\nabla h|^2 \tag{1.7}$$

to the system. In (1.7) κ is the thermal conductivity, also a function of the temperature. The first term on the right hand side of (1.7) reflects the viscous attrition and the second the Joule heating. Let Γ be the boundary of Ω. Assuming the walls of the pipe to be a perfect electrical insulant we have

$$\mathbf{J} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma. \tag{1.8}$$

Moreover, (1.4) reads

$$J_x = \frac{\partial h}{\partial y}, \quad J_y = -\frac{\partial h}{\partial x}, \tag{1.9}$$

thus (1.8) and (1.9) imply that h is constant on Γ (with possibly different values if Ω is not simply connected). We shall study two different boundary value problems for the system (1.5), (1.6), (1.7), more precisely a "Poiseuille" case Pb_P

$$\nabla \cdot (\eta(\theta) \nabla v) + M\frac{\partial h}{\partial x} = -k \quad \text{in} \quad \Omega \quad v = 0 \quad \text{on } \Gamma, \tag{1.10}$$

$$\nabla \cdot (\rho(\theta) \nabla h) + M\frac{\partial v}{\partial x} = 0 \quad \text{in} \quad \Omega \quad h = 0 \quad \text{on } \Gamma, \tag{1.11}$$

$$\begin{gathered} -\nabla \cdot (\kappa(\theta) \nabla \theta) = \eta(\theta) |\nabla v|^2 + \rho(\theta) |\nabla h|^2 \quad \text{in } \Omega \\ \theta = \Theta_b \quad \text{on } \Gamma, \end{gathered} \tag{1.12}$$

and a "Couette" case in which Ω is doubly-connected with boundary consisting of two curves Γ_1 and Γ_2 with $\Gamma = \Gamma_1 \cup \Gamma_2$, and $\Gamma_1 \cap \Gamma_2 = \emptyset$. The external wall of the pipe, of cross-section Γ_2, moves with respect to the internal one with cross-section Γ_1 with constant velocity V and with absence of pressure gradient. This implies $k = 0$ in (1.6). In this way we obtain problem Pb_C

$$\begin{gathered} \nabla \cdot (\eta(\theta) \nabla v) + M\frac{\partial h}{\partial x} = 0 \quad \text{in } \Omega \\ v = 0 \quad \text{on } \Gamma_1, \quad v = V \quad \text{on } \Gamma_2, \\ \nabla \cdot (\rho(\theta) \nabla h) + M\frac{\partial v}{\partial x} = 0 \quad \text{in } \Omega \\ h = 0 \quad \text{on } \Gamma_1, \quad h = H \quad \text{on } \Gamma_2, \\ -\nabla \cdot (\kappa(\theta) \nabla \theta) = \eta(\theta) |\nabla v|^2 + \rho(\theta) |\nabla h|^2 \quad \text{in } \Omega \\ \theta = \Theta_b \quad \text{on } \Gamma, \end{gathered}$$

where H is a given constant. The boundary value of the temperature is supposed to be the trace of a function $\Theta \in H^2(\Omega)$ harmonic in Ω. Moreover we assume Γ to be of class $\mathcal{C}^2$. In Section 2 we prove, using an elliptic regularization, that problems Pb_P and Pb_C have at least one weak solution. A result of uniqueness for problem Pb_C is presented in Section 3.

2. Existence and Uniqueness of Weak Solutions

We assume Γ to be regular (e.g. C^2). For later use we recall the following results.

Lemma 2.1. *Let $a(\mathbf{x})$, $b(\mathbf{x}) \in L^\infty(\Omega)$, $\mathbf{x} = (x, y)$ and*

$$a(\mathbf{x}) \geq a_0 > 0, \quad b(\mathbf{x}) \geq b_0 > 0. \tag{2.1}$$

Then the system

$$v \in H_0^1(\Omega), \quad \int_\Omega \big[a(\mathbf{x})\nabla v \cdot \nabla\varphi - Mh_x\varphi\big] dX = k\int_\Omega \varphi dX, \quad \forall\varphi \in H_0^1(\Omega), \tag{2.2}$$

$$h \in H_0^1(\Omega), \quad \int_\Omega \big[b(\mathbf{x})\nabla h \cdot \nabla\psi - Mv_x\psi\big] dX = 0, \quad \forall\psi \in H_0^1(\Omega), \tag{2.3}$$

has one and only one solution. Moreover,

$$\|v\|_{H_0^1(\Omega)} + \|h\|_{H_0^1(\Omega)} \leq C, \tag{2.4}$$

$$\max_\Omega |v| + \max_\Omega |h| \leq C, \tag{2.5}$$

where the constant C depends only on a_0, b_0, k, M and Ω.

Proof. The bilinear form

$$a((v,h),(\varphi,\psi)) = \int_\Omega \Big[a(\mathbf{x})\nabla v \cdot \nabla\varphi + b(\mathbf{x})\nabla h \cdot \nabla\psi - M\Big(\frac{\partial h}{\partial x}\varphi + \frac{\partial v}{\partial x}\psi\Big)\Big] dX$$

is bounded and coercive in $H_0^1(\Omega) \times H_0^1(\Omega)$, as easily verified. Therefore, by the Lax-Milgram lemma, the system (2.2), (2.3) has one and only one solution which satisfies (2.4). By standard elliptic regularity (see [2]), (2.5) follows. □

The main difficulty in problem Pb_P lies in the quadratic growth in the right hand side of equation (1.12). However, from (1.10) and (1.11) we have

$$\eta(\theta)|\nabla v|^2 + \rho(\theta)|\nabla h|^2 = \nabla\cdot(h\rho(\theta)\nabla h) + \nabla\cdot(v\eta(\theta)\nabla v) + Mh\frac{\partial v}{\partial x} + Mv\frac{\partial h}{\partial x} + kv. \tag{2.6}$$

This suggests the following weak formulation:

$$v \in H_0^1(\Omega), \quad \int_\Omega \Big[\eta(\theta)\nabla v \cdot \nabla\varphi - M\frac{\partial h}{\partial x}\varphi\Big] dX = \int_\Omega k\varphi dX, \quad \forall\varphi \in H_0^1(\Omega), \tag{2.7}$$

$$h \in H_0^1(\Omega), \quad \int_\Omega \Big[\rho(\theta)\nabla h \cdot \nabla\zeta - M\frac{\partial v}{\partial x}\zeta\Big] dX = 0, \quad \forall\zeta \in H_0^1(\Omega), \tag{2.8}$$

$$\begin{aligned} &\theta - \Theta \in H_0^1(\Omega), \quad \int_\Omega \kappa(\theta)\nabla\theta \cdot \nabla\xi dX \\ &= -\int_\Omega h\rho(\theta)\nabla h \cdot \nabla\xi dX + M\int_\Omega h\frac{\partial v}{\partial x}\xi dX - \int_\Omega v\eta(\theta)\nabla v \cdot \nabla\xi dX \\ &\quad + \int_\Omega v\Big(M\frac{\partial h}{\partial x} + k\Big)\xi dX, \quad \forall\xi \in H_0^1(\Omega), \end{aligned} \tag{2.9}$$

where we assume $\eta(u)$, $\rho(u)$, $\kappa(u)$ to be continuous and to satisfy

$$\eta_1 \geq \eta(u) \geq \eta_0 > 0, \quad \rho_1 \geq \rho(u) \geq \rho_0 > 0, \quad \kappa_1 \geq \kappa(u) \geq \kappa_0 > 0. \tag{2.10}$$

To prove existence of a weak solution we consider the following sequence of approximating problems Pb_ϵ:

$$v_\epsilon \in H_0^1(\Omega), \quad \int_\Omega \eta(\theta_\epsilon)\nabla v_\epsilon \cdot \nabla\varphi dX - M\int_\Omega \frac{\partial h_\epsilon}{\partial x}\varphi dX = k\int_\Omega \varphi dX, \quad \forall\varphi \in H_0^1(\Omega) \tag{2.11}$$

$$h_\epsilon \in H_0^1(\Omega), \quad \int_\Omega \rho(\theta_\epsilon)\nabla h_\epsilon \cdot \nabla\zeta dX - M\int_\Omega \frac{\partial v_\epsilon}{\partial x}\zeta dX = 0, \forall\zeta \in H_0^1(\Omega) \tag{2.12}$$

$$\begin{aligned} &\theta_\epsilon - \Theta \in H_0^2(\Omega), \quad \epsilon\int_\Omega \Delta\theta_\epsilon\Delta\xi dX + \int_\Omega \kappa(\theta_\epsilon)\nabla\theta_\epsilon \cdot \nabla\xi dX \\ &= -\int_\Omega h_\epsilon\rho(\theta_\epsilon)\nabla h_\epsilon \cdot \nabla\xi dX + M\int_\Omega h_\epsilon\frac{\partial v_\epsilon}{\partial x}\xi dX - \int_\Omega v_\epsilon\eta(\theta_\epsilon)\nabla v_\epsilon \cdot \nabla\xi dX \\ &\quad + \int_\Omega v_\epsilon\Big(M\frac{\partial h_\epsilon}{\partial x} + k\Big)\xi dX, \quad \forall\xi \in H_0^2(\Omega). \end{aligned} \tag{2.13}$$

Lemma 2.2. *Let $(v_\epsilon, h_\epsilon, \theta_\epsilon)$ be a solution to Pb_ϵ. Then the following "a priori" estimates hold:*

$$\|v_\epsilon\|_{H_0^1(\Omega)} \leq C, \tag{2.14}$$
$$\|h_\epsilon\|_{H_0^1(\Omega)} \leq C, \tag{2.15}$$
$$\|\theta_\epsilon - \Theta|_{H_0^1(\Omega)} \leq C, \tag{2.16}$$
$$\epsilon\|\Delta\theta_\epsilon\|_{L^2(\Omega)}^2 \leq C, \tag{2.17}$$
$$\max_\Omega |v_\epsilon| \leq C, \quad \max_\Omega |h_\epsilon| \leq C \tag{2.18}$$

where the C's denote constants, generally different, depending on η_0, ρ_0, κ_0, Ω, M and k, but not on ϵ

Proof. Setting $\varphi = v_\epsilon$ in (2.11) and $\zeta = h_\epsilon$ in (2.12), we have

$$\int_\Omega \eta(\theta_\epsilon)|\nabla v_\epsilon|^2 dX - M\int_\Omega \frac{\partial h_\epsilon}{\partial x}v_\epsilon dX = k\int_\Omega v_\epsilon dX,$$
$$\int_\Omega \rho(\theta_\epsilon)|\nabla h_\epsilon|^2 dX + M\int_\Omega \frac{\partial h_\epsilon}{\partial x}v_\epsilon dX = 0.$$

Adding and using the Poincarè inequality we obtain (2.14) and (2.15) by (2.10). Applying Lemma 2.1 we get (2.18). Choosing $\xi = \theta_\epsilon - \Theta$ in (2.13) we have

$$\begin{aligned} &\epsilon\int_\Omega \Delta\theta_\epsilon\Delta(\theta_\epsilon - \Theta)dX + \int_\Omega \kappa(\theta_\epsilon)\nabla\theta_\epsilon \cdot \nabla(\theta_\epsilon - \Theta)dX \\ &= -\int_\Omega h_\epsilon\rho(\theta_\epsilon)\nabla h_\epsilon \cdot \nabla(\theta_\epsilon - \Theta)dX + M\int_\Omega h_\epsilon\frac{\partial v_\epsilon}{\partial x}(\theta_\epsilon - \Theta)dX \\ &- \int_\Omega v_\epsilon\eta(\theta_\epsilon)\nabla v_\epsilon \cdot \nabla(\theta_\epsilon - \Theta)dX + \int_\Omega v_\epsilon\Big(M\frac{\partial h_\epsilon}{\partial x} + k\Big)(\theta_\epsilon - \Theta)dX. \end{aligned}$$

Using repeatedly the Hölder inequality and (2.10), (2.14), (2.15) we obtain

$$\epsilon\|\Delta\theta_\epsilon\|_{L^2(\Omega)}^2 + \|\nabla\theta_\epsilon\|_{L^2(\Omega)}^2 \leq C\big(\|\Delta\theta_\epsilon\|_{L^2(\Omega)} + \|\nabla\theta_\epsilon\|_{L^2(\Omega)} + 1\big) \tag{2.19}$$

from which (2.16) and (2.17) follow. ∎

We recall the classical Leray-Schauder fixed point theorem.

Theorem 2.3. *Let $\mathcal{B}$ be a Banach space and $\mathcal{T}$ a continuous and compact mapping from $\mathcal{B}\times[0,1]$ into $\mathcal{B}$ such that $\mathcal{T}(w;0)=\bar{u}$ for all $w\in\mathcal{B}$. If all solutions of the equation*

$$w=\mathcal{T}(w;\lambda),\quad w\in\mathcal{B},\ \lambda\in[0,1]$$

are bounded in $\mathcal{B}$ by a constant not depending on λ, then $\mathcal{T}(w,1)$ has a fixed point in $\mathcal{B}$.

Lemma 2.4. *For every $\epsilon>0$ there exists at least one solution to Pb_ϵ.*

Proof. We omit the dependence of ϵ in v_ϵ, h_ϵ and θ_ϵ. Let $\mathcal{B}=H_0^1(\Omega)$ and define

$$\theta=\mathcal{T}(w,\lambda),\quad \mathcal{T}:\mathcal{B}\times[0,1]\to\mathcal{B}$$

via the linear problem

$$v\in H_0^1(\Omega),\quad \nabla\cdot(\eta(\lambda w)\nabla v)+M\frac{\partial h}{\partial x}=-k, \tag{2.20}$$

$$h\in H_0^1(\Omega),\quad \nabla\cdot(\rho(\lambda w)\nabla h)+M\frac{\partial v}{\partial x}=0, \tag{2.21}$$

$$\begin{aligned}&\theta-\Theta\in H_0^2(\Omega),\quad \epsilon\Delta\Delta\theta+\nabla\cdot(\kappa(\lambda w)\nabla\theta)\\ &=\nabla\cdot(h\rho(\lambda w)\nabla h)+Mh\frac{\partial v}{\partial x}+\nabla\cdot(v\eta(\lambda w)\nabla v)+kv+Mv\frac{\partial h}{\partial x}.\end{aligned} \tag{2.22}$$

Given $w\in\mathcal{B}$ the system (2.20), (2.21) is solvable by Lemma 2.1. Moreover, the right hand side of (2.22) defines a bounded linear functional in $H^2(\Omega)$. Hence (2.22) is solvable with respect to θ and the mapping $(w,\lambda)\to\theta$ is well-defined. Let $(\bar{v},\bar{h},\bar{\theta})$ solve

$$\begin{gathered}\bar{v}\in H_0^1(\Omega),\quad \nabla\cdot(\eta(0)\nabla\bar{v})+M\frac{\partial\bar{h}}{\partial x}=-k,\\ \bar{h}\in H_0^1(\Omega),\quad \nabla\cdot(\rho(0)\nabla\bar{h})+M\frac{\partial\bar{v}}{\partial x}=0,\\ \bar{\theta}-\Theta\in H_0^2(\Omega),\quad \epsilon\Delta\Delta\bar{\theta}+\nabla\cdot(\kappa(0)\nabla\bar{\theta})\\ =\nabla\cdot(\bar{h}\rho(0)\nabla\bar{h})+M\bar{h}\frac{\partial\bar{v}}{\partial x}+\nabla\cdot(\bar{v}\eta(0)\nabla\bar{v})+k\bar{v}+M\bar{v}\frac{\partial\bar{h}}{\partial x}.\end{gathered}$$

We have $\mathcal{T}(w,0)=\bar{\theta}$ for all $w\in\mathcal{B}$. To prove the continuity of $\mathcal{T}(w,\lambda)$, suppose $(w_i,\lambda_i)\to(w^*,\lambda^*)$ in $\mathcal{B}\times[0,1]$ and

$$\theta_i=\mathcal{T}(w_i,\lambda_i),\quad \theta^*=\mathcal{T}(w^*,\lambda^*).$$

Let $(v_i,h_i)\in H_0^1(\Omega)\times H_0^1(\Omega)$ be solution of the system

$$\int_\Omega\eta(\lambda_iw_i)\nabla v_i\cdot\nabla\varphi dX-M\int_\Omega\frac{\partial h_i}{\partial x}\varphi dX=k\int_\Omega\varphi dX,\quad \text{for all }\varphi\in H_0^1(\Omega), \tag{2.23}$$

$$\int_\Omega\rho(\lambda_iw_i)\nabla h_i\cdot\nabla\zeta dX-M\int_\Omega\frac{\partial v_i}{\partial x}\zeta dX=0,\quad \text{for all }\zeta\in H_0^1(\Omega), \tag{2.24}$$

and (v^*,h^*) of the system

$$\int_\Omega\eta(\lambda^*w^*)\nabla v^*\cdot\nabla\varphi dX-M\int_\Omega\frac{\partial h^*}{\partial x}\varphi dX=k\int_\Omega\varphi dX,\quad \text{for all }\varphi\in H_0^1(\Omega), \tag{2.25}$$

$$\int_\Omega \rho(\lambda^* w^*)\nabla h_i \cdot \nabla\zeta dX - M\int_\Omega \frac{\partial v^*}{\partial x}\zeta dX = 0, \quad \text{for all } \zeta \in H_0^1(\Omega). \tag{2.26}$$

Choosing $\varphi = v_i - v^*$ in (2.23) and (2.25) and $\zeta = h_i - h^*$ in (2.24) and (2.26) we have, after simple calculations,

$$\begin{aligned} &\|v_i - v^*\|^2_{H_0^1(\Omega)} + \|h_i - h^*\|^2_{H_0^1(\Omega)} \\ &\le C\Big[\|\eta(w_i\lambda_i) - \eta(\lambda^* w^*)\|_{L^\infty(\Omega)}\|v_i\|_{H_0^1(\Omega)}\|v_i - v^*\|_{H_0^1(\Omega)} \\ &\quad + \|\rho(w_i\lambda_i) - \rho(\lambda^* w^*)\|_{L^\infty(\Omega)}\|h_i\|_{H_0^1(\Omega)}\|h_i - h^*\|_{H_0^1(\Omega)}\Big]. \end{aligned} \tag{2.27}$$

Hence

$$v_i \to v^*, \quad h_i \to h^* \quad \text{in } H_0^1(\Omega). \tag{2.28}$$

Let θ_i and θ^* be given respectively by

$$\begin{aligned} &\theta_i - \Theta \in H_0^2(\Omega), \quad \epsilon\int_\Omega \Delta\theta_i\Delta\xi dX + \int_\Omega \kappa(\theta_i)\nabla\theta_i \cdot \nabla\xi dX \\ &= -\int_\Omega h_i\rho(\lambda_i w_i)\nabla h_i \cdot \nabla\xi dX + k\int_\Omega v_i\xi dX + M\int_\Omega h_i\frac{\partial v_i}{\partial x}\xi dX \\ &+ M\int_\Omega v_i\frac{\partial h_i}{\partial x}\xi dX - \int_\Omega v_i\eta(\lambda_i w_i)\nabla v_i \cdot \nabla\xi dX, \quad \text{for all } \xi \in H_0^2(\Omega), \end{aligned} \tag{2.29}$$

$$\begin{aligned} &\theta^* - \Theta \in H_0^2(\Omega), \quad \epsilon\int_\Omega \Delta\theta^*\Delta\xi dX + \int_\Omega \kappa(\theta^*)\nabla\theta^* \cdot \nabla\xi dX \\ &= -\int_\Omega h^*\rho(\lambda^* w^*)\nabla h^* \cdot \nabla\xi dX + k\int_\Omega v^*\xi dX + M\int_\Omega h^*\frac{\partial v^*}{\partial x}\xi dX \\ &+ M\int_\Omega v^*\frac{\partial h^*}{\partial x}\xi dX - \int_\Omega v^*\eta(\lambda^* w^*)\nabla v^* \cdot \nabla\xi dX, \quad \text{for all } \xi \in H_0^2(\Omega). \end{aligned} \tag{2.30}$$

By difference from (2.29) and (2.30), setting $\xi = \theta_i - \theta^*$ in the resulting equation, using the Poincare inequality and the Sobolev imbedding theorem we have

$$\begin{aligned} &\epsilon\|\Delta(\theta_i - \theta^*)\|^2_{L^2(\Omega)} + \|\nabla(\theta_i - \theta^*)\|^2_{L^2(\Omega)} \\ &\le C\Big[\|h_i - h^*\|_{H_0^1(\Omega)} + \|v_i - v^*\|_{H_0^1(\Omega)} + \|\rho(\lambda_i w_i) - \rho(\lambda^* w^*)\|_{L^\infty(\Omega)} \\ &\quad + \|\eta(\lambda_i w_i) - \eta(\lambda^* w^*)\|_{L^\infty(\Omega)}\Big]\|\Delta(\theta_i - \theta^*)\|_{L^2(\Omega)}. \end{aligned} \tag{2.31}$$

From (2.31) the continuity of $\mathcal{T}(w,\lambda)$ easily follows. The mapping $\mathcal{T}(w,\lambda)$ is also compact, since in dimension 2 bounded subsets of $H_0^2(\Omega)$ are compact in $H_0^1(\Omega)$. Finally, repeating with minor changes the proof of Lemma 2.2, we can prove that all solutions of the equation

$$\theta = \mathcal{T}(\theta,\lambda)$$

are bounded in the $\mathcal{B}$-norm by a constant not depending on λ. Hence problem Pb_ϵ has at least one solution by the Leray-Schauder principle. □

Theorem 2.5. *There exists at least one weak solution to problem Pb_P.*

Proof. By (2.14), (2.15) and (2.16) we can extract from $\{v_\epsilon\}$, $\{h_\epsilon\}$ and $\{\theta_\epsilon\}$ subsequences (not relabelled) such that

$$\begin{gathered} v_\epsilon \to v \quad \text{weakly in } H_0^1(\Omega), \quad h_\epsilon \to h \quad \text{weakly in } H_0^1(\Omega), \\ \theta_\epsilon \to \theta \quad \text{weakly in } H^1(\Omega) \end{gathered} \tag{2.32}$$

and

$$\begin{gathered} \theta_\epsilon \to \theta \quad \text{in } L^2(\Omega), \quad \kappa(\theta_\epsilon) \to \kappa(\theta) \quad \text{in } L^p(\Omega), \\ \rho(\theta_\epsilon) \to \rho(\theta) \quad \text{in } L^p(\Omega), \quad \eta(\theta_\epsilon) \to \eta(\theta) \quad \text{in } L^p(\Omega), \quad 1 \le p < \infty. \end{gathered} \tag{2.33}$$

Letting $\epsilon \to 0$ in (2.11) and (2.12), we have (2.7) and (2.8). It remains to pass to the limit for $\epsilon \to 0$ in (2.13). By (2.17), the first term in the left hand side of (2.13) vanishes when $\epsilon \to 0$. Moreover, by (2.32), (2.33),

$$\int_\Omega \kappa(\theta_\epsilon)\nabla\theta_\epsilon \cdot \nabla\xi dX \to \int_\Omega \kappa(\theta)\nabla\theta \cdot \nabla\xi dX.$$

To pass to the limit in the first term in the right hand side of (2.13) we write

$$\int_\Omega \Big[h_\epsilon\rho(\theta_\epsilon)\nabla h_\epsilon - h\rho(\theta)\nabla h\Big] \cdot \nabla\xi dX = I_1 + I_2 + I_3$$

where

$$I_1 = \int_\Omega \big[(h_\epsilon - h)\rho(\theta_\epsilon)\nabla h_\epsilon\big]\cdot\nabla\xi dX, \quad I_2 = \int_\Omega \Big[h\Big(\rho(\theta_\epsilon) - \rho(\theta)\Big)\nabla h_\epsilon\Big] \cdot \nabla\xi dX,$$
$$I_3 = \int_\Omega h\rho(\theta)(\nabla h_\epsilon - \nabla h) \cdot \nabla\xi dX.$$

We have

$$\begin{gathered} |I_1| \le \rho_1 \|h_\epsilon - h\|_{L^2(\Omega)} \|\nabla h_\epsilon\|_{L^2(\Omega)} \|\nabla\xi\|_{L^\infty(\Omega)}, \\ |I_2| \le \|h\|_{L^p(\Omega)} \|\rho(\theta_\epsilon) - \rho(\theta)\|_{L^p(\Omega)} \|\nabla h_\epsilon\|_{L^2(\Omega)} \|\nabla\xi\|_{L^\infty(\Omega)}, \quad 1/p + 1/q + 1/2 = 1. \end{gathered}$$

Moreover, by (2.32) and (2.33),

$$I_3 = \int_\Omega h\rho(\theta)\big(\nabla h_\epsilon - \nabla h\big) \cdot \nabla\xi dX \to 0, \quad \text{as } \epsilon \to 0.$$

The remaining terms in the right hand side of (2.13) can be dealt with similarly. In the end we obtain (2.9). Thus problem Pb_P has a weak solution. □

3. The Couette Case

Only minor changes are needed to prove that also problem Pb_C has a solution. Uniqueness seems to be, in general, an open question for both problems Pb_P and Pb_C. However, in special cases existence, non-existence and uniqueness can be proved for problem Pb_C, even suppressing the hypothesis of ellipticity (2.10).

Theorem 3.1. *Suppose that in problem Pb_C: $M = 0$, $\Theta_b = 0$, $\kappa(\theta) > 0$, $\rho(\theta) > 0$, $\eta(\theta) > 0$, $\rho(\theta) = \gamma\eta(\theta)$, $\gamma > 0$. Assume that*

$$\int_0^\infty \frac{\kappa(t)}{\rho(t)} dt = l < \infty. \tag{3.1}$$

Then, if $1 + \frac{\gamma V^2}{H^2} < l$ problem Pb_C has one and only one solution. If $l \le 1 + \frac{\gamma V^2}{H^2}$ problem Pb_C has no solution. If, on the contrary,

$$\int_0^\infty \frac{\kappa(t)}{\rho(t)} dt = \infty,$$

then problem Pb_C has one and only one solution.

The proof is based on the transformation

$$\Psi = \frac{1}{2}h^2 + \frac{\gamma}{2}v^2 + \int_0^\theta \frac{\kappa(t)}{\rho(t)}dt, \tag{3.2}$$

which gives the equations

$$\nabla \cdot (\rho(\theta)\nabla\Psi) = 0, \tag{3.3}$$

$$\nabla \cdot (\rho(\theta)\nabla v) = 0, \tag{3.4}$$

$$\nabla \cdot (\rho(\theta)\nabla h) = 0, \tag{3.5}$$

and the boundary conditions

$$\begin{gathered} \Psi = 0 \quad \text{on } \Gamma_1, \quad \Psi = \frac{1}{2}H^2 + \frac{\gamma}{2}V^2 \quad \text{on } \Gamma_2, \\ v = 0 \quad \text{on } \Gamma_1, \quad v = V \quad \text{on } \Gamma_2, \\ h = 0 \quad \text{on } \Gamma_1, \quad h = H \quad \text{on } \Gamma_2. \end{gathered}$$

The system of the three equations (3.3), (3.4), (3.5), together with the functional relation (3.2), can be reduced, quite surprisingly, to the linear Dirichlet problem

$$\Delta\Phi = 0 \quad \text{in } \Omega, \quad \Phi = 0 \quad \text{on } \Gamma_1, \quad \Phi = 1 \quad \text{on } \Gamma_2.$$

For more details, we refer the reader to [1].

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