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ANALYTIC SOLUTIONS OF A FIRST ORDER FUNCTIONAL DIFFERENTIAL EQUATION WITH A STATE DERIVATIVE DEPENDENT DELAY

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Abstract. This article concerns the first-order functional differential equation

$$x'(z) = x(p(z) + bx'(z))$$

with the distinctive feature that the argument of the unknown function depends on the state derivative. An existence theorem is established for analytic solutions and systematic methods for deriving explicit solutions are also given.

1. Introduction

In the past few years there has been a growing interest in studying functional differential equations with state dependent delay. We refer the readers to Eder [1], Elbert [2], Feckan [3], Stanek [6], Wang [7]. Qiu [4] and Stanek [6] considered the equation

$$x'(z) = x\big(p(z) + bx(z)\big),$$

and establish sufficient conditions for the existence of analytic solutions. In this paper, we are concerned with analytic solutions of the first-order functional differential equation

$$x'(z) = x\big(p(z) + bx'(z)\big), \tag{1.1}$$

where b is a nonzero complex number, and $p(z)$ is the given complex function of a complex variable. A distinctive feature of (1.1) is that the argument of the unknown function depends on the state derivative. To construct analytic solution of (1.1) in a systematic manner, we first let

$$y(z) = p(z) + bx'(z). \tag{1.2}$$

Then for any number z_0, we have

$$x(z) = x(z_0) + \frac{1}{b}\int_{z_0}^{z} (y(s) - p(s))ds \tag{1.3}$$

and so

$$x(y(z)) = x(z_0) + \frac{1}{b}\int_{z_0}^{y(z)} (y(s) - p(s))ds.$$

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Therefore, in view of (1.1) and $x'(z) = \frac{1}{b}(y(z) - p(z))$, we have

$$\frac{1}{b}(y(z) - p(z)) = x(z_0) + \frac{1}{b}\int_{z_0}^{y(z)} (y(s) - p(s))ds. \tag{1.4}$$

Furthermore, differentiating both sides of (1.4) with respect to z, we obtain

$$y'(z) - p'(z) = (y(y(z)) - p(y(z)))y'(z). \tag{1.5}$$

To find analytic solutions of (1.5), we first seek an analytic solution $g(z)$ of the auxiliary equation

$$\alpha g'(\alpha z) - p'(g(z))g'(z) = \alpha[g(\alpha^2 z) - p(g(\alpha z))]g'(\alpha z), \tag{1.6}$$

satisfying the initial value conditions

$$g(0) = 0, \quad g'(0) = \eta \neq 0, \tag{1.7}$$

where η is a complex number, and α satisfies the following conditions:

- (H1) $p(z)$ is analytic in a neighborhood of the origin, furthermore, $p(0) = \beta \neq -1$ and $p'(0) = \alpha + \alpha\beta$, where α, β are complex numbers;
- (H2) $0 < |\alpha| < 1$;
- (H3) $|\alpha| = 1, \alpha$ is not a root of unity, and $\ln|\alpha^n - 1|^{-1} \leq k \ln n$, $n = 2, 3, \dots$ for some positive constant k.

Then we show that (1.5) has an analytic solution of the form

$$y(z) = g(\alpha g^{-1}(z)) \tag{1.8}$$

in a neighborhood of the origin.

2. Preparatory Lemmas

We begin with the following preparatory lemma, the proof of which can be found in cites1.

Lemma 2.1. *Assume that* (H3) *holds. Then there is a positive number δ such that $|\alpha^n - 1|^{-1} < (2n)^\delta$ for $n = 1, 2, \dots$. Furthermore, the sequence $\{d_n\}_{n=1}^{\infty}$ defined by $d_1 = 1$ and*

$$d_n = |\alpha^{n-1} - 1|^{-1} \max_{\substack{k_1 + \cdots + k_m = n \\ 0 \leq k_1 \leq \cdots \leq k_m, m \geq 2}} \{d_{k_1} \dots d_{k_m}\}, \quad n = 2, 3, \dots$$

will satisfy

$$d_n \leq N^{n-1} n^{-2\delta}, \quad n = 1, 2, \dots, \tag{2.1}$$

where $N = 2^{5\delta+1}$.

To proceed, we state and prove two preparatory lemmas which will be used in the proof of our main result.

Lemma 2.2. *Suppose* (H1)–(H2) *hold. Then for any nonzero complex number η, equation* (1.6) *has an analytic solution $g(z)$ in a neighborhood of the origin such that $g(0) = 0$ and $g'(0) = \eta$.*

Proof. Because $p(z)$ satisfies (H1),we assume

$$p(z) = \sum_{n=0}^{\infty} p_n z^n, \quad p_0 = \beta, \quad p_1 = \alpha + \alpha\beta. \tag{2.2}$$

Then there exists a positive constant ρ such that

$$|p_n| \le \rho^{n-1}, \quad n = 2, 3, \dots .$$

Introducing new functions

$$G(z) = \rho g(\rho^{-1} z), \quad P(z) = \rho p(\rho^{-1} z),$$

we obtain from $g(0) = 0$ and $g'(0) = \eta$ that $G(0) = 0$ and $G'(0) = \eta$ respectively, and by (1.6) we have

$$\alpha[G(\alpha^2 z) - P(G(\alpha z))]G'(\alpha z) = \alpha G'(\alpha z) - P'(G(z))G'(z),$$

which is again an equation of the form (1.6). Here P is of the form

$$P(z) = \sum_{n=0}^{\infty} P_n z^n,$$

but $|P_n| = |p_n \rho^{1-n}| \le 1$ for $n \ge 2$. Then, without loss of generality,we assume

$$|p_n| \le 1, \quad n = 2, 3, \dots . \tag{2.3}$$

Next, we assume that (1.6) has a formal power series solution

$$g(z) = \sum_{n=1}^{\infty} c_n z^n \tag{2.4}$$

and substitute (2.2) and (2.4) into (1.6), we see that the sequence $\{c_n\}_{n=1}^{\infty}$ is successively determined by the condition

$$\begin{aligned}
&\alpha c_1 + \alpha p_0 c_1 - p_1 c_1 \\
&+ \sum_{n=1}^{\infty} [(n+1) c_{n+1} \alpha^{n+1} + p_0 (n+1) c_{n+1} \alpha^{n+1} - p_1 (n+1) c_{n+1}] z^n \\
&= \sum_{n=1}^{\infty} \Big(\sum_{k=1}^{n} c_k c_{n-k+1} (n-k+1) \alpha^{n+k+1} \Big) z^n \\
&\quad + \sum_{n=1}^{\infty} \Big[\sum_{k=1}^{n} \sum_{\substack{l_1 + \dots + l_m = k \\ m = 1, 2, \dots, k}} ((m+1) p_{m+1} - p_m \alpha^{n+1}) c_{l_1} \dots c_{l_m} (n-k+1) c_{n-k+1} \Big] z^n,
\end{aligned}$$

where $n = 1, 2, \dots$, in a unique manner.

By comparing coefficients in both sides, it is easy to see that the coefficient sequence $\{c_n\}_{n=1}^{\infty}$ satisfies

$$(\alpha + \alpha p_0 - p_1) c_1 = 0, \tag{2.5}$$

and

$$\begin{aligned}
&(n+1)(\alpha^{n+1} + p_0 \alpha^{n+1} - p_1) c_{n+1} \\
&= \sum_{k=1}^{n} c_k c_{n-k+1} (n-k+1) \alpha^{n-k+1} \\
&\quad + \sum_{k=1}^{n} \sum_{\substack{l_1 + \dots + l_m = k \\ m = 1, 2, \dots, k}} [(m+1) p_{m+1} - p_m \alpha^{n+1}] c_{l_1} \dots c_{l_m} (n-k+1) c_{n-k+1},
\end{aligned} \tag{2.6}$$

for n=1,2,. . . .

By (H1), we can ensure $\alpha+\alpha p_0-p_1=\alpha+\alpha\beta-\alpha-\alpha\beta=0$. Hence we know it is suitable for any complex number $c_1=\eta\neq 0$. We may now see by (2.6) that the resulting relation defines c_n $(n=2,3,\dots)$ in a unique manner. Next, we want to prove that the power series (2.4) is convergent in a sufficient small neighborhood of the origin. With lemma conditions, we can prove $\alpha^{n+1}+p_0\alpha^{n+1}-p_1=\alpha(\alpha^n-1)(1+\beta)\neq 0$ for $n=1,2,\dots$. If not, assuming $\alpha(\alpha^n-1)(1+\beta)=0$, we can get $\alpha^n=1$, so $|\alpha|=1$ which contradicts condition (H2). To see this, note that

$$\lim_{n\to\infty}\frac{1}{\alpha(\alpha^n-1)(1+\beta)}=-\frac{1}{\alpha(1+\beta)},\quad 0<|\alpha|<1,$$

thus there exist some positive number M such that $|\frac{1}{\alpha(\alpha^n-1)(1+\beta)}|\leq M$ for $n\geq 1$. By (2.3) and (2.6), we have

$$|c_{n+1}|\leq M(\sum_{k=1}^{n}|c_k||c_{n-k+1}|+\sum_{k=1}^{n}\sum_{\substack{l_1+\dots+l_m=k\\ m=1,2,\dots,k}}(n+2)|c_{l_1}|\dots|c_{l_m}||c_{n-k+1}|),\tag{2.7}$$

where $n=1,2,\dots$. If we now define a positive sequence $\{q_n\}_{n=1}^{\infty}$ by $q_1=|\eta|$ and

$$q_{n+1}=M[\sum_{k=1}^{n}q_kq_{n-k+1}+\sum_{k=1}^{n}\sum_{\substack{l_1+\dots+l_m=k\\ m=1,2,\dots,k}}(n+2)q_{l_1}\dots q_{l_m}q_{n-k+1}],\quad n=1,2,\dots,$$

then it is easily seen that

$$|c_{n+1}|\leq q_{n+1},\quad n=0,1,2,\dots.$$

In other words, the series $\sum_{n=1}^{\infty}q_nz^n$ is a majorant series of $\sum_{n=1}^{\infty}c_nz^n$. So next we want to show that the power series $\sum_{n=1}^{\infty}q_nz^n$ is convergent in a sufficient small neighborhood of the origin. Now if we define $Q(z)=\sum_{n=1}^{\infty}q_nz^n$, then

$$\begin{aligned}
Q(z)&=\sum_{n=1}^{\infty}q_nz^n=\sum_{n=0}^{\infty}q_{n+1}z^{n+1}\\
&=|\eta|z+\sum_{n=1}^{\infty}q_{n+1}z^{n+1}\\
&=|\eta|z+M\sum_{n=1}^{\infty}(\sum_{k=1}^{n}q_kq_{n-k+1})z^{n+1}\\
&\quad+M[\sum_{n=1}^{\infty}(\sum_{k=1}^{n}\sum_{\substack{l_1+\dots+l_m=k\\ m=1,2,\dots,k}}(n+2)q_{l_1}\dots q_{l_m}q_{n-k+1})z^{n+1}]\\
&=|\eta|z+M(\sum_{n=1}^{\infty}q_nz^n)^2+M[\sum_{n=1}^{\infty}(n+2)(Q(z))^n](\sum_{n=1}^{\infty}q_nz^n)\\
&=|\eta|z+M[Q(z)]^2+M[\sum_{n=1}^{\infty}(n+1)(Q(z))^n]Q(z)+M\sum_{n=1}^{\infty}(Q(z))^{n+1}\\
&=|\eta|z+M[Q(z)]^2+M(\sum_{n=1}^{\infty}(Q(z))^{n+1})'Q(z)+M\frac{Q^2(z)}{1-Q(z)}\\
&=|\eta|z+M\frac{4Q^2(z)-4Q^3(z)+Q^4(z)}{[1-Q(z)]^2}.
\end{aligned}$$

So the function $Q = Q(z)$ is the implicit function which defined by the function

$$Q = |\eta| z + M \frac{4Q^2 - 4Q^3 + Q^4}{(1-Q)^2},$$

or

$$F(z, Q) \equiv Q - |\eta| z - M \frac{4Q^2 - 4Q^3 + Q^4}{(1-Q)^2} = 0.$$

Because the function $F(z, Q)$ is continuous in a neighborhood of the origin, furthermore, $F(0,0) = 0$ and $F'_Q(0,0) = 1 \neq 0$. By the implicit function theorem, we see that the $Q = Q(z)$ is analytic on a disk with the origin as the center and with a positive radius. The proof is completed. $\square$

Lemma 2.3. *Suppose* (H1), (H3) *hold. Then equation* (1.6) *has an analytic solution $g(z)$ in a neighborhood of the origin such that $g(0) = 0$ and $g'(0) = \eta \neq 0$.*

Proof. Similar to the proof of Lemma 2.2, we seek a formal power series solution (2.4) of equation (1.6) with $c_1 = \eta$ and

$$\begin{aligned} &(n+1)\alpha(\alpha^n - 1)(1+\beta)c_{n+1} \\ &= \sum_{k=1}^{n} c_k c_{n-k+1}(n+1-k)\alpha^{n+k+1} \\ &\quad + \sum_{k=1}^{n} \sum_{\substack{l_1+\dots+l_m=k \\ m=1,2,\dots,k}} [(m+1)p_{m+1} - p_m \alpha^{n+1}] c_{l_1} \dots c_{l_m} (n-k+1) c_{n-k+1} \end{aligned} \tag{2.8}$$

for $n \geq 1$. So next we want to prove that the power series $\sum_{n=1}^{\infty} c_n z^n$ is convergent in a sufficient small neighborhood of the origin.

The formulation (2.8) can be written as

$$\begin{aligned} |c_{n+1}| \leq \frac{1}{|1+\beta|} |\alpha^n - 1|^{-1} [&\sum_{k=1}^{n} |c_k||c_{n+1-k}| \\ &+ \sum_{k=1}^{n} \sum_{\substack{l_1+\dots+l_m=k \\ m=1,2,\dots,k}} (n+2)|c_{l_1}| \dots |c_{l_m}||c_{n-k+1}|], \quad n = 1, 2, \dots. \end{aligned} \tag{2.9}$$

If we now define a positive sequence $\{v_n\}_{n=1}^{\infty}$ by $v_1 = |\eta|$ and

$$v_{n+1} = M|\alpha^n - 1|^{-1} \Big[\sum_{k=1}^{n} v_k v_{n-k+1} + \sum_{k=1}^{n} \sum_{\substack{l_1+\dots+l_m=k \\ m=1,2,\dots,k}} (n+2) v_{l_1} \cdots v_{l_m} v_{n-k+1} \Big],$$

where $M = 1/(|1+\beta|) > 0$, $n \geq 1$, then it is not difficult to show by induction that

$$|c_{n+1}| \leq v_{n+1}, \quad n = 0, 1, 2, \dots. \tag{2.10}$$

In other words, $V(z) = \sum_{n=0}^{\infty} v_n z^n$ is a majorant series of $g(z) = \sum_{n=0}^{\infty} c_n z^n$. we now only need to show that $V(z)$ has a positive radius of convergence. To see this, we define the positive recursive sequence $\{q_n\}$ by $q_1 = |\eta|$ and

$$q_{n+1} = M \Big[\sum_{k=1}^{n} q_k q_{n-k+1} + \sum_{k=1}^{n} \sum_{\substack{l_1+\dots+l_m=k \\ m=1,2,\dots,k}} (n+2) q_{l_1} \dots q_{l_m} q_{n-k+1} \Big], \quad n = 1, 2, \dots.$$

Then

$$\begin{aligned}
Q(z) &= \sum_{n=1}^{\infty} q_n z^n = \sum_{n=0}^{\infty} q_{n+1} z^{n+1} \\
&= |\eta| z + \sum_{n=1}^{\infty} q_{n+1} z^{n+1} \\
&= |\eta| z + \sum_{n=1}^{\infty} M\Big(\sum_{k=1}^{n} q_k q_{n+1-k} + \sum_{k=1}^{n} \sum_{\substack{l_1+\dots+l_m=k \\ m=1,2,\dots,k}} (n+2) q_{l_1} \dots q_{l_m} q_{n-k+1}\Big) z^{n+1} \\
&= |\eta| z + \frac{4Q^2(z) - 4Q^3(z) + Q^4(z)}{[1-Q(z)]^2}.
\end{aligned} \tag{2.11}$$

Hence by induction, we easily see by Lemma 2.1 that

$$v_{n+1} \le q_{n+1} d_{n+1}, \quad n = 0, 1, 2, \dots, \tag{2.12}$$

where the sequence $\{d_n\}_{n=1}^{\infty}$ is defined by Lemma 2.1.

In fact, if $k = 0$, it holds. Assume by induction that $v_k \le q_k d_k$ for $k = 1, 2, \dots, n-1$. Then

$$\begin{aligned}
v_{n+1} &= M|\alpha^n - 1|^{-1} \Big[\sum_{k=1}^{n} v_k v_{n+1-k} + \sum_{k=1}^{n} \sum_{\substack{l_1+\dots+l_m=k \\ m=1,2,\dots,k}} (n+2) v_{l_1} \dots v_{l_m} v_{n-k+1} \Big] \\
&\le M|\alpha^n - 1|^{-1} \Big[\sum_{k=1}^{n} q_k d_k q_{n+1-k} d_{n+1-k} \\
&\quad + \sum_{k=1}^{n} \sum_{\substack{l_1+\dots+l_m=k \\ m=1,2,\dots,k}} (n+2) q_{l_1} \dots q_{l_m} d_{l_1} \dots d_{l_m} q_{n-k+1} d_{n-k+1} \Big] \\
&\le M\Big[\sum_{k=1}^{n} q_k q_{n-k+1} + \sum_{k=1}^{n} \sum_{\substack{l_1+\dots+l_m=k \\ m=1,2,\dots,k}} (n+2) q_{l_1} \dots q_{l_m} q_{n-k+1} \Big] \\
&\quad \times |\alpha^n - 1|^{-1} \max_{\substack{l_1+\dots+l_m=n+1 \\ m=1,2,\dots,n+1}} \{ d_{l_1} \dots d_{l_m} \} \\
&= q_{n+1} d_{n+1}, \quad n = 0, 1, 2, \dots.
\end{aligned}$$

By equation (2.11), the implicit function of $Q = Q(z)$ is defined by

$$F(z, Q) = Q - |\eta| z - \frac{4Q^2 - 4Q^3 + Q^4}{(1-Q^2)}.$$

In view of $F(0,0) = 0$ and $F'_Q(0,0) = 1 \ne 0$, by virtue of the implicit function theorem there exists a positive constant δ such that the function $Q(z) = \sum_{n=1}^{\infty} q_n z^n$ converges for $|z| < \delta$. So there exists $k > 0$ such that $q_n \le R^n$ for $n = 1, 2, \dots$.

By Lemma 2.1 and (2.12), we finally see that

$$v_n \le R^n N^{n-1} n^{-2\delta}, \quad n = 1, 2, \dots,$$

which implies that the series $\sum_{n=1}^{\infty} v_n z^n$ converges for $|z| < (RN)^{-1}$, therefore, the series (2.4) also converges for $|z| < (RN)^{-1}$. This completes the proof. $\square$

3. Existence of Analytic Solutions to (1.1)

In this section, we state and prove our main result in this article.

Theorem 3.1. *Suppose the conditions of Lemma 2.2 or lemma 2.3 are satisfied. Then* (1.5) *has an analytic solution* $g(z)$ *of the form* (1.8) *in a neighborhood of the origin, where* $g(z)$ *is an analytic solution of* (1.6)*.*

Proof. In view of Lemma 2.2 or lemma 2.3, we may find a sequence $\{c_n\}_{n=1}^{\infty}$ such that the function $g(z)$ of the form (2.4) is analytic solution of (1.6) in a neighborhood of the origin. Since $g'(0) = \eta \neq 0$, the function $g^{-1}(z)$ is analytic in a neighborhood of the point $g(0) = 0$. If we now define $y(z)$ by (1.8), then

$$y'(z) - p'(z) = \frac{\alpha g'(\alpha g^{-1}(z))}{g'(g^{-1}(z))} - p'(z) = \frac{\alpha g'(\alpha g^{-1}(z)) - p'(z) g'(g^{-1}(z))}{g'(g^{-1}(z))},$$

and

$$\begin{aligned}[y(y(z)) - p(y(z))]y'(z) &= [g(\alpha g^{-1}(g\alpha g^{-1}(z))) - p(g(\alpha g^{-1}(z)))]\frac{\alpha g'(\alpha g^{-1}(z))}{g'(g^{-1}(z))} \\ &= \frac{\alpha g'(\alpha g^{-1}(z)) - p'(z) g'(g^{-1}(z))}{g'(g^{-1}(z))}\end{aligned}$$

as required. The proof is completed. □

In the last section, we have shown that under the conditions of lemma 2.2 or lemma 2.3, equation (1.5) has an analytic solution $y(z) = g(\alpha g^{-1}(z))$ in a neighborhood of the point, where g is an analytic solution of (1.6). We now show how to explicitly construct an analytic solution of (1.1) by (1.5). In view of

$$x'(z) = \frac{1}{b}[y(z) - p(z)],$$

we have $x'(0) = \frac{1}{b}[y(0) - p(0)] = -\beta/b$. Furthermore,

$$x'(0) = x(p(0) + bx'(0)) = x(\beta - \beta) = x(0) = -\frac{\beta}{b}.$$

By calculating the derivatives of both sides of (1.1), we obtain successively

$$\begin{gathered} x''(z) = x'(p(z) + bx'(z))(p'(z) + bx''(z)), \\ x'''(z) = x''(p(z) + bx'(z))(p'(z) + bx''(z))^2 + x'(p(z) + bx'(z))(p''(z) + bx'''(z)), \end{gathered}$$

so that

$$\begin{gathered} x''(0) = x'(p(0) + bx'(0))(p'(0) + bx''(0)), \\ x'''(0) = x''(0)(p'(0) + bx''(0))^2 + x'(0)(p''(0) + bx'''(0)); \end{gathered}$$

that is,

$$x''(0) = -\frac{\alpha\beta}{b}, \quad x'''(0) = -\frac{\beta(\alpha^3 + p_2)}{b(1+\beta)}.$$

In general, we can show by induction that

$$\begin{aligned}(x(p(z) + bx'(z)))^{(m)} = \sum_{i=1}^{m} p_{im}(&p'(z) + bx''(z), p''(z) + bx'''(z), \dots, p^{(m)}(z) \\ &+ bx^{(m+1)}(z)) x^{(i)}(p(z) + bx'(z)),\end{aligned}$$

where $m = 1, 2, \dots$, and p_{im} is a polynomial with nonnegative coefficients. Hence

$$\begin{aligned} x^{(m+1)}(0) = \sum_{i=1}^{m} p_{im}(&p'(0) + bx''(0), p''(0) + bx'''(0), \dots, p^{(m)}(0) \\ &+ bx^{(m+1)}(0))x^{(i)}(0) =: \Gamma_m \end{aligned}$$

for $m = 1, 2, \dots$. It is then easy to write out the explicit form of our solution

$$x(z) = -\frac{\beta}{b} - \frac{\beta}{b}z - \frac{\alpha\beta}{2!b}z^2 - \frac{\beta(\alpha^3 + p_2)}{3!b(1+\beta)}z^3 + \sum_{m=3}^{\infty} \frac{\Gamma_m}{(m+1)!} z^{m+1}.$$

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