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MULTIPLE POSITIVE SOLUTIONS FOR A SINGULAR ELLIPTIC EQUATION WITH NEUMANN BOUNDARY CONDITION IN TWO DIMENSIONS

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Abstract. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^2 boundary. In this paper, we are interested in the problem

$$-\Delta u + u = h(x,u)e^{u^2}/|x|^\beta, \quad u > 0 \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial \nu} = \lambda \psi u^q \quad \text{on } \partial\Omega,$$

where $0 \in \partial\Omega$, $\beta \in [0,2)$, $\lambda > 0$, $q \in [0,1)$ and $\psi \geq 0$ is a Hölder continuous function on $\overline{\Omega}$. Here $h(x,u)$ is a $C^1(\overline{\Omega} \times \mathbb{R})$ having superlinear growth at infinity. Using variational methods we show that there exists $0 < \Lambda < \infty$ such that above problem admits at least two solutions in $H^1(\Omega)$ if $\lambda \in (0,\Lambda)$, no solution if $\lambda > \Lambda$ and at least one solution when $\lambda = \Lambda$.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^2 boundary and $0 \in \partial\Omega$. In this work, we study weak solutions $u \in H^1(\Omega)$ of the problem

$$\begin{gathered}-\Delta u + u = h(x,u)e^{u^2}/|x|^\beta, \quad u > 0 \quad \text{in } \Omega,\\ \frac{\partial u}{\partial \nu} = \lambda \psi u^q \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where $\beta \in [0,2)$, $\lambda > 0$, $q \in [0,1)$ and $h(x,t)$ satisfies the following conditions:

- (H1) $h(x,t) \in C^1(\overline{\Omega} \times \mathbb{R})$, $h(x,u) \geq 0$ for all $u \in \mathbb{R}$, $h(x,u) = 0$ if $u < 0$;
- (H2) $\frac{\partial h}{\partial t}(x,t) \geq 0$ for $t > 0$, $h(x,t) \sim t^k$ as $t \to 0$, uniformly in x, for some $k > 1$,
- (H3) $\liminf_{t\to\infty} \frac{h(x,t)}{t} > 0$ and $\limsup_{t\to\infty} \frac{h(x,t)}{t^p} = 0$ for some $p > 1$ uniformly in x;
- (H4) For any $\epsilon > 0$, $\limsup_{t\to\infty} \frac{\partial g}{\partial t}(x,t)e^{-(1+\epsilon)t^2} = 0$.

where $g(x,u) = h(x,u)e^{u^2}/|x|^\beta$ and $G(x,u) = \int_0^u g(x,s)ds$. Associated to the problem (1.1) we have the functional $J_\lambda : H^1(\Omega) \to \mathbb{R}$ defined by

$$J_\lambda(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + |u|^2) - \int_\Omega G(x,u) - \frac{\lambda}{q+1}\int_{\partial\Omega} \psi |u|^{q+1}. \tag{1.2}$$

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The Fréchet derivative of this functional is given by

$$\langle J'_\lambda(u), \phi\rangle = \int_\Omega \nabla u.\nabla\phi + \int_\Omega u\phi - \int_\Omega g(x,u)\phi - \lambda\int_{\partial\Omega} \psi|u|^{q-1}u\phi, \quad \forall\phi \in H^1(\Omega).$$

Clearly, any positive critical point of J_λ is a weak solution of (1.1).

The exponential nature of the nonlinearity $g(x,u)$ is motivated by the following version of Moser-Trudinger inequality (due to Adimurthi-Yadava[2])

$$\sup_{\|u\|_{H^1(\Omega)}\le 1} \int_\Omega e^{2\pi u^2} \le C(|\Omega|), \tag{1.3}$$

where C is a positive constant. The above imbedding immediately implies that the nonlinear map $H^1(\Omega) \ni u \mapsto e^{u^\alpha} \in L^1(\Omega)$ is a continuous map for all $\alpha \in (0,2]$ and is compact if and only if $\alpha \in (0,2)$. The inequality (1.3) is a H^1 version of the following Moser-Trudinger inequality[12],[15]:

$$\sup_{\|u\|_{H_0^1(\Omega)}\le 1} \int_\Omega e^{4\pi u^2} \le C(|\Omega|).$$

In this case non-compactness of the imbedding $H_0^1(\Omega) \ni u \mapsto e^{u^2} \in L^1(\Omega)$ can be shown by using a sequence of functions that are suitable truncations and dilations of the fundamental solution for $-\Delta$ in $\mathbb{R}^2$. These functions are commonly referred to as Moser functions in the literature. In case of the imbedding in (1.3), the non-compactness can be shown by suitably modifying the Moser functions so that they concentrate at a point on the boundary $\partial\Omega$ (see Lemma 2.1 below). The inequality (1.3) cannot be used in our case due to the presence of the singularity $|x|^{-\beta}$. To overcome this, first we prove the following singular version of (1.3) using the methods in [3]:

$$\sup_{\|u\|_{H^1(\Omega)}\le 1} \int_\Omega \frac{e^{\alpha u^2}}{|x|^\beta} \le C(|\Omega|)$$

where C is a positive constant and $\frac{\alpha}{2\pi} + \frac{\beta}{2} < 1$. Moreover, the above inequality does not hold if $\frac{\alpha}{2\pi} + \frac{\beta}{2} > 1$.

We prove the following existence and multiplicity results:

Theorem 1.1. *There exists $\Lambda \in (0,\infty)$ such that (1.1) admits minimal solution u_λ for all $\lambda \in (0,\Lambda)$.*

Theorem 1.2. *There exists $0 < \Lambda < \infty$ such that (1.1) has at least two solutions for all $\lambda \in (0,\Lambda)$, no solutions for $\lambda > \Lambda$ and at least one solution when $\lambda = \Lambda$.*

The minimal solution in the above theorem is obtained using sub-super solution arguments as in [14] and this minimal solution is shown to be a local minimum of J_λ. The second solution is obtained using the generalized mountain-pass theorem of Ghoussoub-Priess[9].

At this point we briefly recall related existence and multiplicity results for elliptic equations. The study of semilinear elliptic problems with critical nonlinearities of Sobolev and Hardy-Sobolev type has recieved considerable interest in recent years. In a recent work [3], authors considered Dirichlet Problem for (1.1) with superlinear type nonlinearity and studied the existence of positive solutions. In [7],[11] authors studied the existence and multiplicity with Hardy-Sobolev critical exponents.

Neumann type Problems are studied in [4], [8],[7] and [14]. The Multiplicity result for Neumann problem with Sobolev critical nonlinearity has been studied in [8] where authors considered the problem

$$\begin{gathered} -\Delta u + u = u^p, \quad u > 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda \psi u^q \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$ and $0 < q < 1 < p \leq \frac{2N}{N-2}$. They proved the following theorem.

Theorem 1.3. *There exists $\tilde{\Lambda}$ such that $(\tilde{P}_\lambda)$ admits at least two solutions for all $\lambda \in (0, \tilde{\Lambda})$, one solution when $\lambda = \tilde{\Lambda}$ and no solution for $\lambda > \tilde{\Lambda}$.*

Subsequently, the problem in two dimensions was considered in [14] where authors proved the Theorem 1.3. In these works authors obtained the minimal solution using sub-super solution method and the second solution by Mountain-pass arguments. The main ingredient is the local minimum nature of the minimal solution which was obtained using H^1 verses C^1 local minimizer arguments introduced in [5].

The special features of this class of problems, considered in this paper, are they involve critical singular growth. In this case main difficulty arise due to the fact that the solutions are not C^1 near origin. So the arguments like C^1 verses H^1 local minimizers cannot be carried out for a singular equation. So to establish the local minimum nature of the minimal of solution we use Perron's method as in [10]. To obtain the mountain-pass type solution, one studies the critical levels and the convergence of Palaise-Smale sequences. The critical levels in our case are different than in [14] where $\beta = 0$ case was studied. We recall the following Hardy-Sobolev inequality, which is used in later sections.

Lemma 1.4. *Let $\Omega \subset \mathbb{R}^2$, then there exists a constant $C > 0$ such that for any $u \in H^1(\Omega)$*

$$\Big(\int_\Omega \frac{|u|^p}{|x|^\beta} dx\Big)^{1/p} \leq C\Big(\int_\Omega (|\nabla u|^2 + u^2) dx\Big)^{1/2} \tag{1.5}$$

for any $p < \infty$, and $\beta < 2$.

Notation. In this paper we make use of the following notation: If $p \in (0, \infty), p'$ denotes the number $p/(p-1)$ so that $p' \in (1, \infty)$ and $1/p + 1/p' = 1$; $L^p(\Omega, |x|^{-\beta})$ denotes the Lebesgue spaces with measure $|x|^{-\beta}dx$ and norm $\|.\|_{L^p(\Omega,|x|^{-\beta})}$; $H^1(\Omega)$ denotes the Sobolev space with norm $\|.\|$; $|A|_n$ denotes the Lebesgue measure of the set $A \subset \mathbb{R}^n$.

2. A singular Moser-Trudinger inequality in $H^1(\Omega)$

In this section we show a singular version of Moser-Trudinger inequality in $H^1(\Omega)$. We recall the following lemma from [2, Lemma 3.3].

Lemma 2.1. *Let $\partial\Omega$ is a smooth manifold. For every $x_0 \in \partial\Omega$, we can find a $L > 0$ such that for each $0 < l < L$ there exists a function $w_l \in H^1(\Omega)$ satisfying*

1. $w_l \geq 0$, $\operatorname{supp}(w_l) \subset B(x_0, L) \cap \overline{\Omega}$
2. $\|w_l\| = 1$
3. *For all $x \in B(x_0, l) \cap \overline{\Omega}, w_l$ is constant and $w_l^2 = \frac{1}{\pi} \log \frac{L}{l} + o(1)$ as $l \to 0$.*

Theorem 2.2. *Let $u \in H^1(\Omega)$. Then for every $\alpha > 0$ and $\beta \in [0,2)$, we have*
$$\int_\Omega \frac{e^{\alpha u^2}}{|x|^\beta} dx < \infty. \tag{2.1}$$
Moreover for any $\beta \in [0,2)$,
$$\sup\big\{\alpha : \sup_{\|u\|\le 1} \int_\Omega \frac{e^{\alpha u^2}}{|x|^\beta} dx < \infty\big\} = \pi(2-\beta) \tag{2.2}$$

Proof. Let $t > 1$ be such that $\beta t < 2$, then by Cauchy-Schwartz inequality,
$$\int_\Omega \frac{e^{\alpha u^2}}{|x|^\beta} dx \le \Big(\int_\Omega e^{\frac{\alpha t}{t-1}u^2} dx\Big)^{(t-1)/t} \Big(\int_\Omega \frac{1}{|x|^{t\beta}}\Big)^{1/t}$$
The above inequality along with (1.2) implies (2.1).

Now Suppose $\sup\{\alpha : \sup_{\|u\|\le 1} \int_\Omega \frac{e^{\alpha u^2}}{|x|^\beta} dx < \infty\} < \pi(2-\beta)$. Then there exists α such that $\frac{\alpha}{2\pi} + \frac{\beta}{2} < 1$.

Again choose $t > 1$ such that $\frac{\alpha}{2\pi} + \frac{\beta t}{2} = 1$, and using Cauchy-Schwartz inequality we obtain,
$$\sup_{\|u\|\le 1} \int_\Omega \frac{e^{\alpha u^2}}{|x|^\beta} dx \le \sup_{\|u\|\le 1} \Big(\int_\Omega e^{2\pi u^2}\Big)^{\frac{\alpha}{2\pi}} \Big(\int_\Omega \frac{1}{|x|^{\frac{2}{t}}} dx\Big)^{\frac{t\beta}{2}} < \infty \tag{2.3}$$
as $1 - \frac{\alpha}{2\pi} = 1 - (1 - \frac{t\beta}{2}) = \frac{\beta t}{2} < 1$ and $\frac{2}{t} < 1$.

Next, we show that (2.3) does not hold if $\frac{\alpha}{2\pi} + \frac{\beta}{2} > 1$. Let w_l be the sequence of Moser functions concentrated around $0 \in \partial\Omega$, as in Lemma 2.1, then
$$\begin{aligned}
\int_\Omega \frac{e^{\alpha w_l^2}}{|x|^\beta} dx &\ge e^{\frac{\alpha}{\pi}\log\frac{R}{l}} \int_{B(l)} \frac{1}{|x|^\beta} dx \\
&= \big(\frac{L}{l}\big)^{\alpha/\pi} \frac{l^{2-\beta}}{2\pi(2-\beta)} \\
&= \frac{2\pi L^{\alpha/2\pi}}{(2-\beta)} \frac{1}{l^{2(\frac{\alpha}{2\pi}+\frac{\beta}{2}-1)}}
\end{aligned}$$
Since $\frac{\alpha}{2\pi} + \frac{\beta}{2} > 1$, the limit $l \to 0$ of right-hand side of the above inequality is infinity. Therefore,
$$\sup_{\|u\|\le 1} \int_\Omega \frac{e^{\alpha w_l^2}}{|x|^\beta} = \infty.$$
Hence the required supremum is $\pi(2-\beta)$. $\square$

3. Existence of minimal solution

In this section, we show that there exists a $\Lambda > 0$ such that J_λ possesses minimal solution for $\lambda \in (0,\Lambda)$. First we show that there exists a solution for λ small. We can show the following strong comparison principle using Hopf lemma and weak comparison arguments.

Lemma 3.1. *Let $u \ne 0$ satisfies $-\Delta u + u \ge 0$ in Ω and $\frac{\partial u}{\partial \nu} \ge 0$ on $\partial\Omega$ then $u > 0$ in $\overline{\Omega}$.*

Lemma 3.2. *There exists $\lambda_0 > 0$, small such that* (1.1) *admits a solution for all $\lambda \in (0,\lambda_0)$.*

Proof. **Step 1:** There exists $\lambda_0, \delta > 0$ and $R_0 > 0$ such that $J_\lambda(u) \geq \delta$ for all $\|u\| = R_0$ and all $\lambda < \lambda_0$. From assumption (H2) for $h(x,u)$ and Hölder's inequality, we obtain that for some $C_1 > 0$,

$$\begin{aligned}\int_\Omega G(x,u) &\leq C_1 \int_\Omega |u|^{k+1} \frac{e^{u^2}}{|x|^\beta} \\ &= \int_\Omega \frac{|u|^{k+1}}{|x|^{\beta/p'}} \frac{e^{u^2}}{|x|^{\beta/p}} \\ &\leq C_1 \|u\|^{k+1}_{L^{p'(k+1)}(\Omega,|x|^{-\beta})} \Big(\int_\Omega e^{p\|u\|^2(\frac{u}{\|u\|})^2}\Big)^{1/p}.\end{aligned}$$

Now since $\beta < 2$ it is possible to choose $p > 1$ and $R > 0$ such that $\frac{pR^2}{2\pi} + \frac{\beta}{2} < 1$. Then, by (2.3) and Hardy-Sobolev inequality(1.5), the last inequality gives for some $C_2 > 0$,

$$\int_\Omega G(x,u) \leq C_2 \|u\|^{k+1}, \quad \forall \, \|u\| \leq R. \tag{3.1}$$

Also, by Hölder's inequality and the trace imbedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ we get for some $C_3 > 0$,

$$\int_{\partial\Omega} \psi |u|^{q+1} \leq C_3 \|\psi\|_{L^\infty(\partial\Omega)} \|u\|^{q+1}_{L^{q+1}(\partial\Omega)} \leq C_3 \|u\|^{q+1}. \tag{3.2}$$

Thus, from (3.1), (3.2) we have that, for $R_0^2 \in (0, 2\pi)$ small enough

$$J_\lambda(u) \geq \frac{1}{2}\|u\|^2 - C_2\|u\|^{k+1} - \lambda C_3 \|u\|^{q+1}, \quad \forall \, \|u\| = R_0. \tag{3.3}$$

Since $k > 1$, we may choose $\lambda_0 > 0$ small enough so that $J_\lambda(u) > \delta$ for some $\delta > 0$ and for all $\lambda \in (0, \lambda_0)$.

Step 2: J_λ possesses a local minimum close to the origin for all $\lambda \in (0, \lambda_0)$. It is easy to see that $J_\lambda(tu) < 0$ for $t > 0$ small enough and any $u \in H^1(\Omega)$. Indeed, $\min_{\|u\| \leq R_0} J_\lambda(u) < 0$ and if this minimum is achieved at some u_λ, then necessarily $\|u_\lambda\| < R_0$ and hence u_λ becomes a local minimum for J_λ. Now let $\{u_n\} \subset \{\|u\| \leq R_0\}$ be a minimizing sequence, then there exists u_λ such that $u_n \rightharpoonup u_\lambda$ in $H^1(\Omega), u_n \to u_\lambda$ strongly in all $L^p(\Omega)$, and pointwise in Ω. Hence using the compact imbedding of $H^1(\Omega)$ into $L^{q+1}(\partial\Omega)$,

$$\int_\Omega |\nabla u_\lambda|^2 \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2, \quad \int_{\partial\Omega} \psi u_n^{q+1} \to \int_{\partial\Omega} \psi u_\lambda^{q+1}.$$

Since R_0^2 and β satisfy $\frac{R_0^2}{2\pi} + \frac{\beta}{2} < 1$, by (2.2) and Vitali's convergence theorem we obtain, $\int_\Omega G(x,u_n) \to \int_\Omega G(x,u_\lambda)$. From these facts it is clear that u_λ is a minimizer for J_λ in $\{\|u\| \leq R_0\}$ and hence is a local minimum. Now by (H1) and maximum principle we get $u > 0$ in Ω. □

Lemma 3.3. *Let $\Lambda = \sup\{\lambda > 0 : (1.1)$ has a solution$\}$. Then $0 < \Lambda < \infty$.*

Proof. Let u_λ be a solution of (1.1). Taking $\phi \equiv 1$ in Ω in $\langle J'(u_\lambda), \phi\rangle = 0$, we get

$$\int_\Omega u_\lambda = \int_\Omega g(u_\lambda) + \lambda \int_{\partial\Omega} \psi u_\lambda^q.$$

Since $\|u_\lambda\|_{L^1(\Omega)} \leq C_1 \|u_\lambda\|_{L^p(\Omega)}$ and $\int_\Omega g(x,u_\lambda) \geq C_2 \|u_\lambda\|^p_{L^p(\Omega)}$ for some $p > 1$, for some constants $C_1, C_2 > 0$, we immediately obtain from above equation that

$\|u_\lambda\|_{L^p(\Omega)}$ is bounded by a constant independent of λ for any $p \geq 1$. Now taking $\phi = u_\lambda^{-q}$ in $\langle J'(u_\lambda), \phi\rangle = 0$ we get,

$$-q\int_\Omega u_\lambda^{-1-q}|\nabla u_\lambda|^2 + \int_\Omega u_\lambda^{1-q} = \int_\Omega u_\lambda^{-q} g(x,u_\lambda) + \lambda\int_{\partial\Omega}\psi.$$

From the above equation it follows that Λ is finite and it is positive by Lemma 3.2. $\square$

Lemma 3.4. *There exists a solution for* (1.1) *for all* $\lambda \in (0,\Lambda)$.

Proof. Let $\lambda \in (0,\Lambda)$, then choose $\lambda_2 \in (0,\Lambda)$ such that $\lambda < \lambda_2$. Let u_{λ_2} be a solutions of (p_{λ_2}). Let $\mu = \min_{\partial\Omega} u_{\lambda_2}$. Let v_λ be the solution of

$$\begin{gathered}
-\Delta u + u = 0, \quad u > 0 \quad \text{in } \Omega,\\
\frac{\partial u}{\partial \nu} = \lambda\psi\mu^q \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.4}$$

Clearly, u_{λ_2} is a super solution of (3.4) above and hence $u_{\lambda_2} > v_\lambda$ in $\overline{\Omega}$ by Lemma 3.2. Let $\tilde{h}_\lambda$ and $\tilde{f}_\lambda$ be the cut-off functions defined as

$$(x\in\Omega, t\in\mathbb{R}) \quad \tilde{g}_\lambda(x,t) = \begin{cases} g(x,v_\lambda(x)) & t < v_\lambda(x),\\ g(x,t) & v_\lambda(x) \leq t \leq u_{\lambda_2}(x),\\ g(u_{\lambda_2}(x)) & t > u_{\lambda_2}(x),\end{cases}$$

$$(x\in\partial\Omega, t\in\mathbb{R}) \quad \tilde{f}_\lambda(x,t) = \begin{cases} \lambda\psi(x)\mu^q & t < v_\lambda(x),\\ \lambda\psi(x)t^q & v_\lambda(x) \leq t \leq u_{\lambda_2}(x),\\ \lambda\psi(x)u_{\lambda_2}^q(x) & t > u_{\lambda_2}(x).\end{cases}$$

Let $\tilde{G}_\lambda(x,u) = \int_0^u \tilde{g}_\lambda(x,t)dt$ $(x\in\Omega)$, $\tilde{F}_\lambda(x,u) = \int_0^u \tilde{f}_\lambda(x,t)dt$, $(x\in\partial\Omega)$. Then the functional $\tilde{J}_\lambda : H^1(\Omega) \to \mathbb{R}$ given by

$$\tilde{J}_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + |u|^2 - \int_\Omega \tilde{G}_\lambda(x,u) - \int_\partial \tilde{F}_\lambda(x,u)$$

is coercive and bounded below. Let u_λ denote the global minimum of $\tilde{J}_\lambda$ on $H^1(\Omega)$. Clearly u_λ is a solution of (1.1). $\square$

Proof of Theorem 1.1. From Lemma 3.2 we know that there exists a solution u_λ for (1.1) for all $\lambda \in (0,\Lambda)$. Let v_λ be the unique solution of

$$\begin{gathered}
-\Delta u + u = 0, \quad u > 0 \quad \text{in } \Omega,\\
\frac{\partial u}{\partial \nu} = \lambda\psi u^q \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.5}$$

Clearly u_λ is super solution of (3.5) and hence by Lemma 3.1 we have $v_\lambda \leq u_\lambda$ in $\overline{\Omega}$. Now Define the sequence $\{u_n\}$ using the monotone iteration

$$\begin{gathered}
u_1 = v_\lambda\\
-\Delta u_{n+1} + u_{n+1} = \frac{g(x,u_n)}{|x|^\beta} \quad \text{in } \Omega\\
\frac{\partial u_{n+1}}{\partial\nu} = \lambda\psi u_n^q \quad \text{on } \partial\Omega,
\end{gathered}$$

for $n = 1,2,3,\dots$. By the comparison theorem in Lemma 3.1, we get that the sequence $\{u_n\}$ is monotone; i.e., $u_1 \leq u_2 \leq \dots \leq u_n \leq u_{n+1} \dots \leq u_\lambda$. By standard monotonicity arguments we obtain a solution u_λ of (1.1) which will be also the minimal solution. $\square$

4. Existence of local minimum for J_λ with $\lambda \in (0, \Lambda)$

In this section we show that the solution u_λ obtained in Theorem 1.1 is a local minimum for J_λ in $H^1(\Omega)$. We adopt the approach of [10] to prove the following theorem.

Theorem 4.1. *The solution u_λ as in Lemma3.2 is a local minimum for J_λ in $H^1(\Omega)$.*

Proof. Let $\lambda_2 > \lambda$ and u_{λ_2} be the minimal solution of (P_{λ_2}). Suppose if u_λ is not a local minimum for J_λ, then there exists a sequence $\{u_n\} \subset H^1(\Omega)$ such that $u_n \to u_\lambda$ strongly in $H^1(\Omega)$ and $J_\lambda(u_n) < J_\lambda(u_\lambda)$. Now define $\underline{u} := v_\lambda$ where v_λ is the unique solution of (3.5) and $\overline{u} := u_{\lambda_2}$, then $\underline{u} < \overline{u}$ in $\overline{\Omega}$. Consider the following cut-off functions

$$v_n(x) = \begin{cases} \underline{u}(x), & u_n(x) \le \underline{u}(x) \\ u_n(x), & \underline{u}(x) \le u_n(x) \le \overline{u}(x) \\ \overline{u}(x), & u_n(x) \ge \overline{u}(x) \end{cases}$$

and define $\overline{w_n} = (u_n - \overline{u})^+$, $\underline{w_n} = (u_n - \underline{u})^-$, $\underline{S_n} = \operatorname{supp}(\underline{w_n}), \overline{S_n} = \operatorname{supp}(\overline{w_n})$. Then $u_n = v_n - \underline{w_n} + \overline{w_n}$, $v_n \in M = \{u \in H^1(\Omega), \underline{u} \le u \le \overline{u}\}$ and

$$J_\lambda(u_n) = J_\lambda(v_n) + A_n + B_n$$

where

$$\begin{aligned} A_n = &\frac{1}{2} \int_{\overline{S}_n} \left[\left(|\nabla u_n|^2 - |\nabla \underline{u}|^2 \right) + \left(|u_n|^2 - |\overline{u}|^2 \right) \right] dx \\ &- \int_{\overline{S}_n} [G(x, u_n) - G(x, \overline{u})]\, dx - \frac{\lambda}{1+q} \int_{\overline{S}_n} \psi \left(u_n^{q+1} - \overline{u}^{q+1} \right) \end{aligned}$$

and

$$\begin{aligned} B_n = &\frac{1}{2} \int_{\underline{S}_n} \left[\left(|\nabla u_n|^2 - |\nabla \underline{u}|^2 \right) + \left(|u_n|^2 - |\underline{u}|^2 \right) \right] dx \\ &- \int_{\underline{S}_n} [G(x, u_n) - G(x, \underline{u})]\, dx - \frac{\lambda}{1+q} \int_{\underline{S}_n} \psi \left(u_n^{q+1} - \underline{u}^{q+1} \right) dx \end{aligned}$$

Since $J_\lambda(u_\lambda) = \inf_{u \in M} J_\lambda(u)$, we have $J_\lambda(u_n) \ge J_\lambda(u_\lambda) + A_n + B_n$. Now since $u_n \to u_\lambda$ strongly in $H^1(\Omega)$ and $\underline{u} < u_\lambda < \overline{u}$ in $\overline{\Omega}$, we have $\operatorname{meas}(\overline{S}_n)$, $\operatorname{meas}(\underline{S}_n) \to 0$ as $n \to \infty$. Therefore,

$$\|\overline{w}_n\|, \|\underline{w}_n\| \to 0 \quad \text{as } n \to \infty. \tag{4.1}$$

Since $\overline{u}$ is super-solution of (1.1), we get

$$\begin{aligned} A_n &= \frac{1}{2} \int_{\overline{S}_n} \left[\left(|\nabla u_n|^2 - |\nabla \underline{u}|^2 \right) + \left(|u_n|^2 - |\overline{u}|^2 \right) \right] dx \\ &\quad - \int_{\overline{S}_n} [G(x, u_n) - G(x, \overline{u})]\, dx - \frac{\lambda}{1+q} \int_{\overline{S}_n} \psi \left(u_n^{q+1} - \overline{u}^{q+1} \right) \\ &= \frac{1}{2} \|\overline{w}_n\|^2 + \int_\Omega (\nabla \overline{u} \nabla \overline{w}_n + \overline{u} \overline{w}_n) \end{aligned}$$

$$
\begin{aligned}
&- \int_{\overline{S}_n} [G(x,\overline{u}+\overline{w}_n) - G(x,\overline{u})]\,dx - \frac{\lambda}{1+q}\int_{\overline{S}_n} \psi\left((\overline{u}+\overline{w}_n)^{q+1} - \overline{u}^{q+1}\right)\\
&= \frac{1}{2}\|\overline{w}_n\|^2 + \int_{\overline{S}_n} (g(x,\overline{u}) - g(x,\overline{u}+\theta\overline{w}_n))\,\overline{w}_n - \lambda\int_{\overline{S}_n\cap\overline{\Omega}} \psi\,[(\overline{u}+\theta\overline{w}_n)^q - \overline{u}^q]\,\overline{w}_n
\end{aligned}
$$

for some $0 < \theta < 1$. It follows from (H4), (2.3) and Hölder's inequalities that for n sufficiently large,

$$
\begin{aligned}
A_n &\geq \frac{1}{2}\|\overline{w}_n\|^2 - \int_{\overline{S}_n} \frac{\partial g}{\partial t}(x,\overline{u}+\theta'\overline{w}_n)\overline{w}_n^2 + o_n(1)\\
&\geq \frac{1}{2}\|\overline{w}_n\|^2 - C_1\int_{\overline{S}_n} \frac{1}{|x|^\beta} e^{(\overline{u}+\overline{w}_n)^2(1+\epsilon)}\overline{w}_n^2 + o_n(1)\\
&\geq \frac{1}{2}\|\overline{w}_n\|^2 - C_2\|\overline{w}_n\|^2|\overline{S}_n| + o_n(1) \geq 0
\end{aligned}
$$

for n large since $|\overline{S}_n| \to 0$ as $n \to \infty$. Similarly, $B_n \geq 0$. Therefore, $J_\lambda(u_n) \geq J_\lambda(u_\lambda)$. This contradicts our assumption $J_\lambda(u_n) < J_\lambda(u_\lambda)$ and hence proves the Theorem. □

5. Existence of Mountain-pass type solution

In this section we show the existence of second solution via Mountain-pass lemma. Throughout this section we fix $\lambda \in (0,\Lambda)$ and u_λ will denote the local minimum for J_λ obtained in Theorem 4.1. Define $\tilde{g_\lambda} : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$
\tilde{g}_\lambda(x,s) = \begin{cases} g(x,s+u_\lambda(x)) - g(x,u_\lambda(x)) & s \geq 0,\\ 0 & s < 0, \end{cases}
$$

and $\tilde{f}_\lambda : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ by

$$
\tilde{f}_\lambda(x,s) = \begin{cases} \lambda\psi(x)((s+u_\lambda(x))^q - u_\lambda^q(x)) & s \geq 0,\\ 0 & s < 0. \end{cases}
$$

Let $\tilde{G}_\lambda(x,s) = \int_0^s \tilde{g}_\lambda(x,t)dt$, $\tilde{F}_\lambda(x,s) = \int_0^s \tilde{f}_\lambda(x,t)dt$. Consider the functional $\tilde{J}_\lambda : H^1(\Omega) \to \mathbb{R}$ defined by

$$
\tilde{J}_\lambda(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 + |v|^2) - \int_\Omega \tilde{G}_\lambda(x,v)dx - \int_{\partial\Omega} \tilde{F}_\lambda(x,v)dx. \tag{5.1}
$$

Since u_λ is local minimum for J_λ, it can be easily checked that 0 is a local minimum for $\tilde{J}_\lambda$. Moreover, any critical point $v_\lambda \geq 0$ of $\tilde{J}_\lambda$ satisfies:

$$
\begin{gathered}
-\Delta v_\lambda + v_\lambda = g(x,v_\lambda+u_\lambda) - g(x,u_\lambda), \quad v_\lambda > 0 \quad \text{in } \Omega,\\
\frac{\partial v_\lambda}{\partial \nu} = \tilde{f}_\lambda(x,v_\lambda) \quad \text{on } \partial\Omega.
\end{gathered} \tag{5.2}
$$

Hence it follows that $w_\lambda = u_\lambda + v_\lambda$ is a solution of (1.1). Therefore, to show the existence of second solution, it is enough to find a $0 < v \in H^1(\Omega)$ which is a critical point of $\tilde{J}_\lambda$. This we can do by using a generalised version of Mountain-pass theorem due to Ghoussoub-Priess[9].

First,we give the following generalized definition of Palais-Smale sequence around a closed set.

Definition 5.1. Let $F \subset H^1(\Omega)$ be a closed set. We say that a sequence $\{v_n\} \subset H^1(\Omega)$ is a Palais-Smale sequence for $\tilde{J}_\lambda$ at the level ρ around F (a $(P.S)_{F,\rho}$ sequence, for short) if

$$\lim_{n\to\infty} \operatorname{dist}(v_n, F) = 0, \quad \lim_{n\to\infty} \tilde{J}_\lambda(v_n) = \rho, \quad \lim_{n\to\infty} \|\tilde{J}'_\lambda(v_\lambda)\|_{(H^1(\Omega))^*} = 0.$$

Remark 5.2. Note that when $F = H^1(\Omega)$, the above definition reduces to the usual definition of a Palaise-Smale sequence at the level ρ.

We can show the following "Compactness result".

Lemma 5.3. *Let $F \subset H^1(\Omega)$ be a closed set, $\rho \in \mathbb{R}$. Let $\{v_n\} \subset H^1(\Omega)$ be a $(P.S)_{F,\rho}$ sequence for $\tilde{J}_\lambda$. Then (up to a subsequence), $v_n \rightharpoonup v_0$ in $H^1(\Omega)$,*

$$\lim_{n\to\infty} \int_\Omega \tilde{g}_\lambda(x, v_n) = \int_\Omega \tilde{g}_\lambda(x, v_0), \quad \lim_{n\to\infty} \int_{\partial\Omega} \tilde{f}_\lambda(x, v_n) = \int_{\partial\Omega} \tilde{f}_\lambda(x, v_0).$$
$$\lim_{n\to\infty} \int_\Omega \tilde{G}_\lambda(x, v_n) = \int_\Omega \tilde{G}_\lambda(x, v_0), \quad \lim_{n\to\infty} \int_{\partial\Omega} \tilde{F}_\lambda(x, v_n) = \int_{\partial\Omega} \tilde{F}_\lambda(x, v_0).$$

Proof. Since $\{v_n\}$ is a $(P.S)_{F,\rho}$ sequence for $\tilde{J}_\lambda$, we have the following relations as $n \to \infty$:

$$\frac{1}{2}\int_\Omega |\nabla v_n|^2 + v_n^2 - \int_\Omega \tilde{G}_\lambda(x, v_n) - \int_{\partial\Omega} \tilde{F}_\lambda(x, v_n) = \rho + o_n(1), \tag{5.3}$$

$$\Big|\int_\Omega \nabla v_n \nabla\phi + \int_\Omega v_n\phi - \int_\Omega \tilde{g}_\lambda(x, v_n)\phi - \int_{\partial\Omega} \tilde{f}_\lambda(x, v_n)\phi\Big| \le o_n(1)\|\phi\| \quad \forall\phi \in H^1(\Omega). \tag{5.4}$$

Step 1: $\sup_n \|v_n\| < \infty$, $\sup_n \int_\Omega \tilde{g}_\lambda(x, v_n)v_n < \infty$, $\sup_n \int_{\partial\Omega} \tilde{f}_\lambda(x, v_n)v_n < \infty$. From (H2), (H3) and the explicit form of $\tilde{f}_\lambda$, given $\epsilon > 0$, there exists $s_\epsilon > 0$ such that $\tilde{G}_\lambda(x, s) \le \epsilon\tilde{g}_\lambda(x, s)s$ for all $s \ge s_\epsilon$. Using (5.3) together with this relation, we get,

$$\begin{aligned}\frac{1}{2}\|v_n\|^2 &\le \int_{\Omega\cap\{v_n\le s_\epsilon\}} \tilde{G}_\lambda(x, v_n) + \epsilon\int_\Omega \tilde{g}_\lambda(x, v_n)v_n + \int_{\partial\Omega} \tilde{f}_\lambda(x, v_n)v_n + \rho + o_n(1)\\ &\le C_\epsilon + \epsilon\int_\Omega \tilde{g}_\lambda(x, v_n)v_n + \int_{\partial\Omega} \tilde{f}_\lambda(x, v_n)v_n + \rho + o_n(1).\end{aligned} \tag{5.5}$$

From (5.4) with $\phi = v_n$ we obtain,

$$\int_\Omega \tilde{g}_\lambda(x, v_n)v_n + \int_{\partial\Omega} \tilde{f}_\lambda(x, v_n)v_n \le \|v_n\|^2 + o_n(1)\|v_n\|. \tag{5.6}$$

From the definition of $\tilde{h}_\lambda$ we have

$$\int_{\partial\Omega} \tilde{f}_\lambda(x, v_n)v_n \le C\|v_n\|^{1+q} \le C + \frac{1}{4}\|v_n\|^2.$$

Hence, plugging the above inequality into (5.5) we get

$$\frac{1}{4}\|v_n\|^2 + \epsilon o_n(1)\|v_n\| \le C_\epsilon + \rho + o_n(1).$$

This shows that $\sup_n \|v_n\| < \infty$ and hence by (5.6) the claim in Step 1 follows.

Since $\{v_n\} \subset H^1(\Omega)$ is bounded, up to a subsequence, $v_n \rightharpoonup v_0$ in $H^1(\Omega)$ for some $v_0 \in H^1(\Omega)$.

Step 2:

$$\lim_{n\to\infty}\int_\Omega \tilde g_\lambda(x,v_n)=\int_\Omega \tilde g_\lambda(x,v_0),\quad \lim_{n\to\infty}\int_{\partial\Omega}\tilde f_\lambda(x,v_n)=\int_{\partial\Omega}\tilde f_\lambda(x,v_0), \tag{5.7}$$

$$\lim_{n\to\infty}\int_\Omega \tilde G_\lambda(x,v_n)=\int_\Omega \tilde G_\lambda(x,v_0),\quad \lim_{n\to\infty}\int_{\partial\Omega}\tilde F_\lambda(x,v_n)=\int_{\partial\Omega}\tilde F_\lambda(x,v_0). \tag{5.8}$$

Let $\mu(A)=\int_A |x|^{-\beta}dx$ and $|B|_1$ denote the one dimensional Lebesgue measure of a set $B\subset\partial\Omega$. We first show that $\{\tilde g_\lambda(.,v_n)\}$ and $\{\tilde f_\lambda(.,v_n)\}$ are equi-integrable families in $L^1(\Omega)$ and $L^1(\partial\Omega)$ respectively, i.e., given $\epsilon>0$, there exists a $\delta>0$ such that for any $A\subset\Omega, B\subset\partial\Omega$, with $\mu(A)+|B|_1<\delta$, we have $\sup_n\int_A|\tilde g_\lambda(x,v_n)|+\int_B\tilde f_\lambda(x,v_n)\le\epsilon$. Once this is shown, (5.7) follows from Vitali's convergence theorem. Relation (5.8) follows from (5.7) by (H3) and the fact that the trace imbedding $H^1(\Omega)\hookrightarrow L^2(\partial\Omega)$ is compact.

Let $\tilde C=\sup_n\Big(\int_\Omega\tilde g_\lambda(x,v_n)v_n+\int_{\partial\Omega}\tilde f_\lambda(x,v_n)v_n\Big)$. By Step 1, $\tilde C<\infty$. Given $\epsilon>0$, define

$$\mu_\epsilon^1=\max_{x\in\overline\Omega,|s|\le\frac{4\tilde C}{\epsilon}}|\tilde g_\lambda(x,s)||x|^\beta,\ \mu_\epsilon^2=\max_{x\in\partial\Omega,|s|\le\frac{4\tilde C}{\epsilon}}|\tilde f_\lambda(x,s)|.$$

Then for any $A\subset\Omega, B\subset\partial\Omega$ with $\mu(A)\le\frac{\epsilon}{4\mu_\epsilon^1}, |B|_1\le\frac{\epsilon}{4\mu_\epsilon^1}$ we get,

$$\begin{aligned}
&\int_A|\tilde g_\lambda(x,v_n)|+\int_B|\tilde f(x,v_n)|\\
&\le\int_{A\cap\{|v_n|\ge\frac{4\tilde C}{\epsilon}\}}\frac{|\tilde g_\lambda(x,v_n)v_n|}{|v_n|}+\int_{A\cap\{|v_n|\le\frac{4\tilde C}{\epsilon}\}}|\tilde g_\lambda(x,v_n)|\\
&\quad+\int_{B\cap\{|v_n|\ge\frac{4\tilde C}{\epsilon}\}}\frac{|\tilde f_\lambda(x,v_n)v_n|}{|v_n|}+\int_{B\cap\{|v_n|\le\frac{4\tilde C}{\epsilon}\}}|\tilde f_\lambda(x,v_n)|\\
&\le\frac{\epsilon}{2}+\mu(A)\mu_\epsilon^1+|B|_1\mu_\epsilon^2\le\epsilon
\end{aligned}$$

This completes Step 2 and the proof of the Lemma. □

Now we note that $\tilde J_\lambda(0)=0$ and $v=0$ is a local minimum for $\tilde J_\lambda$. It is also clear that $\lim_{t\to\infty}\tilde J_\lambda(tv)=-\infty$ for any $v\in H^1(\Omega)\backslash\{0\}$. Hence, we may fix $e\in H^1(\Omega)$ such that $\tilde J_\lambda(e)<0$. Let $\Gamma=\{\gamma:[0,1]\to H^1(\Omega);\gamma$ is continuous, $\gamma(0)=0,\gamma(1)=e\}$. We define the mountain-pass level

$$\rho_0=\inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}\tilde J_\lambda(\gamma(t)).$$

It follows that $\rho_0\ge0$. Let $R_0=\|e\|$. If $\rho_0=0$ we obtain that $\inf\{\tilde J_\lambda|\|v\|=R\}=0$ for all $R\in(0,R_0)$. We now let $F=H^1(\Omega)$ if $\rho_0>0$ and $F=\{\|v\|=\frac{R_0}{2}\}$ if $\rho_0=0$. We can now prove the following upper bound for ρ_0.

Lemma 5.4. $\rho_0<\frac{\pi}{2}(2-\beta)$.

Proof. Let $\{w_n\}$ be the sequence as in Lemma 2.1 by taking $n=\frac{1}{l}$. We now suppose $\rho_0\ge\frac{\pi}{2}(2-\beta)$ and derive a contradiction. This means that (thanks to Lemma 3.1 in [13]) for some $t_n>0$,

$$\tilde J_\lambda(t_nw_n)=\sup_{t>0}\tilde J_\lambda(t_nw_n)\ge\frac{\pi}{2}(2-\beta),\quad\forall n.$$

Then the above inequality gives,

$$\frac{t_n^2}{2} - \int_\Omega \tilde{G}_\lambda(x, t_n w_n) - \int_{\partial\Omega} \tilde{F}_\lambda(x, t_n w_n) \geq \frac{\pi}{2}(2-\beta), \quad \forall n. \tag{5.9}$$

In particular,

$$t_n^2 \geq \pi(2-\beta), \quad \text{for all large } n. \tag{5.10}$$

Since t_n yields is a maximum for the map $t \mapsto \tilde{J}_\lambda(tw_n)$ on $(0,\infty)$, $\frac{d}{dt}(\tilde{J}_\lambda(tw_n))|_{t=t_n} = 0$. That is,

$$t_n^2 = \int_\Omega \tilde{g}_\lambda(x, t_n w_n) t_n w_n + \int_{\partial\Omega} \tilde{f}_\lambda(x, t_n w_n) t_n w_n. \tag{5.11}$$

We note that $\inf_{x\in\overline{\Omega}} \tilde{g}_\lambda(x,s) \geq e^{s^2}$ for s large. Since $t_n w_n \to \infty$ on $\{|x| \leq \frac{\delta}{n}\}$ we get from (5.11),

$$\begin{aligned}
t_n^2 &\geq \int_{\{|x|\leq\frac{\delta}{n}\}\cap\overline{\Omega}} \tilde{g}_\lambda(x, t_n w_n) t_n w_n \\
&\geq \int_{\{|x|\leq\frac{\delta}{n}\}} \frac{e^{t_n^2 w_n^2}}{|x|^\beta} t_n w_n \\
&= \frac{\sqrt{\pi}\delta^2}{\sqrt{2}} e^{t_n^2 \frac{\log n}{2\pi}} (t_n (\log n)^{1/2}) \Big(\int_{|x|\leq\delta/n\cap\overline{\Omega}} |x|^{-\beta} dx \Big) \\
&= C e^{\frac{t_n^2}{\pi}\log n} (\frac{\delta}{n})^{(2-\beta)} t_n (\log n)^{1/2} \\
&= C e^{(\frac{t_n^2}{\pi} - (2-\beta))\log n} t_n (\log n)^{1/2}
\end{aligned}$$

Clearly the above inequality implies that $\{t_n\}$ is bounded sequence. Using (5.10) in the above inequality,

$$t_n^2 \geq \sqrt{\frac{\pi}{2}} \delta^2 e^{\epsilon \log n} t_n (\log n)^{1/2}, \quad \text{for some } \epsilon > 0$$

which implies $t_n \to \infty$ as $n \to \infty$, a contradiction. This contradiction shows that $\rho_0 < \frac{\pi}{2}(2-\beta)$. □

Lemma 5.5. *$\tilde{J}_\lambda$ has a critical point v_λ of mountain-pass type with $v_\lambda > 0$ in Ω.*

Proof. Let $\{v_n\} \subset H^1(\Omega)$ be a $(P.S)_{F,\rho}$ sequence for $\tilde{J}_\lambda$(for the existence of such a sequence, see [9]). Then, by Lemma 5.3, up to a subsequence, $v_n \rightharpoonup v_\lambda$ in $H^1(\Omega)$ for some $v_\lambda \in H^1(\Omega)$. Clearly $v_n \to v_\lambda$ point wise a.e. in Ω. If $v_\lambda \not\equiv 0$,then it is easy to see from (5.2) using weak maximum principle that v_λ has no nonpositive local minimum in Ω. By an application of Hopf maximum principle and the fact that $\frac{\partial v_\lambda}{\partial \nu} \geq 0$ on $\partial\Omega$, we conclude that $\min_{\overline{\Omega}} v_\lambda$ is achieved inside Ω. Hence $v_\lambda > 0$ in $\overline{\Omega}$. So it is enough to show that $v_\lambda \not\equiv 0$. We divide the proof into steps:

Case 1: $\rho_0 = 0$ In this case, from (5.3) we get

$$\begin{aligned}
o_n(1) = J(v_n) &= \frac{1}{2}\int_\Omega \big(|\nabla v_n|^2 + v_n^2\big) - \int_\Omega \tilde{G}(x, v_n)dx - \int_{\partial\Omega} \tilde{F}(x, v_n) \\
&= \frac{1}{2}\|v_n\|^2 + o_n(1)
\end{aligned}$$

Case 2: $\rho_0 \in (0, \frac{\pi}{2}(2-\beta))$ Since $J(v_n) \to \rho_0$, we obtain, $\|v_n\| \to 2\rho_0$ as $n \to \infty$. This and Lemma 4.4 immediately imply that $\|v_n\| \leq \pi(2-\beta) - \epsilon_0$ for some $\epsilon_0 > 0$.

Let $0 < \delta < \frac{\epsilon_0}{\pi(2-\beta)-\epsilon_0}$ and let $q_0 = \frac{\pi(2-\beta)}{(1+\delta)(\pi(2-\beta)-\epsilon_0)}$. Then $q_0 > 1$. Now we may choose q such that $1 < q < q_0$. Now from the definition of $\tilde{g}$ we get

$$\sup_{x\in\overline{\Omega}} \tilde{g}(x,s)|x|^\beta \le Ce^{(1+\delta)s^2}$$

for some constant C_1. Hence using the compact imbedding of $H^1(\Omega)$ in $L^p(\Omega, |x|^\beta)$, we get

$$\begin{aligned}\int_\Omega v_n \tilde{g}(x, v_n) &\le C \int_\Omega \frac{v_n}{|x|^\beta} e^{(1+\delta)v_n^2}\\ &\le C_2 \|v_n\|_{L^{q'}(\Omega,|x|^\beta)} \int_\Omega \frac{v_n}{|x|^\beta} e^{(1+\delta)(\frac{v_n}{\|v_n\|})^2\|v_n\|^2} = o_n(1)\end{aligned}$$

since $\frac{(1+\delta)q[\pi(2-\beta)-\epsilon_0]}{2\pi} + \frac{\beta}{2} < 1$. This implies

$$o_n(1)\|v_n\| = \langle J'(v_n), v_n\rangle = \|v_n\|^2 + o_n(1)$$

This contradicts the fact that $\|v_n\| \to 2\rho_0 > 0$. Hence $v_\lambda \not\equiv 0$. □

Proof of Theorem 1.2. From Lemma 5.5 and the arguments in section 5, we obtain that apart from u_λ we obtain a second solution $\tilde{u}_\lambda$ for all $\lambda \in (0, \Lambda)$, and by definition of Λ, (1.1) has no solution for $\lambda > \Lambda$. When $\lambda = \Lambda$, from Lemma 5.4 it is clear that $J_\lambda(u_\lambda) \le J_\lambda(v_\lambda) < 0$. Let $\{\lambda_n\}$ be a sequence such that $\lambda_n \to \Lambda$ and $\{u_{\lambda_n}\}$ be the corresponding sequence of solutions to (P_{λ_n}). Then,

$$\limsup_{n\to\infty} J_{\lambda_n}(u_{\lambda_n}) \le 0, \quad J'_{\lambda_n}(u_{\lambda_n}) = 0. \tag{5.12}$$

Now (5.12) implies that $\{u_{\lambda_n}\}$ is a bounded sequence in $H^1(\Omega)$. Hence there exists u_Λ such that $u_{\lambda_n} \rightharpoonup u_\Lambda$ in $H^1(\Omega)$. Now it is easy to verify that u_Λ is a weak solution of (P_Λ). □

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