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EXPONENTIAL CONVERGENCE OF SOLUTIONS OF SICNNS WITH MIXED DELAYS

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Abstract. In this paper, we discuss shunting inhibitory cellular neural networks (SICNNs) with mixed delays and time-varying coefficients. We establish conditions for all solutions of SICNNs to converge exponentially to zero. Our theorem improve some known results and allow for more general activation functions.

1. Introduction

In this article, we study the following shunting inhibitory cellular neural networks with mixed delays and time-varying coefficients:

$$\begin{aligned} x_{ij}'(t) = -a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)f[x_{kl}(t-\tau_{ij}(t))]x_{ij}(t) \\ - \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t)\int_0^\infty k_{ij}(u)g[x_{kl}(t-u)]du\cdot x_{ij}(t) + L_{ij}(t), \end{aligned} \tag{1.1}$$

where $i=1,2,\dots,m,\ j=1,2,\dots,n$; C_{ij} denotes the cell at the (i,j) position of the lattice; x_{ij} is the activity of the cell C_{ij}; the r-neighborhood $N_r(i,j)$ of C_{ij} is defined as

$$N_r(i,j) = \{C_{kl} : \max(|k-i|,|l-j|)\le r, 1\le k\le m, 1\le l\le n\}$$

and $N_q(i,j)$ is similarly defined; $L_{ij}(t)$ is the external input to C_{ij}; $a_{ij}>0$ represents the passive decay rate of the cell activity; $C_{ij}^{kl}\ge 0$ and $B_{ij}^{kl}\ge 0$ are the connection or coupling strength of postsynaptic activity of the cell C_{kl} transmitted to the cell C_{ij}; the activation functions f,g are continuous functions representing the output or firing rate of the cell C_{kl}; and $\tau_{ij}(t)\ge 0$ are the transmission delays.

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Recall that in 1990s, Bouzerdoum and Pinter [1, 2, 3] introduced and analyzed the networks commonly called shunting inhibitory cellular neural networks (SICNNs). Now, SICNNs have been extensively applied in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing (see, e.g., [4, 5] and references therein).

It is well known that analysis of dynamic behaviors is very important for design of neural networks. Therefore, there has been of great interest for many authors to study all kinds of dynamic behaviors for SICNNs and its variants (see, e.g., [6, 11, 7, 8, 13, 12, 9, 10]). Especially, there are many interesting and important works about exponential convergence behavior of solutions to SICNNs. For example, in [13], the authors studied the following SICNNs with delays

$$x'_{ij}(t) = -a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t) f[x_{kl}(t-\tau(t))]x_{ij}(t) + L_{ij}(t), \tag{1.2}$$

where $i = 1, 2, \dots, m, j = 1, 2, \dots, n$, and established a theorem which ensure that all the solutions of (1.2) converge exponentially to zero. Also, in [7], the authors considered the same problem for the the following SICNNs with distributed delays

$$x'_{ij}(t) = -a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\int_0^\infty k_{ij}(u) f[x_{kl}(t-u)]du\cdot x_{ij}(t) + L_{ij}(t),$$

where $i = 1, 2, \dots, m, j = 1, 2, \dots, n$. In addition, the authors in [8] studied the convergence behavior of solutions for the SICNNs (1.1).

In [7, 8, 13], the activity functions f and g are assumed to be bounded. Recently, in [11], the assumption is weakened into

(H0) There exist constants $m \ge 1$, $n \ge 1$, L_f and L_g such that for all $u \in \mathbb{R}$,

$$|f(u)| \le L_f|u|^m, \quad |g(u)| \le L_g|u|^n.$$

In this paper, we allow for more general activity functions f and g; i.e., we only assume that

(H1) f and g are continuous functions on $\mathbb{R}$.

In addition, we do not need the restrictive condition used in [11] (see remark 2.3).

Throughout this paper, for $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, $k_{ij} : [0, +\infty) \to \mathbb{R}$ are continuous integrable functions, $a_{ij}, C_{ij}^{kl}, B_{ij}^{kl}, \tau_{ij}$ are continuous functions, and L_{ij} are continuous bounded functions. Moreover, for real functions $u(t)$ and $v(t)$, we write $u(t) = O(v(t))$ if there exists a constant $M \ge 0$ such that for some $N > 0$,

$$|u(t)| \le M|v(t)|, \quad \forall t \ge N.$$

Since f and g are continuous functions, we define the following functions on $[0, +\infty)$:

$$F(x) = \max_{|t|\le x}|f(t)|, \quad G(x) = \max_{|t|\le x}|g(t)|.$$

2. Main results

In the proof of our results, we will use the following assumptions:

(H2) There exist constants $\eta > 0$ and $\lambda > 0$ such that

$$[\lambda - a_{ij}(t)] + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)F(\beta) + \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t)G(\beta)\int_0^\infty |k_{ij}(u)|du < -\eta,$$

for all $t > 0$, $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$, where

$$\beta = \frac{\max_{(i,j)}\{\sup_{t\geq 0} |L_{ij}(t)|\}}{\eta}.$$

(H3) $L_{ij}(t) = O(e^{-\lambda t})$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$.

Lemma 2.1. *Assume that* (H1) *and* (H2) *hold. Then, for every solution*

$$\{x_{ij}(t)\} = (x_{11}(t), \dots, x_{1n}(t), \dots, x_{i1}(t), \dots, x_{in}(t), \dots, x_{m1}(t), \dots, x_{mn}(t)),$$

of (1.1) *with initial condition* $\sup_{-\infty<s\leq 0} \max_{(i,j)} |x_{ij}(s)| < \beta$*, there holds*

$$|x_{ij}(t)| \leq \beta, \tag{2.1}$$

for all $t \in \mathbb{R}$ *and* $ij \in \{11, 12, \dots, mn\}$.

Proof. Assume that (2.1) does not hold. Then there exist $i_0 \in \{1, 2, \dots, m\}$ and $j_0 \in \{1, 2, \dots, n\}$ such that

$$\{t > 0 : |x_{i_0 j_0}(t)| > \beta\} \neq \emptyset. \tag{2.2}$$

For each $k \in \{1, 2, \dots, m\}$ and $l \in \{1, 2, \dots, n\}$, let

$$T_{kl} = \begin{cases} \inf\{t > 0 : |x_{kl}(t)| > \beta\} & \{t > 0 : |x_{kl}(t)| > \beta\} \neq \emptyset, \\ +\infty & \{t > 0 : |x_{kl}(t)| > \beta\} = \emptyset. \end{cases}$$

Then $T_{kl} > 0$ and

$$|x_{kl}(t)| \leq \beta, \quad \forall t \leq T_{kl},\ k = 1, 2, \dots, m,\ l = 1, 2, \dots, n. \tag{2.3}$$

We denote $T_0 = T_{ij} = \min_{(k,l)} T_{kl}$, where $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$. In view of (2.2), we have $0 < T_0 < +\infty$. It follows from (2.3) that

$$|x_{kl}(t)| \leq \beta, \quad \forall t \leq T_0,\ k = 1, 2, \dots, m,\ l = 1, 2, \dots, n. \tag{2.4}$$

In addition, noticing that $T_0 = \inf\{t > 0 : |x_{ij}(t)| > \beta\}$, we obtain

$$|x_{ij}(T_0)| = \beta, \quad D^+(|x_{ij}(s)|)|_{s=T_0} \geq 0. \tag{2.5}$$

Combing (H2), (2.4) and (2.5), we have

$$
\begin{aligned}
&D^+(|x_{ij}(s)|)|_{s=T_0}\\
&= \operatorname{sgn}(x_{ij}(T_0))\Big\{ - a_{ij}(T_0)x_{ij}(T_0) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(T_0) f[x_{kl}(T_0-\tau_{ij}(T_0))]x_{ij}(T_0)\\
&\quad - \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(T_0)\int_0^\infty k_{ij}(u) g[x_{kl}(T_0-u)]du\cdot x_{ij}(T_0) + L_{ij}(T_0)\Big\}\\
&\le -a_{ij}(T_0)\cdot |x_{ij}(T_0)| + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(T_0)F(\beta)\cdot |x_{ij}(T_0)|\\
&\quad + \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(T_0)G(\beta)\int_0^\infty |k_{ij}(u)|du\cdot |x_{ij}(T_0)| + |L_{ij}(T_0)|\\
&\le \Big\{ - a_{ij}(T_0) + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(T_0)F(\beta)\\
&\quad + \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(T_0)G(\beta)\int_0^\infty |k_{ij}(u)|du\Big\}\cdot \beta + |L_{ij}(T_0)|\\
&< -\eta\cdot\beta + |L_{ij}(T_0)|\\
&= -\max_{(i,j)}\{\sup_{t\ge 0} |L_{ij}(t)|\} + |L_{ij}(T_0)| \le 0.
\end{aligned}
$$

This contradicts $D^+(|x_{ij}(s)|)|_{s=T_0} \ge 0$. Thus, (2.1) holds. $\square$

Theorem 2.2. *Let* (H1)–(H3) *hold. Then, for every solution*

$$\{x_{ij}(t)\} = (x_{11}(t),\dots,x_{1n}(t),\dots,x_{i1}(t),\dots,x_{in}(t),\dots,x_{m1}(t),\dots,x_{mn}(t))$$

of (1.1) *with initial condition* $\sup_{-\infty<s\le 0}\max_{(i,j)}|x_{ij}(s)| < \beta$, *there holds*

$$x_{ij}(t) = O(e^{-\lambda t}),\quad ij = 11,12,\dots,mn.$$

Proof. It follows from (H3) that there exist constants $M>0$ and $T>0$ such that

$$|L_{ij}(t)| < \frac{1}{2}\eta M e^{-\lambda t},\quad \forall t\ge T,\ ij = 11,12,\dots,mn. \tag{2.6}$$

Let $\{x_{ij}(t)\} = (x_{11}(t),\dots,x_{1n}(t),\dots,x_{i1}(t),\dots,x_{in}(t),\dots,x_{m1}(t),\dots,x_{mn}(t))$ be a solution of (1.1) with initial condition $\sup_{-\infty<s\le 0}\max_{(i,j)}|x_{ij}(s)| < \beta$.

Set

$$V_{ij}(t) = \max_{s\le t}\{e^{\lambda s}|x_{ij}(s)|\},\quad ij = 11,12,\dots,mn.$$

It is easy to prove that each $V_{ij}(t)$ is continuous. For any given $t_0\ge T$ and $ij\in\{11,12,\dots,mn\}$, we consider three cases.

Case (i) $V_{ij}(t_0) > e^{\lambda t_0}|x_{ij}(t_0)|$. It follows form the continuity of $x_{ij}(t)$ that there exists $\delta_1>0$ such that

$$e^{\lambda t}|x_{ij}(t)| < V_{ij}(t_0),\quad \forall t\in(t_0,t_0+\delta_1).$$

Thus, we can conclude

$$V_{ij}(t) = V_{ij}(t_0),\quad \forall t\in(t_0,t_0+\delta_1).$$

Case (ii) $V_{ij}(t_0) = e^{\lambda t_0}|x_{ij}(t_0)| < M$. Also, by the continuity of $x_{ij}(t)$, there exists $\delta_2>0$ such that

$$e^{\lambda t}|x_{ij}(t)| < M,\quad \forall t\in(t_0,t_0+\delta_2).$$

Therefore,

$$V_{ij}(t) < M, \quad \forall t \in (t_0, t_0 + \delta_2).$$

Case (iii) $V_{ij}(t_0) = e^{\lambda t_0}|x_{ij}(t_0)| \geq M$. By Lemma 2.1, $|x_{kl}(t)| \leq \beta$ for all $t \in \mathbb{R}$ and $kl \in \{11, 12, \dots, mn\}$. In view of this and (H2), (2.6), we have

$$\begin{aligned}
&D^+(e^{\lambda s}|x_{ij}(s)|)|_{s=t_0} \\
&= \lambda e^{\lambda t_0}|x_{ij}(t_0)| + e^{\lambda t_0}\operatorname{sgn}(x_{ij}(t_0))\Big\{ - a_{ij}(t_0)x_{ij}(t_0) \\
&\quad - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t_0) f[x_{kl}(t_0 - \tau_{ij}(t_0))]x_{ij}(t_0) \\
&\quad - \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t_0)\int_0^\infty k_{ij}(u)g[x_{kl}(t_0-u)]du\cdot x_{ij}(t_0) + L_{ij}(t_0)\Big\} \\
&\leq e^{\lambda t_0}|x_{ij}(t_0)|\Big\{\lambda - a_{ij}(t_0) + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t_0)F(\beta) \\
&\quad + \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t_0)G(\beta)\int_0^\infty |k_{ij}(u)|du\Big\} + \frac{1}{2}\eta M \\
&\leq e^{\lambda t_0}|x_{ij}(t_0)|\cdot(-\eta) + \frac{1}{2}\eta M \leq -\eta M + \frac{1}{2}\eta M \\
&= -\frac{1}{2}\eta M < 0.
\end{aligned}$$

Since $D^+(e^{\lambda s}|x_{ij}(s)|)|_{s=t_0} < 0$, there exists $\delta_3 > 0$ such that

$$e^{\lambda t}|x_{ij}(t)| < e^{\lambda t_0}|x_{ij}(t_0)| = V_{ij}(t_0), \quad \forall t \in (t_0, t_0 + \delta_3).$$

Then, we conclude that

$$V_{ij}(t) = V_{ij}(t_0), \quad \forall t \in (t_0, t_0 + \delta_3).$$

In summary, for any given $ij \in \{11, 12, \dots, mn\}$, for all $t_0 \geq T$, there exists $\delta = \min\{\delta_1, \delta_2, \delta_3\} > 0$ such that

$$V_{ij}(t) \leq \max\{V_{ij}(t_0), M\}, \quad \forall t \in (t_0, t_0 + \delta).$$

Now, take $t_0 = T$. Then there exists $\delta' > 0$ such that

$$V_{ij}(t) \leq \max\{V_{ij}(T), M\}, \quad \forall t \in (T, T + \delta').$$

Since V_{ij} is continuous, we have

$$V_{ij}(t) \leq \max\{V_{ij}(T), M\}, \quad \forall t \in (T, T + \delta'].$$

Take $t_0 = T + \delta'$. Then there exists $\delta'' > 0$ such that

$$V_{ij}(t) \leq \max\{V_{ij}(T+\delta'), M\} \leq \max\{V_{ij}(T), M\}, \quad \forall t \in (T + \delta', T + \delta' + \delta'').$$

Then

$$V_{ij}(t) \leq \max\{V_{ij}(T), M\}, \quad \forall t \in (T, T + \delta' + \delta'').$$

Continuing the above step, at last, we get a maximal interval (T, α_{ij}) such that

$$V_{ij}(t) \leq \max\{V_{ij}(T), M\}, \quad \forall t \in (T, \alpha_{ij}).$$

Also, we have $\alpha_{ij} = +\infty$. In fact, if $\alpha_{ij} < +\infty$, then we have

$$V_{ij}(t) \leq \max\{V_{ij}(T), M\}, \quad \forall t \in (T, \alpha_{ij}].$$

Take $t_0 = \alpha_{ij}$. Then there exists $\delta^* > 0$ such that

$$V_{ij}(t) \le \max\{V_{ij}(T), M\}, \quad \forall t \in (T, \alpha_{ij} + \delta^*).$$

This is a contradiction. Therefore,

$$V_{ij}(t) \le \max\{V_{ij}(T),\ M\}, \quad \forall t > T.$$

It follows that

$$e^{\lambda t}|x_{ij}(t)| \le \max\{V_{ij}(T),\ M\}, \quad \forall t > T,$$

whichimplies $x_{ij}(t) = O(e^{-\lambda t})$. □

Remark 2.3. In [11], it is assume that $\beta < 1$. But in Theorem 2.2, we do not need this condition. In addition, it is not difficult to show that Theorem [11, Theorem 2.1] is a corollary of Theorem 2.2.

3. Examples

In this section, we give an example to illustrate our results.

Example 3.1. Consider the SICNNs:

$$x'_{ij}(t) = -a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_1(i,j)} C_{ij}^{kl}(t) f[x_{kl}(t-\tau_{ij}(t))]x_{ij}(t) + L_{ij}(t), \tag{3.1}$$

where $i = 1,2,3$, $j = 1,2,3$, $\tau_{ij}(t) = |\frac{1}{2}t\sin(i+j)t|$, $f(x) = e^x$,

$$\begin{pmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{pmatrix} = \begin{pmatrix} 5+\sin^2 t & 5+|\sin t| & 7+\sin t \\ 6+\sin t & 7+|\sin t| & 6+\sin t \\ 7+\sin t & 8+\sin t & 8+\sin t \end{pmatrix},$$

$$\begin{pmatrix} c_{11}(t) & c_{12}(t) & c_{13}(t) \\ c_{21}(t) & c_{22}(t) & c_{23}(t) \\ c_{31}(t) & c_{32}(t) & c_{33}(t) \end{pmatrix} = \begin{pmatrix} 0.1|\sin t| & 0.1\sin^2 t & 0.2|\sin t| \\ 0 & 0.2\sin^2 t & 0 \\ 0.1\sin^2 t & 0.1|\sin t| & 0.2\sin^2 t \end{pmatrix},$$

$$\begin{pmatrix} L_{11}(t) & L_{12}(t) & L_{13}(t) \\ L_{21}(t) & L_{22}(t) & L_{23}(t) \\ L_{31}(t) & L_{32}(t) & L_{33}(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{-t} & e^{-2t} & 2e^{-2t} \\ e^{-2t} & e^{-t} & e^{-2t} \\ e^{-2t} & e^{-t} & e^{-2t} \end{pmatrix}.$$

Obviously, (H1) holds. By some calculations, it is easy to obtain that for all $t \in \mathbb{R}$,

$$a_{ij}(t) \ge 5. \qquad \sum_{C_{kl}\in N_1(i,j)} C_{ij}^{kl}(t) \le 1.$$

In addition,

$$\max_{(i,j)}\{\sup_{t\ge 0}|L_{ij}(t)|\} = 2, \quad F(x) = e^x.$$

Let $\lambda = 0.2$ and $\eta = 2$. Then $\beta = 1$ and

$$[\lambda - a_{ij}(t)] + \sum_{C_{kl}\in N_1(i,j)} C_{ij}^{kl}(t)F(\beta) \le 0.2 - 5 + e < -2 = -\eta,$$

for all $t > 0$, $i = 1,2,3$ and $j = 1,2,3$. Therefore, (H2) holds.

It is easy to verify that (H3) holds for $\lambda = 0.2$. Now, by Theorem 2.2, all the solutions of (3.1) with initial condition

$$\sup_{-\infty<s\le 0}\max_{(i,j)}|x_{ij}(s)| < 1$$

converge exponentially to zero when $t \to +\infty$.

Remark 3.2. In the above example, f is neither bounded nor satisfies (H0). Therefore, the results in [11, 7, 13, 8] can not be applied to this equation.

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