Electronic Journal of Differential Equations, Vol. 2009(2009), No. 34, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ASYMPTOTIC BEHAVIOR OF SOLUTIONS ON A THIN PLASTIC PLATE

ABDELAZIZ AIT MOUSSA, JAMAL MESSAHO

Abstract. In the present work, we study the asymptotic behavior of solutions to a plasticity problem in a containing structure, a thin plastic plate of thickness that tends to zero. To find the limit problems with interface conditions we use the epiconvergence method.

1. Introduction

The study of the inclusion between two elastic bodies involves introducing a very thin third body between them. A very similar situation occurs when taking into account the effects of a thin layer which has been bonded onto the surface of a body to prevent wear caused by the contact with another solid. It is, therefore of interest to study the asymptotic behavior of thin layer between the two bodies, assuming various contact laws between them. In the case of a thin plate, the thermal conductivity problems were treated by Brillard et al and Sanchez-Palencia et al in [8, 15]. The elasticity problems, linear and nonlinear case, were widely studied by Ait Moussa et al, Ait moussa, Brillard et al, Geymonat et al and Lenci et al in [2, 3, 12, 13, 14]. In the case of an oscillating layer, we have treated the scalar case for a thermal conductivity problem in Messaho et al in [5]. In the present work, we consider a structure containing a thin plastic plate of thickness depending on a parameter ε intended to tend towards 0. The aim of this work is to study the asymptotic behavior of the solution of a plasticity problem posed on a such structure.

This paper is organized in the following way. In section 2, we express the problem to study, and we give some notation and we define functional spaces for this study in the section 3. In the section 4, we study the problem (4.1). The section 5 is reserved to the determination of the limits problems and our main result.

2. Statement of the problem

We consider a structure constituted of two linear elastics bodies, joined together by a thin plastic plate of thickness ε, the latter obeys to a nonlinear plastic law of

2000 *Mathematics Subject Classification.* 35B40, 82B24, 76M50.
Key words and phrases. Asymptotic behavior; plasticity problem; epiconvergence method; limit problems.

Submitted July 30, 2008. Published February 23, 2009.

power type. More precisely the stress field is related to the displacement's field by

$$\sigma^\varepsilon = \lambda|e(u^\varepsilon)|^{-1}e(u^\varepsilon), \quad \lambda > 0.$$

The structure occupies the regular domain $\Omega = B_\varepsilon \cup \Omega_\varepsilon$, where B_ε is given by $B_\varepsilon = \{x = (x', x_3)/|x_3| < \frac{\varepsilon}{2}\}$, and $\Omega_\varepsilon = \Omega \setminus B_\varepsilon$ represent the regions occupied by the thin plate and the two elastic bodies (see figure 1). ε being a positive parameter intended to approach 0.

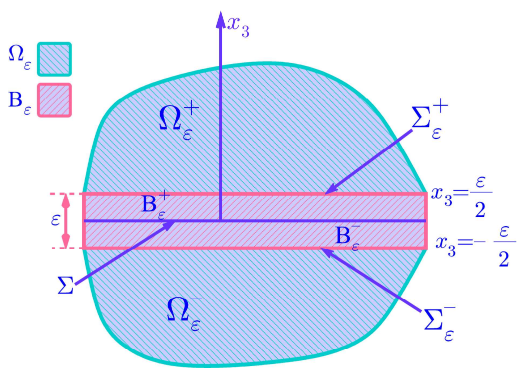

FIGURE 1. The domain Ω.

The structure is subjected to a density of forces of volume f, $f : \Omega \to \mathbb{R}^3$, and it is fixed on the boundary $\partial\Omega$. Equations which relate the stress field σ^ε, $\sigma^\varepsilon : \Omega \to \mathbb{R}^9_S$, and the field of displacement u^ε, $u^\varepsilon : \Omega \to \mathbb{R}^3$ are

$$\begin{gathered}
\operatorname{div}\sigma^\varepsilon + f = 0 \quad \text{in } \Omega, \\
\sigma^\varepsilon_{ij} = a_{ijkh}e_{kh}(u^\varepsilon) \quad \text{in } \Omega_\varepsilon, \\
\sigma^\varepsilon = \lambda|e(u^\varepsilon)|^{-1}e(u^\varepsilon) \quad \text{in } B_\varepsilon, \\
u^\varepsilon = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{2.1}$$

Where a_{ijkh} are the elasticity coefficients and $\mathbb{R}^9_S$ the vector space of the square symmetrical matrices of order three. $e_{ij}(u)$ are the components of the linearized tensor of deformation $e(u)$. In the sequel, we assume that the elasticity coefficients a_{ijkh} satisfy to the following hypotheses:

$$a_{ijkh} \in L^\infty(\Omega), \tag{2.2}$$

$$a_{ijkh} = a_{jikh} = a_{khij}, \tag{2.3}$$

$$a_{ijkh}\tau_{ij}\tau_{kh} \geq C\tau_{ij}\tau_{ij}, \quad \forall \tau \in \mathbb{R}^9_S. \tag{2.4}$$

3. Notation and functional setting

Here is the notation that will be used in the sequel:
$x = (x', x_3)$ where $x' = (x_1, x_2)$, $\tau \otimes \zeta = (\tau_i \zeta_j)_{1 \leq i,j \leq 3}$ and $\tau \otimes_S \zeta = \frac{\tau \otimes \zeta + \zeta \otimes \tau}{2}$ for all $\tau, \zeta \in \mathbb{R}^3$.

In the following C will denote any constant with respect to ε. Also, we use the convention $0.(+\infty) = 0$.

Functional setting. First, we introduce the space

$$\begin{aligned} V^\varepsilon = \big\{ &u \in L^1(\Omega, \mathbb{R}^3) : e(u) \in L^2(\Omega_\varepsilon, \mathbb{R}^9_S),\ u \in BD(B_\varepsilon), \\ &[u]^\varepsilon = 0 \text{ in } \Sigma^\pm_\varepsilon \text{ and } u = 0 \text{ in } \partial\Omega \big\}, \end{aligned}$$

where $[u]^\varepsilon$ is the jump of u on $\Sigma^\pm_\varepsilon$ defined by

$$[u]^\varepsilon = \pm u_{|_{\Omega^\pm_\varepsilon}} \mp u_{|_{B^\pm_\varepsilon}},$$

$$BD(B_\varepsilon) = \big\{ u \in L^1(\Omega, \mathbb{R}^3) : e(u) \in M_1(B_\varepsilon, \mathbb{R}^9_S) \big\},$$

$$BD_0(\Omega) = \big\{ u \in BD(\Omega, \mathbb{R}^3) : u = 0 \text{ in } \partial\Omega \big\},$$

and $M_1(.)$ is a bounded measure space, for more information we can refer the reader to [16]. We show easily that V^ε is a Banach space with the norm

$$u \to \|e(u)\|_{L^2(\Omega_\varepsilon, \mathbb{R}^9_S)} + \|e(u)\|_{M_1(B_\varepsilon, \mathbb{R}^9_S)}.$$

Where

$$\|e(u)\|_{M_1(B_\varepsilon, \mathbb{R}^9_S)} = \int_{B_\varepsilon} |e(u)| = \sup_{\tau \in \mathcal{C}^\infty_0(B_\varepsilon),\, |\tau(x)| \leq 1.} \langle e(u), \tau \rangle.$$

We remark that $V^\varepsilon \subset BD_0(\Omega)$.

Our goal in this work is to study the problem (2.1), and its limit behavior when ε tends to zero.

4. Study of problem (2.1)

Problem (2.1) is equivalent to the minimization problem

$$\inf_{v \in V^\varepsilon} \big\{ \frac{1}{2} \int_{\Omega_\varepsilon} a_{ijhk} e_{hk}(v) e_{ij}(v) dx + \lambda \int_{B_\varepsilon} |e(v)| - \int_\Omega f v dx \big\} \tag{4.1}$$

To study problem (2.1), we will study the minimization problem (4.1). The existence and uniqueness of solutions to (4.1) is given in the following proposition.

Proposition 4.1. *Under the hypotheses* (2.2), (2.3), (2.4) *and for* $f \in L^\infty(\Omega, \mathbb{R}^3)$, *problem* (4.1) *admits an unique solution* u^ε *in* V^ε.

Proof. From (2.2) and (2.4), we show easily that the energy functional in (4.1) is weakly lower semicontinuous, strictly convex and coercive over V^ε. Since V^ε is not reflexive, so we may not apply directly result given in Dacorogna [9, theorem 1.1 p.48], but we can follow our proof by using the compact imbedding of Sobolev for the BD space, for more information we can refer the reader to [9]. Indeed, let u_n be a minimizing sequence for (4.1), to simplify the writing let

$$\mathbb{F}^\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} a_{ijhk} e_{hk}(u) e_{ij}(u) dx + \lambda \int_{B_\varepsilon} |e(u)| - \int_\Omega f u dx,$$

so, we have $\mathbb{F}^\varepsilon(u_n) \to \inf_{v\in V^\varepsilon} \mathbb{F}^\varepsilon(v)$. Using the coercivity of $\mathbb{F}^\varepsilon$, we may then deduce that there exists a constant $C > 0$, independent of n, such that

$$\|u_n\|_{V^\varepsilon} \le C,$$

according to the reflexivity of $H^1(\Omega_\varepsilon)$ and using the given result in [16, p.158] for $BD(B_\varepsilon)$, then for a subsequence of u_n, still denoted by u_n, there exists $u_0 \in V^\varepsilon$ such that $u_n \rightharpoonup u_0$ in V^ε. The weak lower semi-continuity and the strict convexity of $\mathbb{F}^\varepsilon$ imply then the result. □

Lemma 4.1. *Assuming that for any sequence $(u^\varepsilon)_{\varepsilon>0} \subset V^\varepsilon$, there exists a constant $C > 0$ such that $\mathbb{F}^\varepsilon(u^\varepsilon) \le C$, under (2.2), (2.4) and for $f \in L^\infty(\Omega,\mathbb{R}^3)$, $(u^\varepsilon)_{\varepsilon>0}$ satisfies*

$$\|e(u^\varepsilon)\|^2_{L^2(\Omega_\varepsilon,\mathbb{R}^9_S)} \le C, \tag{4.2}$$

$$\|e(u^\varepsilon)\|_{M_1(B_\varepsilon,\mathbb{R}^9_S)} \le C, \tag{4.3}$$

moreover u^ε is bounded in $BD_0(\Omega,\mathbb{R}^3)$.

Proof. Since $\mathbb{F}^\varepsilon(u^\varepsilon) \le C$, we have

$$\frac{1}{2}\int_{\Omega_\varepsilon} a_{ijhk} e_{hk}(u^\varepsilon) e_{ij}(u^\varepsilon) dx + \lambda \int_{B_\varepsilon} |e(u^\varepsilon)| - \int_\Omega f u^\varepsilon dx \le C\,.$$

Then

$$\frac{1}{2}\int_{\Omega_\varepsilon} a_{ijhk} e_{hk}(u^\varepsilon) e_{ij}(u^\varepsilon) dx + \lambda \int_{B_\varepsilon} |e(u^\varepsilon)| \le C + \int_\Omega f u^\varepsilon dx\,.$$

According to (2.4), Hölder and Young the inequalities, we have

$$\begin{aligned}\|e(u^\varepsilon)\|^2_{L^2(\Omega_\varepsilon,\mathbb{R}^9_S)} + \int_{B_\varepsilon} |e(u^\varepsilon)| &\le C + C\int_\Omega f u^\varepsilon dx,\\ &\le C + C\|e(u^\varepsilon)\|_{L^2(\Omega_\varepsilon,\mathbb{R}^9_S)} + \int_{B_\varepsilon} f u^\varepsilon dx,\end{aligned}$$

since $BD(\Omega) \hookrightarrow L^q(\Omega,\mathbb{R}^3)$ for all $q \in [1, \frac{3}{2}]$, (with a continuous imbedding, see for example [16]). In particular $BD(\Omega) \hookrightarrow L^{q_0}(\Omega,\mathbb{R}^3)$ with $1 < q_0 \le \frac{3}{2}$, according to the Hölder inequality, we then have

$$\begin{aligned}\int_{B_\varepsilon} f u^\varepsilon &\le \|f\|_{L^{q_0'}(B_\varepsilon,\mathbb{R}^3)} \|u^\varepsilon\|_{L^{q_0}(B_\varepsilon,\mathbb{R}^3)},\\ &\le C\varepsilon^{1/q_0'} \int_\Omega |e(u^\varepsilon)|,\\ &\le C\varepsilon^{1/q_0'}\Big(\|e(u^\varepsilon)\|_{L^2(\Omega_\varepsilon,\mathbb{R}^9_S)} + \int_{B_\varepsilon} |e(u^\varepsilon)|\Big),\end{aligned}$$

so for $\varepsilon < \left(\frac{1}{1+C}\right)^{q_0'}$, let $\widetilde{C} = \frac{C}{1+C}$, we then have

$$\begin{aligned}\|e(u^\varepsilon)\|^2_{L^2(\Omega_\varepsilon,\mathbb{R}^9_S)} + \int_{B_\varepsilon} |e(u^\varepsilon)| &\le C + C\|e(u^\varepsilon)\|_{L^2(\Omega_\varepsilon,\mathbb{R}^9_S)} + \widetilde{C}\int_{B_\varepsilon} |e(u^\varepsilon)|, \\ &\le C + \frac{1}{2}\|e(u^\varepsilon)\|^2_{L^2(\Omega_\varepsilon,\mathbb{R}^9_S)} + \widetilde{C}\int_{B_\varepsilon} |e(u^\varepsilon)|,\end{aligned}$$

so that

$$\frac{1}{2}\|e(u^\varepsilon)\|^2_{L^2(\Omega_\varepsilon,\mathbb{R}^9_S)} + (1-\widetilde{C})\int_{B_\varepsilon} |e(u^\varepsilon)| \le C.$$

Therefore, we will have (4.2) and (4.3). According to (4.2) and (4.3) and for a small enough ε the sequence (u^ε) is bounded in $BD_0(\Omega,\mathbb{R}^3)$. □

Remark 4.2. The solution u^ε of the problem (4.1) satisfy to the lemma 4.1.

To apply the epiconvergence method, we need to characterize the topological spaces containing any cluster point of the solution of the problem (4.1) with respect to the used topology, therefore the weak topology to use is insured by the lemma 4.1. So the topological spaces characterization is given in the following proposition.

Proposition 4.3. *The solution u^ε of the problem* (4.1) *possess a cluster point u^* in $BD_0(\Omega) \cap H^1(\Omega \setminus \Sigma, \mathbb{R}^3)$ with respect to the weak topology of $BD_0(\Omega)$.*

Proof. According to the remark 4.2 and lemma 4.1, for a small enough ε, u^ε is bounded in $BD_0(\Omega)$, so for a subsequences of u^ε, still denoted by u^ε, there exists $u^* \in BD_0(\Omega)$, (see [16, p. 158]), such that

$$u^\varepsilon \rightharpoonup u^* \text{in } BD_0(\Omega,\mathbb{R}^3), \tag{4.4}$$

so that

$$\lim_{\varepsilon\to 0}\int_\Omega v e(u^\varepsilon) = \int_\Omega v e(u^*), \quad \forall v \in \mathcal{C}_0^\infty(\Omega,\mathbb{R}^9_S). \tag{4.5}$$

For ε a small enough, let $\eta > 0$ and $\Omega^\eta = \{x \in \Omega : |x_3| > \eta\}$, such that $\varepsilon < \eta$. From (4.2), we then have

$$\|e(u^\varepsilon)\|^2_{L^2(\Omega^\eta,\mathbb{R}^9_S)} \le C,$$

Therefore, $e(u^\varepsilon)$ is bounded in $L^2(\Omega^\eta,\mathbb{R}^9_S)$, so for a subsequence of $e(u^\varepsilon)$, still denoted by $e(u^\varepsilon)$, there exists $w \in L^2(\Omega^\eta,\mathbb{R}^9_S)$, such that

$$e(u^\varepsilon) \rightharpoonup w \quad \text{in } L^2(\Omega^\eta,\mathbb{R}^9_S),$$

according (4.4) and (4.5) remains true in $\mathcal{C}_0^\infty(\Omega^\eta,\mathbb{R}^9_S)$, we then deduce $e(u^*) = w$, hence $e(u^*) \in L^2(\Omega^\eta,\mathbb{R}^9_S)$ for all $\eta > 0$, so by passing to the limit ($\eta \to 0$), we then have $e(u^*) \in L^2(\Omega \setminus \Sigma,\mathbb{R}^9_S)$. According to the classical result [16, proposition 1.2, p. 16], we have $u^* \in H^1(\Omega \setminus \Sigma,\mathbb{R}^3)$. □

In the following, let

$$\mathbb{H}^1_0 = \{u \in H^1(\Omega \setminus \Sigma,\mathbb{R}^3) : u = 0 \text{ on } \partial\Omega\}.$$
$$\mathbb{C}^\infty_0 = \{u \in \mathcal{C}^\infty(\Omega \setminus \Sigma,\mathbb{R}^3) : u = 0 \text{ on } \partial\Omega\}.$$

Remark 4.4. Proposition 4.3 remains valid for any weak cluster point u of a sequence u_ε in V^ε,that satisfies (4.2) and (4.3).

To study the limit behavior of the solution of the problem (4.1), we will use the epiconvergence method, (see Annex, definition 6.1).

5. Limit behavior

Let

$$F^\varepsilon(u)=\begin{cases}\frac{1}{2}\int_{\Omega_\varepsilon} a_{ijkh}e_{kh}(u)e_{ij}(u)dx+\lambda\int_{B_\varepsilon}|e(u)| & \text{if } u\in V^\varepsilon,\\ +\infty & \text{if } u\in BD_0(\Omega)\setminus V^\varepsilon.\end{cases} \tag{5.1}$$

$$G(u)=-\int_\Omega fudx,\quad \forall u\in BD_0(\Omega).$$

We design by τ_f the weak topology on the space $BD_0(\Omega)$. In the sequel, we shall characterize, the epi-limit of the energy functional given by (5.1) in the following theorem.

Theorem 5.1. *Under (2.2), (2.3), (2.4) and for $f\in L^\infty(\Omega,\mathbb{R}^3)$, there exists a functional $F\colon BD_0(\Omega)\to\mathbb{R}\cup\{+\infty\}$ such that*

$$\tau_f-lim_e F^\varepsilon{=}F \quad \text{in } BD_0(\Omega),$$

where F is given by

$$F(u)=\begin{cases}\frac{1}{2}\int_\Omega a_{ijkh}e_{kh}(u)e_{ij}(u)+\lambda\int_\Sigma|[u]\otimes_S e_3| & \text{if } u\in\mathbb{H}^1_0,\\ +\infty & \text{if } u\in BD_0(\Omega)\setminus\mathbb{H}^1_0.\end{cases}$$

Proof. (a) We are now in position to determine the upper epi-limit.
Let $u\in\mathbb{H}^1_0\subset BD_0(\Omega)$, so there exists a sequence (u^n) in $\mathbb{C}^\infty_0$ such that $u^n\to u$ in $\mathbb{H}^1_0$when $n\to+\infty$, so $u^n\rightharpoonup u$ weakly in $BD_0(\Omega)$. Let us consider the sequence

$$u^{\varepsilon,n}=\begin{cases}u^n(x',x_3) & \text{if } |x_3|>\frac{\varepsilon}{2},\\ \frac{1}{2}\big(u^n(x',\frac{\varepsilon}{2})+u^n(x',-\frac{\varepsilon}{2})\big) & \\ +\frac{x_3}{\varepsilon}\big(u^n(x',\frac{\varepsilon}{2})-u^n(x',-\frac{\varepsilon}{2})\big) & \text{if } |x_3|<\frac{\varepsilon}{2}.\end{cases}$$

We have $u^{\varepsilon,n}\in V^\varepsilon$ and we prove easily that $u^{\varepsilon,n}\rightharpoonup u^n$ in $\mathbb{H}^1_0$ when $\varepsilon\to 0$. As

$$F^\varepsilon(u^{\varepsilon,n})= \quad \frac{1}{2}\int_{\Omega_\varepsilon}a_{ijkh}e_{kh}(u^{\varepsilon,n})e_{ij}(u^{\varepsilon,n})+\lambda\int_{B_\varepsilon}|e(u^{\varepsilon,n})|.$$

It implies that

$$F^\varepsilon(u^{\varepsilon,n})=\frac{1}{2}\int_{\Omega_\varepsilon}a_{ijkh}e_{kh}(u^n)e_{ij}(u^n)+\lambda\int_{B_\varepsilon}|e(u^{\varepsilon,n})|=:S_1+S_2.$$

So that

$$\lim_{\varepsilon\to 0}S_1=\frac{1}{2}\int_\Omega a_{ijkh}e_{kh}(u^n)e_{ij}(u^n).$$

we have

$$S_2=\lambda\int_{B_\varepsilon}|e(u^{\varepsilon,n})|, \tag{5.2}$$

As in [3] we show that

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon}|e(u^{\varepsilon,n})-\frac{1}{\varepsilon}[u^n]\otimes_S e_3|=0.$$

Consequently,

$$\limsup_{\varepsilon\to 0}F^\varepsilon(u^{\varepsilon,n})=\frac{1}{2}\int_\Omega a_{ijkh}e_{kh}(u^n)e_{ij}(u^n)+\lambda\int_\Sigma|[u^n]\otimes_S e_3|.$$

Since $u^n \to u$ in $\mathbb{H}_0^1$ when $n \to +\infty$, therefore according to a classic result, diagonalization's lemma, (see, [6, Lemma 1.15 p. 32]), there exists a function $n(\varepsilon) : \mathbb{R}^+ \to \mathbb{N}$ increasing to $+\infty$ when $\varepsilon \to 0$ such that $u^{\varepsilon,n(\varepsilon)} \rightharpoonup u$ in $\mathbb{H}_0^1$ when $\varepsilon \to 0$. and while $n \to +\infty$, consequently we have

$$\begin{aligned}\limsup_{\varepsilon\to 0} F^\varepsilon(u^{\varepsilon,n(\varepsilon)}) &\le \limsup_{n\to+\infty}\limsup_{\varepsilon\to 0} F^\varepsilon(u^{\varepsilon,n}),\\ &\le \frac{1}{2}\int_\Omega a_{ijkh}e_{kh}(u)e_{ij}(u) + \lambda\int_\Sigma |[u]\otimes_S e_3|.\end{aligned}$$

For $u \in BD_0(\Omega,\mathbb{R}^3)\setminus \mathbb{H}_0^1$, so for any sequence $u^\varepsilon \rightharpoonup u$ in $BD_0(\Omega)$,we obtain

$$\limsup_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) \le +\infty.$$

(b) We are now in position to determine the lower epi-limit. Let $u \in \mathbb{H}_0^1$ and $(u^\varepsilon) \subset V^\varepsilon$ such that $u^\varepsilon \rightharpoonup u$ in $BD_0(\Omega)$. If $\liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) = +\infty$, there is nothing to prove, because

$$\frac{1}{2}\int_\Omega a_{ijkh}e_{kh}(u)e_{ij}(u) + \lambda\int_\Sigma |[u]\otimes_S e_3| \le +\infty.$$

otherwise, $\liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) < +\infty$, there exists a subsequence of $F^\varepsilon(u^\varepsilon)$, still denoted by $F^\varepsilon(u^\varepsilon)$ and a constant $C > 0$, such that $F^\varepsilon(u^\varepsilon) \le C$, which implies that

$$\begin{gathered}\|e(u^\varepsilon\|_{L^2(\Omega_\varepsilon,\mathbb{R}^9_S)} \le C,\\ \int_{B_\varepsilon} |e(u^\varepsilon)| \le C,\end{gathered}$$

then $\chi_{\Omega_\varepsilon} e(u^\varepsilon)$ is bounded in $L^2(\Omega,\mathbb{R}^9_S)$, so for a subsequence of $\chi_{\Omega_\varepsilon} e(u^\varepsilon)$, still denoted by $\chi_{\Omega_\varepsilon} e(u^\varepsilon)$, we then show easily, like in the proof of the above proposition, that

$$\chi_{\Omega_\varepsilon} e(u^\varepsilon) \rightharpoonup e(u) \quad \text{in } L^2(\Omega,\mathbb{R}^9_S)) \tag{5.3}$$

From the subdifferentiability's inequality of $u \to \frac{1}{2}\int_{\Omega_\varepsilon} a_{ijkh}e_{kh}(u)e_{ij}(u)$, and passing to the lower limit, we obtain

$$\liminf_{\varepsilon\to 0}\frac{1}{2}\int_{\Omega_\varepsilon} a_{ijkh}e_{kh}(u^\varepsilon)e_{ij}(u^\varepsilon) \ge \frac{1}{2}\int_\Omega a_{ijkh}e_{kh}(u)e_{ij}(u).$$

For $\eta < \varepsilon/2$, let us set

$$B^\eta = \big\{x \in \Omega : |x_3| < \eta\big\}.$$

According to the diagonalization's lemma [6, Lemma 1.15 p. 32], there exists a function $\eta(\varepsilon) : \mathbb{R}^+ \to \mathbb{R}^+$ decreasing to 0 when $\varepsilon \to 0$ such that

$$\liminf_{\varepsilon\to 0}\int_{B^{\eta(\varepsilon)}} |e(u^\varepsilon)| \ge \liminf_{\eta\to 0}\liminf_{\varepsilon\to 0}\int_{B^\eta} |e(u^\varepsilon)|. \tag{5.4}$$

Since

$$\int_{B^\eta} |e(u^\varepsilon)| \ge \int_{B^\eta} \phi\big(e(u^\varepsilon) - e(u)\big) + \int_{B^\eta} \phi e(u), \quad \forall \phi \in \mathcal{C}_0^\infty(B^\eta,\mathbb{R}^9_S),$$

it follows that

$$\liminf_{\varepsilon\to 0}\int_{B^\eta} |e(u^\varepsilon)| \ge \int_{B^\eta} \phi e(u), \quad \forall \phi \in \mathcal{C}_0^\infty(B^\eta,\mathbb{R}^9_S).$$

Therefore,
$$\liminf_{\varepsilon\to 0}\int_{B^\eta}|e(u^\varepsilon)|\geq\int_{B^\eta}|e(u)|.$$
According to a classic result [16, Lemma 2.2 p. 145]), we then have
$$\liminf_{\varepsilon\to 0}\int_{B^\eta}|e(u^\varepsilon)|\geq\int_{B^\eta}\phi e(u)+\int_\Sigma\phi[u]\otimes_S e_3dx',\quad\forall\phi\in\mathcal{C}_0^\infty(\Omega,\mathbb{R}_S^9).$$
By passing to the limit, $(\eta\to 0)$, we have
$$\liminf_{\eta\to 0}\liminf_{\varepsilon\to 0}\int_{B^\eta}|e(u^\varepsilon)|\geq\int_\Sigma|[u]\otimes_S e_3|dx'.$$
According to the definition of B^η and (5.4), we deduce that
$$\liminf_{\varepsilon\to 0}\int_{B_\varepsilon}|e(u^\varepsilon)|\geq\int_\Sigma|[u]\otimes_S e_3|dx'.$$
Hence
$$\liminf_{\varepsilon\to 0}F^\varepsilon(u^\varepsilon)\geq\frac{1}{2}\int_\Omega a_{ijkh}e_{kh}(u)e_{ij}(u)+\int_\Sigma|[u]\otimes_S e_3|dx'.$$
For $u\in BD_0(\Omega)\setminus\mathbb{H}_0^1$ and $u^\varepsilon\in V^\varepsilon$, such that $u^\varepsilon\rightharpoonup u$ in $BD_0(\Omega)$. Assume that
$$\liminf_{\varepsilon\to 0}F^\varepsilon(u^\varepsilon)<+\infty.$$
So there exists a constant $C>0$ and a subsequence of $F^\varepsilon(u^\varepsilon)$, still denoted by $F^\varepsilon(u^\varepsilon)$, such that
$$F^\varepsilon(u^\varepsilon)<C. \tag{5.5}$$
So u^ε verifies the following evaluations (4.2) and (4.3), as $u^\varepsilon\rightharpoonup u$ in $BD_0(\Omega)$, thanks to the remark 4.4, we have $u\in\mathbb{H}_0^1$, what contradicts the fact that $u\in BD_0(\Omega)\setminus\mathbb{H}_0^1$, consequently we have
$$\liminf_{\varepsilon\to 0}F^\varepsilon(u^\varepsilon)=+\infty.$$
Hence the proof is complete. □

In the sequel, we determine the limit problem linked to (4.1), when ε approaches to zero. Thanks to the epi-convergence results, (see Annex, theorem 6.3, proposition 6.2) and the theorem 5.1, according to the τ_f-continuity of the functional G in $BD_0(\Omega)$, we have $F^\varepsilon+G$ τ_f-epiconverges to $F+G$ in $BD_0(\Omega)$.

Proposition 5.2. *For any $f\in L^2(\Omega,\mathbb{R}^3)$, there exists $u^*\in BD_0(\Omega)$ satisfying: $u^\varepsilon\rightharpoonup u^*$ in $BD_0(\Omega)$ and*
$$F(u^*)+G(u^*)=\inf_{v\in\mathbb{H}_0^1}\{F(v)+G(v)\}.$$

Proof. Thanks to lemma 4.1, the family $(u^\varepsilon)_\varepsilon$ is bounded in $BD_0(\Omega)$, therefore it possess a τ_f-cluster point u^* in $BD_0(\Omega)$. And thanks to a classical epi-convergence result, theorem 6.3, it follows that u^* is a solution of the problem: Find
$$\inf_{v\in BD_0(\Omega)}\big\{F(v)+G(v)\big\}. \tag{5.6}$$
Since $F=+\infty$ on $BD_0(\Omega)\setminus\mathbb{H}_0^1$, so (5.6) becomes
$$\inf_{v\in\mathbb{H}_0^1}\big\{F(v)+G(v)\big\}.$$
According to the uniqueness of solutions of problem (5.6), so u^ε admits an unique τ_f-cluster point u^*, and therefore $u^\varepsilon\rightharpoonup u^*$ in $BD_0(\Omega)$. □

Conclusion. Using the epiconvergence method, we showed that the question of finding the limit problem, composed of a classical linear elasticity problem posed over $\Omega \setminus \Sigma$, contains an interface condition which depends on the displacement field jump. We found the same result of Ait Moussa, with $p = 1$, in [4].

6. Annex

Definition 6.1 ([6, Definition 1.9]). Let $(\mathbb{X}, \tau)$ be a metric space and $(F^\varepsilon)_\varepsilon$ and F be functionals defined on $\mathbb{X}$ and with value in $\mathbb{R} \cup \{+\infty\}$. F^ε epi-converges to F in $(\mathbb{X}, \tau)$, noted $\tau - lim_e F^\varepsilon = F$, if the following assertions are satisfied

- For all $x \in \mathbb{X}$, there exists x^0_ε, $x^0_\varepsilon \xrightarrow{\tau} x$ such that $\limsup_{\varepsilon \to 0} F^\varepsilon(x^0_\varepsilon) \le F(x)$.
- For all $x \in \mathbb{X}$ and all x_ε with $x_\varepsilon \xrightarrow{\tau} x$, $\liminf_{\varepsilon \to 0} F^\varepsilon(x_\varepsilon) \ge F(x)$.

We have the following stability result for epi-convergence.

Proposition 6.2 ([6, p. 40]). *Suppose that F^ε epi-converges to F in $(\mathbb{X}, \tau)$ and that $G\colon \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$, is $\tau -$ continuous. Then $F^\varepsilon + G$ epi-converges to $F + G$ in $(\mathbb{X}, \tau)$*

This epi-convergence is a special case of the $\Gamma-$convergence introduced by De Giorgi (1979) [11]. It is well suited to the asymptotic analysis of sequences of minimization problems since one has the following fundamental result.

Theorem 6.3 ([6, theorem 1.10]). *Suppose that*

(1) *F^ε admits a minimizer on $\mathbb{X}$,*
(2) *The sequence $(\overline{u}^\varepsilon)$ is τ-relatively compact,*
(3) *The sequence F^ε epi-converges to F in this topology τ.*

Then every cluster point $\overline{u}$ of the sequence $(\overline{u}^\varepsilon)$ minimizes F on $\mathbb{X}$ and

$$\lim_{\varepsilon' \to 0} F^{\varepsilon'}(\overline{u}^{\varepsilon'}) = F(\overline{u}),$$

where $(\overline{u}^{\varepsilon'})_{\varepsilon'}$ denotes any subsequence of $(\overline{u}^\varepsilon)_\varepsilon$ which converges to $\overline{u}$.

References

[1] E. Acerbi, G. Buttazzo and D. Percivale; Thin inclusions in linear elasticity : a variational approach. *J. rein angew. Math.*, **386** (1988), 99-115.
[2] A. Ait Moussa et C. Licht; Comportement asymptotique d'une plaque mince non linéaire. *J. Math du Maroc*, No. 2 (1994), 1-16.
[3] A. Ait Moussa; Comportement asymptotique d'une bande elastique mince. *Les Annales Magrébines de l'Ingénieur* Vol. 10, No. 2 (1996), 7-23.
[4] A. Ait Moussa; Comportement Asymptotique des Solutions d'un Problème de Bandes Minces, Thèse de Doctorat (1995), Oujda, Maroc.
[5] A. Ait Moussa and J.Messaho; Limit behavior of an oscillating thin layer. *Electronic Journal of Differential Equations*, Conference 14 (2006), 21-33.
[6] H. Attouch; Variational convergence for functions and operators. *Pitman Advance Publishing Program*, 1984.
[7] H. Brezis; Analyse fonctionnelle, Théorie et Applications. Masson(1992).
[8] A. Brillard; Asymptotic analysis of nonlinear thin layers. *International Series of Numerical Mathematics*, Vol. 123 (1997).
[9] B. Dacorognat; Direct Methods in Calculus of Variations. *Applied Mathematical sciences*, No. 78, Springer-Verlag, Berlin, 1989.
[10] I. Ekeland et R. Temam; Analyse convexe et problèmes variationnels. *Dunod Gauthier-Villars Paris*, 1974.

[11] E. De Giorgi; Convergence problems for functions and operators. *Proceedings of the International Congress on "Recent Methods in Nonlinear analysis"* 1978. De Giorgi, Mosco, Spagnolo Eds. Pitagora Editrice (Bologna) 1979.
[12] J. F. Ganghoffer, A. Brillard and J. Schultz; Modelling of mechanical behaviour of joints bonded by a nonlinear incompressible elastic adhesive. *European Journal of Mechanics. A/Solids* Vol. 16 (1997), No. 2, 255-276.
[13] G. Geymonat, F. Krasucki and S. Lenci; Annalyse asymptotique du comportement d'un assemblage collé. *C. R. Acad. Sci. Paris*, t. 322 (1996), série 1, 1107-1112.
[14] S. Lenci, F. Krasucki and G. Geymonat; Mathematical analysis of a bonded joint with a soft adhesive. *Mathematics and Mechanics of Solid*, **4** (1999), 201-225.
[15] H. Phum Huy and E. Sanchez-Palencia; Phénoménes de transmission à travers des couches minces de conductivité élevée. *Journal of Mathematical Analysis and Applications*, **47** (1974), 284-309.
[16] R. Temam; Problémes mathématiques en plasticité. *Ed. Gauthier Villars*, Paris, 1983.

Abdelaziz Ait Moussa
Département de mathématiques et informatique, Faculté des sciences, Université Mohammed 1er, Oujda, Morocco
E-mail address: a_aitmoussa@yahoo.fr

Jamal Messaho
Département de mathématiques et informatique, Faculté des sciences, Université Mohammed 1er, Oujda, Morocco
E-mail address: j_messaho@yahoo.fr