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UNIQUENESS OF A SYMMETRIC POSITIVE SOLUTION TO AN ODE SYSTEM

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In memory of Jack K. Hale (1928–2009)

Abstract. In this article, we prove uniqueness of symmetric positive solutions of the variational ODE system

$$-w'' + aw - wv = 0$$
$$-v'' + bv - \frac{w^2}{2} = 0,$$

where a and b are positive constants.

1. Introduction and Statement of the Result

In this article, we prove uniqueness of symmetric positive solutions of the variational ODE system

$$\begin{aligned} -w'' + aw - wv &= 0 \\ -v'' + bv - \frac{w^2}{2} &= 0 \end{aligned} \tag{1.1}$$

where a and b are positive constants. The solutions under consideration are defined for all $x \in \mathbb{R}$ and have finite energy.

To show how (1.1) arises, we consider the so-called χ^2 SHG equations

$$\begin{aligned} i\frac{\partial w}{\partial t} + r\frac{\partial^2 w}{\partial x^2} - \theta w + w^* v &= 0 \\ i\sigma\frac{\partial v}{\partial t} + s\frac{\partial^2 v}{\partial x^2} - \alpha v + \frac{w^2}{2} &= 0 \end{aligned} \tag{1.2}$$

where r, s, σ, θ are positive real parameters and $w(x)$ and $v(x)$ are complex functions. This system governs phenomena in nonlinear optics (see [5] for instance).

A solitary wave is a solution of (1.2) of the form

$$(w(x)e^{i\gamma t}, v(x)e^{2i\gamma t}).$$

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Hence, (w,v) satisfies

$$\begin{aligned} -rw'' + (\theta+\gamma)w - w^* v &= 0 \\ -sv'' + (\alpha + 2\sigma\gamma)v - \frac{w^2}{2} &= 0. \end{aligned} \tag{1.3}$$

The solutions of (1.3) are critical points of $E+\gamma I$ where E and I are the following conserved quantities for (1.2)

$$E(w,v) = \int_{-\infty}^{+\infty} (r|w'|^2 + s|v'|^2 + \theta|w|^2 + \alpha|v|^2 - \mathrm{Re}(w^2 v^*))\,dx, \tag{1.4}$$

$$I(w,v) = \int_{-\infty}^{+\infty} (|w|^2 + 2\sigma|v|^2)\,dx. \tag{1.5}$$

If w and v are real solutions of (1.3) then it solves

$$\begin{aligned} -rw'' + (\theta+\gamma)w - wv &= 0 \\ -sv'' + (\alpha + 2\sigma\gamma)v - \frac{w^2}{2} &= 0. \end{aligned} \tag{1.6}$$

Replacing (w,v) by $(k_1 w, k_2 v)$ in (1.6), with $k_2 = r$ and $k_1^2 = rs$, we get

$$\begin{gathered} -w'' + \frac{(\theta+\gamma)}{r} w - wv = 0 \\ -v'' + \frac{(\alpha+2\sigma\gamma)}{s} v - \frac{w^2}{2} = 0. \end{gathered}$$

Therefore, we consider the real variational ODE system

$$-w'' + aw - wv = 0 \tag{1.7}$$

$$-v'' + bv - \frac{w^2}{2} = 0 \tag{1.8}$$

and we will be interested in solutions that have finite energy (or equivalently, tend to zero as $|x|$ tends to infinity). The existence of positive solutions of (1.7)-(1.8) has been proved in [6]. Briefly the argument goes as follows. We define $H = H^1(\mathbb{R}) \times H^1(\mathbb{R})$ equipped with the norm

$$\int_{-\infty}^{+\infty} (w'^2(x) + v'^2(x) + aw^2(x) + bv^2(x))\,dx.$$

We consider the functionals

$$E(w,v) = \int_{-\infty}^{+\infty} (w'^2(x) + v'^2(x) - w^2(x)v(x))\,dx,$$

$$I(w,v) = \int_{-\infty}^{+\infty} (aw^2(x) + bv^2(x))\,dx.$$

Using the method of concentration-compactness ([3]), we minimize $E(w,v)$ under $I(w,v) = 1$ in the space H. If we replace $(w(x), v(x))$ by $(|w(x)|, |v(x)|)$ then E does not increase. Therefore, any minimizer is nonnegative and solves the Euler-Lagrange system

$$-w'' + \mu aw - wv = 0 \tag{1.9}$$

$$-v'' + \mu bv - \frac{w^2}{2} = 0 \tag{1.10}$$

with $\mu \geq 0$ (because (w, v) is a minimizer). On the other hand, it is easy to see that any solution $(w, v) \in H$ of (1.9)-(1.10) with $\mu = 0$ is the solution identically zero. Therefore, we must have $\mu > 0$. Defining a new pair $(k_1 w(k_3 x), k_2 v(k_3 x))$ with $k_3^2 = 1/\mu, k_1 = k_2 = 1/\mu$, we see that this new pair satisfies (1.7)-(1.8).

In [4] the symmetry of any positive solution of (1.7)-(1.8) has been proved using a result of [1]. However, as pointed out in [1], their proof works for $N \geq 2$. Since we are in dimension one, we need the following modified version given in [2].

Theorem 1.1. *Consider the system*

$$\begin{aligned} w'' + f(w, v) = 0 \\ v'' + g(w, v) = 0 \end{aligned} \tag{1.11}$$

where $f(w, v)$ and $g(w, v)$ are C^1 functions satisfying the conditions:

$$f(0,0) = 0 = g(0,0), \quad \frac{\partial f(w,v)}{\partial v}, \frac{\partial g(w,v)}{\partial w} \geq 0.$$

Suppose that there exist $\epsilon > 0$ and $\delta > 0$ such that $w > 0$, $v > 0$, $w^2 + v^2 < \epsilon$ imply

$$\frac{\partial f(w,v)}{\partial w}, \frac{\partial g(w,v)}{\partial v} < -\delta, \quad 0 < \frac{\partial f(w,v)}{\partial v}, \frac{\partial g(w,v)}{\partial w} < \delta.$$

Then, except for translations, any positive solution of (1.11) *is even and decreasing.*

We conclude that, except for translations, any positive solution of (1.7)-(1.8) is symmetric and decreasing.

In [4] we have also proved the following result.

Theorem 1.2. *The linearized operator of* (1.7)-(1.8) *at any positive symmetric solution has zero as a simple eigenvalue with odd eigenfunctions (w_x, v_x) and it has exactly one negative eigenvalue.*

The fact that zero is a simple eigenvalue of the linearized operator is not a proof of uniqueness of symmetric positive solution, but it may suggest it. Our main result is that this is indeed the case.

Theorem 1.3. *For $a, b > 0$, the positive symmetric decreasing solution of* (1.7)-(1.8) *is unique.*

Several interesting numerical experiments concerning system (1.7)-(1.8) are presented in [6]. They indicate uniqueness of positive solution (which is confirmed by Theorem 1.3) and that (1.7)-(1.8) may have solutions that change sign.

2. Proof of main result

First we establish the following abstract uniqueness result.

Theorem 2.1. *Let X be a Banach space and $F : X \times [0, 1] \to X$ be a continuous functions with continuous Frechet derivative with respect to the first variable. Also assume that*

(i) *the set of the solutions (u, λ) of $F(u, \lambda) = 0$, $u \in X, \lambda \in [0, 1]$ is precompact;*
(ii) *for any solution of $F(u, \lambda) = 0$, the derivative $F_u(u, \lambda)$ is invertible;*
(iii) *the equation $F(u, 0) = 0$ has a unique solution.*

Then the equation $F(u, \lambda) = 0$ has a unique solution for $\lambda \in [0, 1]$.

Proof. First we claim that there is a $\lambda_0 > 0$ such that the solution of $F(u, \lambda) = 0$ is unique for $0 \le \lambda < \lambda_0$. In fact, otherwise, there is a sequence $0 < \lambda_n \to 0$ such that $F(u, \lambda_n) = 0$ has at least two distinct solution u_n and v_n. In view of assumption (i) and passing to a subsequence if necessary, we can assume that u_n converges to u and v_n converges to v. In view of (iii), we must have $u = v$. However, by (ii) and the implicit function theorem, in a neighborhood of u, for small λ, the solution of $F(u, \lambda) = 0$ is unique. This contradiction proves the claim. The same argument shows that the set A of λ, $0 \le \lambda \le 1$, for which the solution of $F(u, \mu) = 0$ is unique for $0 \le \mu \le \lambda$ is open. Since by ii) A is clearly closed, A has to be the whole interval $[0, 1]$ and the theorem is proved. □

Remark. If we take $u \in \mathbb{R}$ and $F(u, \lambda) = u(\lambda u - 1) = \lambda u^2 - u$, we have $F_u(u, \lambda) = 2\lambda - 1$. We see that, except for assumption i), all the others are satisfied but the conclusion of the theorem does not hold. This is so because there is the branch $u = 1/\lambda$ of solutions bifurcating from infinity.

Theorem 1.3 will be a consequence of Theorem 2.1. To verify all its assumptions, we start with the following result.

Lemma 2.2. *The system*

$$\begin{aligned} -w'' + aw - wv &= 0 \\ -v'' + av - \frac{w^2}{2} &= 0 \end{aligned} \tag{2.1}$$

($a = b$ in (1.7)-(1.8)) has a unique positive solution with finite energy.

Proof. Defining $z(x) = w(x) - \sqrt{2}v(x)$, multiplying the second equation by $\sqrt{2}$ and subtracting we get

$$-z'' + z + \frac{w}{\sqrt{2}} z = 0.$$

Multiplying this last equation by z and integrating we get

$$\int_{-\infty}^{+\infty} (z'^2(x) + z^2(x) + \frac{w}{\sqrt{2}} z(x)^2)\, dx = 0$$

and this implies $z \equiv 0$ (because w is a positive). Therefore, each component of the solution of (2.1) solves a single second order equation and this implies uniqueness and the lemma is proved. □

To verify the other assumptions of Theorem 2.1, we establish a chain of estimates. Since we wish to find estimates for solutions of (1.7)-(1.8) which remain uniform for a and b in a certain interval, we fix two constants $0 < c_1 < c_2$ and we assume

$$c_1 \le a, b \le c_2. \tag{2.2}$$

In the sequel, d_i, $1 \le i \le$ will indicate constants depending on c_1 and c_2 only. Let $(w(x), v(x))$ be as in Theorem 1.3. Since

$$T(w, v, w', v') \hat{=} - w'^2 - v'^2 + aw^2 + bv^2 - w^2 v \tag{2.3}$$

is a first integral for (1.7)-(1.8), we must have

$$-w'^2(x) - v'^2(x) + aw^2(x) + bv^2(x) - w^2(x)v(x) = 0 \tag{2.4}$$

for any x.

Bound for $v(0)$. Using the fact the $(w(x), v(x))$ is symmetric, if we set $x = 0$ in (2.4) we get

$$w^2(0) = \frac{bv^2(0)}{v(0) - a}\cdot \tag{2.5}$$

In particular $v(0) > a$. Moreover, $v''(0) \le 0$ (because $v(x)$ has a maximum at $x = 0$) and then the second equation (1.8) yields

$$bv(0) \le \frac{w^2(0)}{2}\,. \tag{2.6}$$

This together with (2.5) implies

$$bv(0) \le \frac{1}{2}\frac{bv^2(0)}{(v(0) - a)} \tag{2.7}$$

and finally $v(0) \le 2a$ because $v(0) > a$.

Bound for $v'(x)$. Multiplying the second equation (1.8) by $v'(x)$, then for $x \ge 0$ we get:

$$\frac{d}{dx}(-v'(x)^2 + bv^2(x)) = w^2(x)v'(x) \le 0.$$

Therefore $-v'(x)^2 + bv^2(x)$ is decreasing and, since it vanishes at $+\infty$, we get

$$-v'(x)^2 + bv^2(x) \ge 0$$

and then

$$v'(x)^2 \le bv^2(x) \le bv^2(0) \le 4a^2b. \tag{2.8}$$

Bound for $w'(x)$. We know $w'(x) \le 0$ and that $w'(x)$ reaches its minimum when $w''(x) = 0$. By the first equation (1.7), this occurs when $v(x) = a$ and then, from (2.4),

$$w'(x)^2 + v'(x)^2 = bv^2(x) \le bv^2(0) \le 4a^2b.$$

We conclude

$$|w'(x)| = -w'(x) \le 2a\sqrt{b}. \tag{2.9}$$

Bound for $w(0)$. Suppose $w(0) = M$ and $w(x_0) = M/2$ for some $x_0 > 0$. Since

$$w(0) - w(x_0) = -\int_0^{x_0} w'(s)\,ds,$$

then, in view of (2.9), we have $\frac{M}{2} \le 2a\sqrt{b}x_0$ and this implies

$$x_0 \ge \frac{M}{4a\sqrt{b}}. \tag{2.10}$$

Moreover, the solution of the linear equation

$$-v''(x) + bv(x) = h(x) \tag{2.11}$$

is given by

$$v(x) = \frac{1}{2\sqrt{b}}\int_{-\infty}^{+\infty} e^{-\sqrt{b}|x-y|}h(y)\,dy, \tag{2.12}$$

and then, the second equation (1.8) and (2.10) give

$$\begin{aligned} v(0) &= \frac{1}{4\sqrt{b}} \int_{-\infty}^{+\infty} e^{-\sqrt{b}|y|} w^2(y)\,dy \\ &= \frac{1}{2\sqrt{b}} \int_{0}^{+\infty} e^{-\sqrt{b}y} w^2(y)\,dy \\ &\ge \frac{1}{2\sqrt{b}} \int_{0}^{x_0} e^{-\sqrt{b}y} w^2(y)\,dy \\ &\ge \frac{M^2}{8\sqrt{b}} \int_{0}^{x_0} e^{-\sqrt{b}y}\,dy \\ &= \frac{M^2}{8b}(1-e^{-\sqrt{b}x_0}) \\ &\ge \frac{M^2}{8b}(1-e^{-\frac{M}{4a}}). \end{aligned}$$

Therefore,

$$2a \ge v(0) \ge \frac{M^2}{8b}(1-e^{-\frac{M}{4a}})$$

and this gives that $M = w(0) \le d_1$, for some constant d_1. In view of (2.5), this gives also that $v(0) \ge d_2 > a$, for some constant d_2, and also gives a lower bound for $w(0) \ge d_3$.

Bound for the length of the interval for which $v(x) \ge a$. By the first equation in (1.7) and the previous estimates for $v(0)$ and $w(0)$, we have $w''(0) \le -d_4 < 0$ and $|w'''(x)| \le d_5$. Defining $X = -\frac{w''(0)}{2d_5}$ then, for $0 \le x \le X$ we have

$$w''(x) - w''(0) = \int_0^x w'''(s)\,ds \le d_5 X = -w''(0)/2,$$

and then $w''(x) \le w''(0)/2 \le -d_4/2$ for $0 \le x \le X$. Moreover,

$$w'(X) = w'(0) + \int_0^X w''(s)\,ds \le \int_0^X \frac{w''(0)}{2}\,ds = X\frac{w''(0)}{2} = -\frac{w''(0)^2}{4d_5} \le -d_6.$$

Since, by (1.7), $w''(x) \le 0$ whenever $v(x) \ge a$, we have $w'(x) \le -d_6$ whenever $v(x) \ge a$ and $x \ge X$. Furthermore,

$$-w(0) \le -w(X) \le w(x) - w(X) = \int_X^x w'(s)\,ds \le -d_6(x-X).$$

Therefore, defining $X_1 = w(0)/d_6 + X$, we see that we must have $v(X_1) \le a$.

Estimate for the time $v(x)$ stays close (and less) than a. Let $x_0 \le X_1$ be such that $v(x_0) = a$ and let $d_7 > 0$ and $d_8 < a$ be such that

$$(a-v)w^2 + bv^2 \ge d_7^2$$

whenever $d_8 \le v \le a, w \le d_1$. Then, if $d_8 \le v(x) \le a$ for $x_0 \le x \le x_0 + X_2$, by (2.4) we have $-w'(x) - v'(x) \ge d_7$ and then

$$w(x_0) + v(x_0) \ge -w(x) + w(x_0) - v(x) + v(x_0) \ge d_7 X_2$$

and this gives a uniform upper bound for X_2.

Exponential decay for $w(x)$ and for $v(x)$. Since

$$\frac{d}{dx}(-w'(x)^2 + aw^2(x) - w^2(x)v(x)) = -w^2(x)v'(x) \geq 0$$

the function $-w'(x)^2 + (a - v(x))w^2(x)$ is increasing and then $-w'(x)^2 + (a - v(x))w^2(x) \leq 0$ for all $x \geq 0$ because it vanishes at infinity. Now, for $x \geq X_3 \hat{=} X_1 + X_2$ we have $-w'(x)^2 + d_8 w(x)^2 \leq 0$ and then $w'(x) + d_9 w(x) \leq 0$ and then $\frac{d}{dx} e^{d_9 x} w(x) \leq 0$ and finally, $w(x) \leq e^{-d_9(x - X_3)} w(X_3)$ for $x \geq X_3$ and this implies

$$w(x) \leq d_{10} e^{-d_9 x}, \quad x \geq 0. \tag{2.13}$$

From the second equation (1.8) we get

$$v(x) = \frac{1}{4\sqrt{b}} \int_{-\infty}^{+\infty} e^{-\sqrt{b}|x-y|} w^2(y)\, dy$$

and this together with (2.13) and elementary calculation gives a similar exponential decay

$$v(x) \leq d_{11} e^{-d_{12} x}, \quad x \geq 0 \tag{2.14}$$

for $v(x)$.

Proof of Theorem 1.3. Using (2.12) to invert the linear operators $-w'' + aw$ and $-v'' + bv$, we see that system (1.7)-(1.8) can be written as

$$\begin{aligned} w(x) &= \frac{1}{2\sqrt{a}} \int_{-\infty}^{+\infty} e^{-\sqrt{a}|x-y|} w(y)v(y)\, dy \\ v(x) &= \frac{1}{4\sqrt{b}} \int_{-\infty}^{+\infty} e^{-\sqrt{b}|x-y|} w^2(y)\, dy. \end{aligned} \tag{2.15}$$

Defining u as the pair (w, v), system (2.15) can be viewed as the equation

$$F(u, \lambda) = 0 \tag{2.16}$$

where λ, say, is b, with a kept fixed. We denote by $H^1_{ev} \subset H^1(\mathbb{R})$ the subspace of the even functions. If we take $X = H^1_{ev} \times H^1_{ev}$, then $F : X \to X$ is a well defined very smooth function. In view of Theorem 1.2, assumption ii) of Theorem 2.1 is satisfied because X consists of even functions. Uniqueness for $\lambda = a$ is given by Lemma 2.2. To verify assumption (i) of Theorem 2.1, we recall that a subset K of X is precompact if and only if the following conditions are satisfied:

(1) for each n the restriction of the functions of K to the interval $[-n, n]$ is precompact;
(2) for every $\epsilon > 0$, there is an $x(\epsilon) > 0$ such that for all $u \in K$ we have

$$\int_{|x| \geq x(\epsilon)} (|u'|^2(x) + |u(x)|^2)\, dx < \epsilon.$$

To verify these conditions we first notice that we have obtained uniform bound for the $H^1(\mathbb{R})$ norm of the solution (w, v) of (1.7)-(1.8). This implies uniform bound for the H^2 norm of such solutions and this verifies condition (1) for precompactness. The uniform exponential decay (2.13) and (2.14) for $w(x)$ and $v(x)$ together with (2.3) gives the uniform exponential decay also for the derivatives. This implies that condition (2) for precompactness is satisfies; therefore, Theorem 1.3 is proved. □

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