Electronic Journal of Differential Equations, Vol. 2009(2009), No. 152, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF ALMOST PERIODIC SOLUTIONS FOR HOPFIELD NEURAL NETWORKS WITH CONTINUOUSLY DISTRIBUTED DELAYS AND IMPULSES

YONGKUN LI, TIANWEI ZHANG

Abstract. By means of a Cauchy matrix, we prove the existence of almost periodic solutions for Hopfield neural networks with continuously distributed delays and impulses. An example is employed to illustrate our results.

1. Introduction

Let $\mathbb{R}$ be the set of real numbers $\mathbb{R}^+ = [0, \infty)$, $\Omega \subset \mathbb{R}$, $\Omega \neq \emptyset$. The set of sequences that are unbounded and strictly increasing is denoted by $\mathbb{B} = \{\{\tau_k\} \in \mathbb{R} : \tau_k < \tau_{k+1}, k \in \mathbb{Z}, \lim_{k\to\pm\infty} \tau_k = \pm\infty\}$.

Recently, Stamov [1] investigated the generalized impulsive Lasota-Wazewska model

$$\begin{gathered} x'(t) = -a(t)x(t) + \sum_{i=1}^{n} \beta_i(t) e^{-\gamma_i(t)x(t-\xi)}, \quad t \neq \tau_k, \\ \Delta x(\tau_k) = \alpha_k x(\tau_k) + \nu_k, \end{gathered} \tag{1.1}$$

where $t \in \mathbb{R}$, $\alpha(t), \beta_i(t), \gamma_i(t) \in C(\mathbb{R}, \mathbb{R}^+)$, $i = 1, 2, \dots, n$, ξ is a positive constant, $\{\tau_k\} \in \mathbb{B}$, with $\alpha_k, \nu_k \in \mathbb{R}$ for $k \in \mathbb{Z}$. By means of the Cauchy matrix he obtained sufficient conditions for the existence and exponential stability of almost periodic solutions for (1.1). In this paper, we consider a more general model; that is, the following impulsive Hopfield neural networks with continuously distributed delays

$$\begin{aligned} x_i'(t) &= -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t-\xi)) \\ &\quad + \sum_{j=1}^{n} b_{ij}(t) \int_0^\infty K_{ij}(u) g_j(x_j(t-u))\,\mathrm{d}u + I_i(t), \quad t \neq \tau_k, \\ \Delta x_i(\tau_k) &= \alpha_{ik} x_i(\tau_k) + \nu_{ik}, \end{aligned} \tag{1.2}$$

where $i = 1, 2, \dots, n$, $k \in \mathbb{Z}$, $x_i(t)$ denotes the potential (or voltage) of cell i at time t; $c_i(t) > 0$ represents the rate with which the ith unit will reset its potential to the

---

2000 *Mathematics Subject Classification.* 34K14; 34K45; 92B20.
Key words and phrases. Almost periodic solution; Hopfield neural networks; impulses; Cauchy matrix.

Submitted October 28, 2009. Published November 25, 2009.
Supported by grant 10971183 from the National Natural Sciences Foundation of China.

resting state in isolation when disconnected from the network and external inputs at time t; $a_{ij}(t)$ and $b_{ij}(t)$ are the connection weights between cell i and j at time t; ξ is a constant and denotes the time delay; $K_{ij}(t)$ corresponds to the transmission delay kernels; f_j and g_j are the activation functions; $I_i(t)$ is an external input on the ith unit at time t. Furthermore, $\{\tau_k\} \in \mathbb{B}$, with the constants $\alpha_{ik} \in \mathbb{R}$, $\gamma_{ik} \in \mathbb{R}$, $k \in \mathbb{Z}$, $i = 1, 2, \dots, n$.

Remark 1.1. If $i = 1$, $f_j(x_j(t-\xi)) = e^{-\gamma_j(t)x(t-\xi)}$, $b_{ij}(t) = I_i(t) = 0$, $j = 1, 2, \dots, n$, then (1.2) reduces to (1.1).

Our main am of this paper is to investigate the existence of almost periodic solutions of system (1.2). Let $t_0 \in \mathbb{R}$. Introduce the following notation:

$PC(t_0)$ is the space of all functions $\phi : [-\infty, t_0] \to \Omega$ having points of discontinuity at $\theta_1, \theta_2, \dots \in (-\infty, t_0)$ of the first kind and left continuous at these points.

For $J \subset \mathbb{R}$, $PC(J, \mathbb{R})$ is the space of all piecewise continuous functions from J to $\mathbb{R}$ with points of discontinuity of the first kind τ_k, at which it is left continuous.

The initial conditions associated with system (1.2) are of the form

$$x_i(s) = \phi_i(s), \ s \in (-\infty, t_0],$$

where $\phi_i \in PC(t_0)$, $i = 1, 2, \dots, n$.

The remainder of this article is organized as follows: In Section 2, we will introduce some necessary notations, definitions and lemmas which will be used in the paper. In Section 3, some sufficient conditions are derived ensuring the existence of the almost periodic solution. At last, an illustrative example is given.

2. Preliminaries

In this section, we introduce necessary notations, definitions and lemmas which will be used later.

Definition 2.1 ([2])**.** The set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k, j \in \mathbb{Z}$, $\{\tau_k\} \in \mathbb{B}$ is said to be uniformly almost periodic if for arbitrary $\epsilon > 0$ there exists a relatively dense set of ϵ-almost periods common for any sequences.

Definition 2.2 ([2])**.** A function $x(t) \in PC(\mathbb{R}, \mathbb{R})$ is said to be almost periodic, if the following hold:

(a) The set of sequences $\{\tau_k^j\}, \tau_k^j = \tau_{k+j} - \tau_k, k, j \in \mathbb{Z}$, $\{\tau_k\} \in \mathbb{B}$ is uniformly almost periodic.

(b) For any $\epsilon > 0$ there exists a real number $\delta > 0$ such that if the points t' and t'' belong to one and the same interval of continuity of $x(t)$ and satisfy the inequality $|t' - t''| < \delta$, then $|x(t') - x(t'')| < \epsilon$.

(c) For any $\epsilon > 0$ there exists a relatively dense set T such that if $\tau \in T$, then $|x(t+\tau) - x(t)| < \epsilon$ for all $t \in \mathbb{R}$ satisfying the condition $|t - \tau_k| > \epsilon$, $k \in \mathbb{Z}$.

The elements of T are called ϵ-almost periods.

Together with the system (1.2) we consider the linear system

$$\begin{gathered} x_i'(t) = -c_i(t)x_i(t), \quad t \neq \tau_k, \\ \Delta x_i(\tau_k) = \alpha_{ik}x_i(\tau_k), \quad k \in \mathbb{Z}, \end{gathered} \tag{2.1}$$

where $t \in \mathbb{R}$, $i = 1, 2, \dots, n$. Now let us consider the equations

$$x_i'(t) = -c_i(t)x_i(t), \quad \tau_{k-1} < t \le \tau_k, \quad \{\tau_k\} \in \mathbb{B}$$

and their solutions
$$x_i(t) = x_i(s)\exp\Big\{-\int_s^t c_i(\sigma)\,\mathrm{d}\sigma\Big\}$$
for $\tau_{k-1} < s < t \le \tau_k$, $i = 1, 2, \dots, n$.

As in [3], the Cauchy matrix of the linear system (2.1) is
$$\begin{aligned} &W_i(t,s)\\ &= \begin{cases} \exp\big\{-\int_s^t c_i(\sigma)\,\mathrm{d}\sigma\big\}, & \tau_{k-1} < s < t < \tau_k;\\ \prod_{j=m}^{k+1}(1+\alpha_{ij})\exp\big\{-\int_s^t c_i(\sigma)\,\mathrm{d}\sigma\big\}, & \tau_{m-1} < s \le \tau_m < \tau_k < t \le \tau_{k+1}. \end{cases}\end{aligned}$$
The solutions of system (2.1) are of the form
$$x_i(t;t_0;x_i(t_0)) = W_i(t,t_0)x_i(t_0), \quad t_0 \in \mathbb{R},\ i = 1, 2, \dots, n.$$
For convenience, we introduce the notation
$$\overline{f} = \sup_{t\in\mathbb{R}}|f(t)|, \quad \underline{f} = \inf_{t\in\mathbb{R}}|f(t)|.$$
In this article, we use the following hypotheses:

(H1) $c_i(t) \in C(\mathbb{R}, \mathbb{R}^+)$ is almost periodic and there exists a positive constant c such that $c < c_i(t)$, $t \in \mathbb{R}$, $i = 1, 2, \dots, n$.

(H2) The set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k \in \mathbb{Z}$, $j \in \mathbb{Z}$, $\{\tau_k\} \in \mathbb{B}$ is uniformly almost periodic and there exists $\theta > 0$ such that $\inf_{k\in\mathbb{Z}} \tau_k^1 = \theta > 0$.

(H3) The sequence $\{\alpha_{ik}\}$ is almost periodic and $1 - e^2 \le \alpha_{ik} \le e^2 - 1$, $k \in \mathbb{Z}$, $i = 1, 2, \dots, n$.

(H4) The sequence $\{\nu_{ik}\}$ is almost periodic and $\gamma = \sup_{k\in\mathbb{Z}} |\nu_{ik}|$, $k \in \mathbb{Z}$, $i = 1, 2, \dots, n$.

(H5) The functions $a_{ij}(t)$, $b_{ij}(t)$ and $I_i(t)$ are almost periodic in the sense of Bohr and $|I_i(t)| < \infty$, $t \in \mathbb{R}$, $i, j = 1, 2, \dots, n$.

(H6) The functions $f_j(t)$ and $g_j(t)$ are almost periodic in the sense of Bohr and $f_j(0) = g_j(0) = 0$, $j = 1, 2, \dots, n$. There exist positive bounded functions $L_f(t)$ and $L_g(t)$ such that for u, $v \in \mathbb{R}$
$$\max_{1\le j\le n}|f_j(u) - f_j(v)| \le L_f(t)|u - v|, \quad \max_{1\le j\le n}|g_j(u) - g_j(v)| \le L_g(t)|u - v|.$$

(H7) The delay kernels $K_{ij} \in C(\mathbb{R}, \mathbb{R})$ and there exists a positive constant K such that
$$\int_0^{+\infty}|K_{ij}(s)|\,\mathrm{d}s \le K, \quad i, j = 1, 2, \dots, n.$$

Lemma 2.3 ([2]). *Assume* (H1)-(H6). *Then for each $\epsilon > 0$, there exist ϵ_1, $0 < \epsilon_1 < \epsilon$, relatively dense sets T of real numbers and Q of whole numbers, such that the following relations are fulfilled:*

(a) $|c_i(t+\tau) - c_i(t)| < \epsilon$, $t \in \mathbb{R}$, $\tau \in T$, $i = 1, 2, \dots, n$;
(b) $|a_{ij}(t+\tau) - a_{ij}(t)| < \epsilon$, $t \in \mathbb{R}$, $\tau \in T$, $|t - \tau_k| > \epsilon$, $k \in \mathbb{Z}$, $i, j = 1, 2, \dots, n$;
(c) $|b_{ij}(t+\tau) - b_{ij}(t)| < \epsilon$, $t \in \mathbb{R}$, $\tau \in T$, $|t - \tau_k| > \epsilon$, $k \in \mathbb{Z}$, $i, j = 1, 2, \dots, n$;
(d) $|I_i(t+\tau) - I_i(t)| < \epsilon$, $t \in \mathbb{R}$, $\tau \in T$, $|t - \tau_k| > \epsilon$, $k \in \mathbb{Z}$, $i = 1, 2, \dots, n$;
(e) $|f_j(t+\tau) - f_j(t)| < \epsilon$, $t \in \mathbb{R}$, $\tau \in T$, $|t - \tau_k| > \epsilon$, $k \in \mathbb{Z}$, $j = 1, 2, \dots, n$;
(f) $|g_j(t+\tau) - g_j(t)| < \epsilon$, $t \in \mathbb{R}$, $\tau \in T$, $|t - \tau_k| > \epsilon$, $k \in \mathbb{Z}$, $j = 1, 2, \dots, n$;
(g) $|\alpha_{i(k+q)} - \alpha_{ik}| < \epsilon$, $q \in Q$, $k \in \mathbb{Z}$, $i = 1, 2, \dots, n$;

(h) $|\nu_{i(k+q)} - \nu_{ik}| < \epsilon$, $q \in Q$, $k \in \mathbb{Z}$, $i = 1, 2, \dots, n$;
(i) $|\tau_k^q - \tau| < \epsilon_1$, $q \in Q$, $\tau \in T$, $k \in \mathbb{Z}$, $i = 1, 2, \dots, n$.

Lemma 2.4 ([2]). *Let $\{\tau_k\} \in \mathbb{B}$ and the condition* (H2) *hold. Then for $1 > 0$ there exists a positive integer A such that on each interval of length 1, we have no more than A elements of the sequence $\{\tau_k\}$, i.e.,*

$$i(s,t) \le A(t-s) + A,$$

where $i(s,t)$ is the number of the points τ_k in the interval (s,t).

Lemma 2.5. *Assume* (H1)-(H3). *Then for the Cauchy matrix $W_i(t,s)$ of system* (2.1), *we have*

$$|W_i(t,s)| \le e^{2A} e^{-\alpha(t-s)}, \quad t \ge s,\ t, s \in \mathbb{R},\ i = 1, 2, \dots, n,$$

where $\alpha = c - 2A$, A is determined in Lemma 2.4.

Proof. Since the sequence $\{\alpha_{ik}\}$ is almost periodic, then it is bounded and from (H3) it follows that $|1 + \alpha_{ik}| \le e^2$, $k \in \mathbb{Z}$, $i = 1, 2, \dots, n$. From the expression of $W_i(t,s)$ and the above inequality it follows that

$$\begin{aligned} |W_i(t,s)| &= |1 + \alpha_{ik}|^{i(s,t)} e^{-\int_s^t c_i(\theta)\,\mathrm{d}\theta} \\ &\le |1 + \alpha_{ik}|^{A(t-s)+A} e^{-c(t-s)} \\ &\le e^{2A} e^{-(c-2A)(t-s)} \\ &= e^{2A} e^{-\alpha(t-s)}, \end{aligned}$$

where $t \ge s$, $t, s \in \mathbb{R}$, $i = 1, 2, \dots, n$. The proof is complete. □

From [3, Lemma 3], we obtain the following lemma.

Lemma 2.6. *Assume* (H1)-(H3) *and the condition*

(H8) $\alpha = c - 2A > 0$.

Then for any $\epsilon > 0$, $t \ge s$, $t, s \in \mathbb{R}$, $|t - \tau_k| > \epsilon$, $|s - \tau_k| > \epsilon$, $k \in \mathbb{Z}$ there exists a relatively dense set T of the function $c_i(t)$ and a positive constant Γ such that for $\tau \in T$ it follows that

$$|W_i(t+\tau, s+\tau) - W_i(t,s)| \le \epsilon \Gamma e^{-\frac{\alpha}{2}(t-s)}, \quad t \ge s,\ t, s \in \mathbb{R},\ i = 1, 2, \dots, n.$$

3. Main results

Let

$$P = \max_{1 \le i \le n} \Big\{ \frac{\overline{I}_i e^{2A}}{\alpha} + \frac{\gamma e^{2A}}{1 - e^{-\alpha}} \Big\}.$$

Theorem 3.1. *Assume* (H1)-(H8) *and*

(H9) $r = \max_{1 \le i \le n} \Big\{ \frac{e^{2A}}{\alpha} \Big(\sum_{j=1}^n \overline{a}_{ij} \overline{L}_f + \sum_{j=1}^n \overline{b}_{ij} \overline{L}_g K \Big) \Big\} < 1$.

Then (1.2) *has a unique almost periodic solution.*

Proof. Set $\mathbb{X} = \{\varphi(t) \in PC(\mathbb{R}, \mathbb{R}^n) : \varphi(t) = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T$, where $\varphi_i(t)$ is a almost periodic function satisfying $\|\varphi\| = \max_{1 \le i \le n}\{\sup_{t \in \mathbb{R}} |\varphi_i(t)|\} \le N = \frac{P}{1-r}$, $i = 1, 2, \dots, n\}$ with the norm $\|\varphi\| = \max_{1 \le i \le n}\{\sup_{t \in \mathbb{R}} |\varphi_i(t)|\}$. We define an map Φ on $\mathbb{X}$ by

$$(\Phi\varphi)(t) = ((\Phi_1\varphi)(t), (\Phi_2\varphi)(t), \dots, (\Phi_n\varphi)(t))^T,$$

where $t \in \mathbb{R}$,

$$\begin{aligned}(\Phi_i\varphi)(t) \\ &= \int_{-\infty}^{t} W_i(t,s)\Big(\sum_{j=1}^{n} a_{ij}(s) f_j(\varphi_j(s-\xi)) \\ &\quad + \sum_{j=1}^{n} b_{ij}(s)\int_0^{\infty} K_{ij}(u) g_j(\varphi_j(s-u))\,\mathrm{d}u + I_i(s)\Big)\,\mathrm{d}s + \sum_{\tau_k<t} W_i(t,\tau_k)\nu_{ik},\end{aligned} \tag{3.1}$$

where $k \in \mathbb{Z}$, $i = 1, 2, \dots, n$. And let $\mathbb{X}^*$ be a subset of $\mathbb{X}$ defined by

$$\mathbb{X}^* = \Big\{\varphi \in \mathbb{X} : \|\varphi - \phi\| \le \frac{rP}{1-r}\Big\},$$

where $\phi = (\phi_1, \phi_2, \dots, \phi_n)^T$ and

$$\phi_i = \int_{-\infty}^{t} W_i(t,s) I_i(s)\,\mathrm{d}s + \sum_{\tau_k<t} W_i(t,\tau_k)\nu_{ik}, \quad k \in \mathbb{Z},\ i = 1, 2, \dots, n.$$

We have

$$\begin{aligned}\|\phi\| &= \max_{1\le i\le n}\Big\{\sup_{t\in\mathbb{R}}\Big|\int_{-\infty}^{t} W_i(t,s) I_i(s)\,\mathrm{d}s + \sum_{\tau_k<t} W_i(t,\tau_k)\nu_{ik}\Big|\Big\} \\ &\le \max_{1\le i\le n}\Big\{\sup_{t\in\mathbb{R}}\Big(\int_{-\infty}^{t} |W_i(t,s)||I_i(s)|\,\mathrm{d}s + \sum_{\tau_k<t} |W_i(t,\tau_k)||\nu_{ik}|\Big)\Big\} \\ &\le \max_{1\le i\le n}\Big\{\sup_{t\in\mathbb{R}}\Big(\int_{-\infty}^{t} e^{2A}e^{-\alpha(t-s)}\overline{I}_i\,\mathrm{d}s + \sum_{\tau_k<t} e^{2A}e^{-\alpha(t-\tau_k)}\nu_{ik}\Big)\Big\} \\ &\le \max_{1\le i\le n}\Big\{\frac{\overline{I}_i e^{2A}}{\alpha} + \frac{\gamma e^{2A}}{1-e^{-\alpha}}\Big\} = P.\end{aligned} \tag{3.2}$$

Then for arbitrary $\varphi \in \mathbb{X}^*$ from (3.1) and (3.2) we have

$$\|\varphi\| \le \|\varphi - \phi\| + \|\phi\| \le \frac{rP}{1-r} + P = \frac{P}{1-r}.$$

Now we prove that Φ is self-mapping from $\mathbb{X}^*$ to $\mathbb{X}^*$. For arbitrary $\varphi \in \mathbb{X}^*$ it follows that

$$\begin{aligned}\|\Phi\varphi - \phi\| &= \max_{1\le i\le n}\Big\{\sup_{t\in\mathbb{R}}\Big|\int_{-\infty}^{t} W_i(t,s)\Big(\sum_{j=1}^{n} a_{ij}(s) f_j(\varphi_j(s-\xi)) \\ &\quad + \sum_{j=1}^{n} b_{ij}(s)\int_0^{\infty} K_{ij}(u) g_j(\varphi_j(s-u))\,\mathrm{d}u\Big)\,\mathrm{d}s\Big|\Big\} \\ &\le \max_{1\le i\le n}\Big\{\sup_{t\in\mathbb{R}}\Big(\int_{-\infty}^{t} e^{2A}e^{-\alpha(t-s)}\Big(\sum_{j=1}^{n}\overline{a}_{ij}\overline{L}_f + \sum_{j=1}^{n}\overline{b}_{ij}\overline{L}_g K\Big)\,\mathrm{d}s\Big)\Big\}\|\varphi\| \\ &\le \max_{1\le i\le n}\Big\{\frac{e^{2A}}{\alpha}\Big(\sum_{j=1}^{n}\overline{a}_{ij}\overline{L}_f + \sum_{j=1}^{n}\overline{b}_{ij}\overline{L}_g K\Big)\Big\}\|\varphi\| \\ &= r\|\varphi\| \le \frac{rP}{1-r}.\end{aligned} \tag{3.3}$$

Moreover, we get

$$\begin{aligned}
\|\Phi\varphi\| &= \max_{1\le i\le n}\Big\{\sup_{t\in\mathbb{R}}\Big|\int_{-\infty}^{t} W_i(t,s)\Big(\sum_{j=1}^{n} a_{ij}(s)f_j(\varphi_j(s-\xi))\\
&\quad+\sum_{j=1}^{n} b_{ij}(s)\int_0^\infty K_{ij}(u)g_j(\varphi_j(s-u))\,\mathrm{d}u+I_i(s)\Big)\,\mathrm{d}s+\sum_{\tau_k<t}W_i(t,\tau_k)\nu_{ik}\Big|\Big\}\\
&\le \frac{rP}{1-r}+P\\
&=\frac{P}{1-r}=N.
\end{aligned} \tag{3.4}$$

On the other hand, let $\tau\in T$, $q\in Q$, where the sets T and Q are determined in Lemma 2.3. Then

$$\begin{aligned}
&|(\Phi_i\varphi)(t+\tau)-(\Phi_i\varphi)(t)|\\
&=\Big|\int_{-\infty}^{t+\tau} W_i(t+\tau,s)\Big(\sum_{j=1}^{n} a_{ij}(s)f_j(\varphi_j(s-\xi))\\
&\quad+\sum_{j=1}^{n} b_{ij}(s)\int_0^\infty K_{ij}(u)g_j(\varphi_j(s-u))\,\mathrm{d}u+I_i(s)\Big)\,\mathrm{d}s\\
&\quad+\sum_{\tau_k<t+\tau}W_i(t+\tau,\tau_k)\nu_{ik}-\sum_{\tau_k<t}W_i(t,\tau_k)\nu_{ik}\\
&\quad-\int_{-\infty}^{t} W_i(t,s)\Big(\sum_{j=1}^{n} a_{ij}(s)f_j(\varphi_j(s-\xi))\\
&\quad+\sum_{j=1}^{n} b_{ij}(s)\int_0^\infty K_{ij}(u)g_j(\varphi_j(s-u))\,\mathrm{d}u+I_i(s)\Big)\,\mathrm{d}s\Big|\\
&\le\Big|\int_{-\infty}^{t} W_i(t+\tau,s+\tau)\Big(\sum_{j=1}^{n} a_{ij}(s+\tau)f_j(\varphi_j(s+\tau-\xi))\\
&\quad+\sum_{j=1}^{n} b_{ij}(s+\tau)\int_0^\infty K_{ij}(u)g_j(\varphi_j(s+\tau-u))\,\mathrm{d}u+I_i(s+\tau)\Big)\,\mathrm{d}s\\
&\quad+\sum_{\tau_k<t}W_i(t+\tau,\tau_{k+q})\nu_{i(k+q)}-\sum_{\tau_k<t}W_i(t,\tau_k)\nu_{ik}\\
&\quad-\int_{-\infty}^{t} W_i(t,s)\Big(\sum_{j=1}^{n} a_{ij}(s)f_j(\varphi_j(s-\xi))\\
&\quad+\sum_{j=1}^{n} b_{ij}(s)\int_0^\infty K_{ij}(u)g_j(\varphi_j(s-u))\,\mathrm{d}u+I_i(s)\Big)\,\mathrm{d}s\Big|\\
&\le\int_{-\infty}^{t}|W_i(t+\tau,s+\tau)-W_i(t,s)|\Big|\sum_{j=1}^{n} a_{ij}(s+\tau)f_j(\varphi_j(s+\tau-\xi))
\end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n} b_{ij}(s+\tau)\int_0^\infty K_{ij}(u)g_j(\varphi_j(s+\tau-u))\,\mathrm{d}u + I_i(s+\tau)\Big|\,\mathrm{d}s\\
&+\int_{-\infty}^{t}|W_i(t,s)|\Big|\sum_{j=1}^{n} a_{ij}(s+\tau)f_j(\varphi_j(s+\tau-\xi))\\
&-\sum_{j=1}^{n} a_{ij}(s)f_j(\varphi_j(s-\xi)) - \sum_{j=1}^{n} b_{ij}(s)\int_0^\infty K_{ij}(u)g_j(\varphi_j(s-u))\,\mathrm{d}u\\
&+\sum_{j=1}^{n} b_{ij}(s+\tau)\int_0^\infty K_{ij}(u)g_j(\varphi_j(s+\tau-u))\,\mathrm{d}u\\
&+I_i(s+\tau)-I_i(s)\Big|\,\mathrm{d}s + \sum_{\tau_k<t}|W_i(t,\tau_k)||\nu_{i(k+q)}-\nu_{ik}|\\
&+\sum_{\tau_k<t}|W_i(t+\tau,\tau_{k+q})-W_i(t,\tau_k)||\nu_{i(k+q)}|\\
&\le C\epsilon,
\end{aligned}
$$

where

$$
\begin{aligned}
C=\max_{1\le i\le n}\Big\{&\frac{1}{\alpha}\sum_{j=1}^{n}(2\Gamma\overline{a}_{ij}\overline{L}_f+2\Gamma\overline{b}_{ij}\overline{L}_gK+\overline{L}_fe^{2A}+\overline{L}_ge^{2A}K)N+\frac{e^{2A}+2\Gamma\overline{I}_i}{\alpha}\\
&+\frac{e^{2A}}{\alpha}\sum_{j=1}^{n}(\overline{a}_{ij}\overline{L}_f+\bar{b}_{ij}\overline{L}_gK)+\frac{\gamma\Gamma}{1-e^{-\frac{\alpha}{2}}}+\frac{e^{2A}}{1-e^{-\alpha}}\Big\}.
\end{aligned}
$$

From (3.3)-(3.5), we obtain that $\Phi\varphi\in\mathbb{X}^*$. Let $\varphi\in\mathbb{X}^*$, $\psi\in\mathbb{X}^*$. We have

$$
\begin{aligned}
\|\Phi\varphi-\Phi\psi\| &= \max_{1\le i\le n}\Big\{\sup_{t\in\mathbb{R}}\Big|\int_{-\infty}^{t}W_i(t,s)\Big(\sum_{j=1}^{n}a_{ij}(s)f_j(\varphi_j(s-\xi))\\
&\quad+\sum_{j=1}^{n}b_{ij}(s)\int_0^\infty K_{ij}(u)g_j(\varphi_j(s-u))\,\mathrm{d}u\Big)\,\mathrm{d}s\\
&\quad-\int_{-\infty}^{t}W_i(t,s)\Big(\sum_{j=1}^{n}a_{ij}(s)f_j(\psi_j(s-\xi))\\
&\quad+\sum_{j=1}^{n}b_{ij}(s)\int_0^\infty K_{ij}(u)g_j(\psi_j(s-u))\,\mathrm{d}u\Big)\,\mathrm{d}s\Big|\Big\}\\
&\le \max_{1\le i\le n}\Big\{\frac{e^{2A}}{\alpha}\Big(\sum_{j=1}^{n}\overline{a}_{ij}\overline{L}_f+\sum_{j=1}^{n}\overline{b}_{ij}\overline{L}_gK\Big)\Big\}\|\varphi-\psi\|\\
&= r\|\varphi-\psi\|\\
&< \|\varphi-\psi\|.
\end{aligned}
$$

From this inequality, it follows that Φ is contracting operator in $\mathbb{X}^*$. So (1.2) has a unique almost periodic solution. This completes the proof. $\square$

Remark 3.2. In [1], $\alpha_k, k\in\mathbb{Z}$ are required to take values in $[-1,0]$, which is a more strict requirement (H2) in this article.

4. An example

Consider the impulsive Hopfield neural network

$$\begin{gathered} x'(t) = -c(t)x(t) + f(x(t-\frac{1}{2})) + \frac{1}{20}\int_0^{\infty} K(u)g(x(t-u))\,\mathrm{d}u + I(t), \quad t \neq \tau_k, \\ \Delta x(\tau_k) = \alpha_k x(\tau_k) + \nu_k, \quad k \in \mathbb{Z}, \end{gathered} \tag{4.1}$$

where (H2) and (H4) hold with $A = 2$, $c(t) = e^8 + \cos t$, $f(t) = \frac{1}{2}|t|$, $K(t) = e^{-4t}$, $g(t) = \frac{1}{4}\sin^2 t$, $I(t) = 2 + \sin t$, the sequence $\{\alpha_k\}$ is almost periodic and $1 - e^2 \leq \alpha_k \leq e^2 - 1$, $k \in \mathbb{Z}$. Obviously, $c = e^8 - 1$, $\overline{a} = 1$, $\overline{b} = \frac{1}{20}$, $\overline{L}_f = \overline{L}_g = \frac{1}{2}$, $K = \frac{1}{4}$. Then $\alpha = e^8 - 5 > 0$, $r = \frac{e^4}{e^8-5}(1 \times \frac{1}{2} + \frac{1}{20} \times \frac{1}{4} \times 5) < 1$, so (H8)-(H9) hold. It is easy to verify that (H1)-(H7) is satisfied. According to Theorem 3.1, (4.1) has one unique almost periodic solution.

References

[1] G. T. Stamov; *On the existence of almost periodic solutions for the impulsive Lasota-Wazewska model*, Appl. Math. Lett. 22 (2009) 516-520.

[2] A. M. Samoilenko, N. A. Perestyuk; *Differential Equations with Impulse Effect*, World Scientific, Singapore, 1995.

[3] V. Lakshmikantham, D. D. Bainov, P. S. Simeonov; *Theory of Impulsive Differential Equations*, World Scientific, Singapore, New Jersey, London, 1989.

Yongkun Li
Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, China
E-mail address: `yklie@ynu.edu.cn`

Tianwei Zhang
Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, China
E-mail address: `1200801347@stu.ynu.edu.cn`