Electronic Journal of Differential Equations, Vol. 2009(2009), No. 150, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

TRIPLE SOLUTIONS FOR MULTI-POINT BOUNDARY-VALUE PROBLEM WITH p-LAPLACE OPERATOR

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Abstract. Using a fixed point theorem due to Avery and Peterson, this article shows the existence of solutions for multi-point boundary-value problem with p-Laplace operator and parameters. Also, we present an example to illustrate the results obtained.

1. Introduction

During the previous two decades, boundary-value problems for second-order differential equations with p-Laplace operator have been extensively studied and a lot of excellent results have been established by using fixed point index theory, upper and lower solution arguments, fixed point theorem like Leggett-Williams multiple fixed point theorem and so on (see [2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and references therein). For example, Ma, Du and Ge [10] studied the following boundary-value problem (BVP, for short) with p-Laplace operator

$$(\varphi_p(u'(t)))' + q(t)f(t, u(t)) = 0, \quad t \in (0, 1);$$
$$u'(0) = \sum_{i=1}^{n} \alpha_i u'(\xi_i), u(1) = \sum_{i=1}^{n} \beta_i u(\xi_i),$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\varphi_p^{-1} = \varphi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, and $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$. The nonlinearity f is not depending on u'. Using the upper and lower solutions method, they obtained sufficient conditions for the existence of one positive solution.

Lv, O'Regan and Zhang [9] considered the following boundary-value problem (BVP) with p-Laplace operator

$$(\varphi_p(y'(t)))' + q(t)f(y(t)) = 0, \quad t \in [0, 1];$$
$$y(0) = y(1) = 0.$$

By Leggett-Williams multiple fixed point theorem, they provided sufficient conditions for the existence of multiple (at least three) positive solutions.

2000 *Mathematics Subject Classification.* 34B10, 34B15.
Key words and phrases. Triple solutions; p-Laplace operator; fixed point theorem; multi-point boundary-value problem.

Submitted April 22, 2009. Published November 25, 2009.
Supported by grants 209072 from the Key Project of Chinese Ministry of Education, and J08LI10 from the Science and by Technology Development Funds of Shandong Education Committee.

Recently Ji, Tian and Ge [8] studied the following boundary-value problem, in which the nonlinearity contains u',

$$\begin{gathered} (\varphi_p(u'(t)))' + \lambda f(t, u(t), u'(t)) = 0, \quad t \in (0,1); \\ u'(0) = \sum_{i=1}^{n} \alpha_i u'(\xi_i), \quad u(1) = \sum_{i=1}^{n} \beta_i u(\xi_i). \end{gathered} \tag{1.1}$$

Applying Krasnosel'skii fixed point theorem, they obtained the existence of at least one positive solution.

Wang and Ge [13] studied the multi-point boundary-value problem

$$\begin{gathered} (\varphi_p(u'(t)))' + q(t) f(t, u(t), u'(t)) = 0, \quad t \in (0,1); \\ u(0) = \sum_{i=1}^{n} \alpha_i u(\xi_i), \quad u(1) = \sum_{i=1}^{n} \beta_i u(\xi_i). \end{gathered}$$

Using the fixed point theorem due to Avery and Peterson, they provided sufficient conditions for the existence of multiple positive solutions.

Motivated by [8, 13], we investigate (1.1). We study boundary value conditions that are different from those in [9, 13]. We obtain three solutions by the fixed point theorem due to Avery and Peterson, which is different from the methods in [8, 9, 10]. To the best of our knowledge, (1.1) has not been studied via this fixed point theorem.

This article is organized as follows. Section 2 gives some preliminaries. Section 3 is devoted to the existence of triple solutions for (1.1). Finally an example is shown to illustrate the results obtained. Now, we give some notation which will be used later.

Let $X = C^1[0,1]$ be a Banach space with the norm

$$\|u\| = \max\big\{ \max_{t\in[0,1]} |u(t)|, \max_{t\in[0,1]} |u'(t)| \big\}.$$

A function $u(t)$ is called a positive solution of (1.1) if $u \in X$, satisfies (1.1) and $u(t) > 0$ for $t \in (0,1)$. Let

$$\begin{gathered} C^*[0,1] = \{u \in X : u(t) \ge 0, u'(t) \le 0, u'(t) \text{ is nonincreasing for } t \in [0,1]\}, \\ P = \{u \in X : u(t) \ge 0, u'(t) \le 0, u'(t) \text{ is concave on } t \in [0,1]\}. \end{gathered}$$

It is easy to see P is a cone of X.

In this paper, we assume the following hypotheses:

- (H1) $\alpha_i, \beta_i \ge 0$, $0 < \sum_{i=1}^{n} \alpha_i$, $\sum_{i=1}^{n} \beta_i < 1$.
- (H2) $f \in C([0,1] \times [0,+\infty) \times (-\infty, 0], [0,+\infty))$.

2. Preliminaries

In this section, we provide some background definitions from the study of cone in Banach spaces; see for example [4].

Let $(E, \|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $P \subseteq E$ is said to be a cone provided the following two conditions are satisfied:

(a) if $y \in P$ and $\lambda \ge 0$, then $\lambda y \in P$;
(b) if $y \in P$ and $-y \in P$, then $y = 0$.

If $P \subseteq E$ is a cone, we denote the order induced by P on E by $\leq$, that is, $x \leq y$ if and only if $y - x \in P$.

A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E, provided that $\alpha : P \to [0, +\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Similarly, we say a map β is a nonnegative continuous convex functional on a cone P of a real Banach space E, provided that $\beta : P \to [0, +\infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. Then for positive real numbers a, b, c and d, we define the following convex sets:

$$P(\gamma, d) = \{x \in P | \gamma(x) < d\},$$
$$P(\gamma, \alpha, b, d) = \{x \in P | b \leq \alpha(x), \gamma(x) \leq d\},$$
$$P(\gamma, \theta, \alpha, b, c, d) = \{x \in P | b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq d\},$$

and a closed set

$$R(\gamma, \psi, a, d) = \{x \in P | a \leq \psi(x), \gamma(x) \leq d\}.$$

The following fixed point theorem is fundamental in the proofs of our main results.

Lemma 2.1 ([1]). *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \leq \lambda\psi(x)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers L and d,*

$$\alpha(x) \leq \psi(x) \quad \text{and} \quad \|x\| \leq L\gamma(x), \forall x \in \overline{P(\gamma, d)}.$$

Suppose $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous, and there exist positive numbers a, b, and c with $a < b$ such that

- (S1) *$\{x \in P(\gamma, \theta, \alpha, b, c, d) | \alpha(x) > b\} \neq \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\gamma, \theta, \alpha, b, c, d)$*
- (S2) *$\alpha(Tx) > b$ for $x \in P(\gamma, \alpha, b, d)$ with $\theta(Tx) > c$*
- (S3) *$0 \notin R(\gamma, \psi, a, d)$ and $\psi(Tx) < a$ for $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$.*

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ such that $\gamma(x_i) \leq d$ for $i = 1, 2, 3$; $b < \alpha(x_1)$; $a < \psi(x_2)$ with $\alpha(x_2) < b$; $\psi(x_3) < a$.

To prove the main results in this paper, we will employ the following lemmas.

Lemma 2.2 ([8]). *Assume* (H1)-(H2), *and let*

$$k = \frac{\varphi_p(\sum_{i=1}^{n} \alpha_i)}{1 - \varphi_p(\sum_{i=1}^{n} \alpha_i)}.$$

For $x \in C^*[0, 1]$, if $u(t)$ is a solution of the problem

$$(\varphi_p(u'(t)))' + \lambda f(t, x(t), x'(t)) = 0, \quad t \in (0, 1);$$

$$u'(0)=\sum_{i=1}^{n}\alpha_i u'(\xi_i),\quad u(1)=\sum_{i=1}^{n}\beta_i u(\xi_i),$$

then

$$\begin{aligned} u(t)=&-\frac{\sum_{i=1}^{n}\beta_i\int_{\xi_i}^{1}\varphi_q(Ax-\int_0^s\lambda f(r,x(r),x'(r))dr)ds}{1-\sum_{i=1}^{n}\beta_i}\\ &-\int_t^1\varphi_q(Ax-\int_0^s\lambda f(r,x(r),x'(r))dr)ds, \end{aligned}\tag{2.1}$$

where $Ax\in[-k\lambda\int_0^1 f(s,x(s),x'(s))ds,0]$ *is unique and satisfies*

$$\varphi_q(Ax)=\sum_{i=1}^{n}\alpha_i\varphi_q(Ax-\int_0^{\xi_i}\lambda f(s,x(s),x'(s))ds),\tag{2.2}$$

Define the operator T by

$$\begin{aligned} (Tu)(t)=&-\frac{\sum_{i=1}^{n}\beta_i\int_{\xi_i}^{1}\varphi_q(Au-\int_0^s\lambda f(r,u(r),u'(r))dr)ds}{1-\sum_{i=1}^{n}\beta_i}\\ &-\int_t^1\varphi_q(Au-\int_0^s\lambda f(r,u(r),u'(r))dr)ds. \end{aligned}$$

Then by Lemma 2.2 it is easy to see $u(t)$ is a solution of (1.1) if and only if $u(t)=(Tu)(t)$.

Lemma 2.3 ([8]). *For each $\lambda>0$, the operator $T:P\to P$ is completely continuous.*

Now we give an important property of Ax defined by (2.1).

Lemma 2.4. *Assume* (H1) *holds. Then for each $x\in C^*[0,1]$, $\tau\in(0,\xi_1)$,*

$$\frac{\varphi_p(\sum_{i=1}^{n}\alpha_i)}{1-\varphi_p(\sum_{i=1}^{n}\alpha_i)}\int_\tau^{\xi_1}\lambda f(r,x(r),x'(r))dr\le -Ax\le k\int_0^1\lambda f(r,x(r),x'(r))dr.\tag{2.3}$$

Proof. By (2.1), we have

$$\begin{aligned} \varphi_q(Ax)&=\sum_{i=1}^{n}\alpha_i\varphi_q(Ax-\int_0^{\xi_i}\lambda f(s,x(s),x'(s))ds)\\ &\ge\sum_{i=1}^{n}\alpha_i\varphi_q(Ax-\int_0^{1}\lambda f(s,x(s),x'(s))ds), \end{aligned}$$

and

$$\begin{aligned} \varphi_q(Ax)&=\sum_{i=1}^{n}\alpha_i\varphi_q(Ax-\int_0^{\xi_i}\lambda f(s,x(s),x'(s))ds)\\ &\le\sum_{i=1}^{n}\alpha_i\varphi_q(Ax-\int_\tau^{\xi_1}\lambda f(s,x(s),x'(s))ds). \end{aligned}$$

From the increasing property of φ_q and the two inequalities above, it is easy to get the conclusion. ◻

Set

$$m := \frac{2^{\frac{1}{q-1}}+1}{\xi_1}, \quad l := \frac{\frac{(m+1)\xi_1}{2^{\frac{1}{q-1}}}+\xi_1^{-m}}{2},$$
$$N := \frac{\sum_{i=1}^n \beta_i(1-\xi_i)+(1-\sum_{i=1}^n \beta_i)(1-\xi_1)}{1-\sum_{i=1}^n \beta_i}.$$

Choose an $\tau \in (0, \xi_1)$ such that $l^{-1/m} < \tau < \xi_1$, and define the functionals:

$$\gamma(x) = \psi(x) := \|x\|, \quad \theta(x) := \max_{t\in[0,\tau]} |x'(t)|, \quad \alpha(x) := \min_{t\in[\tau,\xi_1]} x(t), \quad \forall x \in P. \tag{2.4}$$

Then it is easy to get the following lemma.

Lemma 2.5. *The four functionals defined by* (2.4) *satisfy Lemma 2.1. In addition, for each* $x \in P$, $\theta(x) = -x'(\tau)$, $\alpha(x) = x(\xi_1)$, $\gamma(x) = \psi(x) = x(0)$.

3. Main Results

First we state the following hypotheses to be used in this article.

(H3) There exists a positive constant H such that
$$f(t,u,v) < lt^m\varphi_p(|u|+|v|),$$
for $t \in [0,1]$ and $(u,v) \in \mathbb{R}^2$ satisfying $0 \le |u|+|v| \le H$.

(H4) There exist positive constants b, d such that
$$\max\{\frac{1}{1-\xi_1}, \frac{1}{2l^{q-1}}, \frac{1}{N}\}b < d \le \frac{1}{2}H,$$
$$f(t,u,v) > \varphi_p(b), \quad \text{for } (t,u,v) \in [\tau,\xi_1]\times[b,d]\times[-d,0].$$

Now we are ready to state our main results.

Theorem 3.1. *Assume* (H1)-(H4). *Let*
$$M = \frac{1-\sum_{i=1}^n \beta_i\xi_i}{(1-\sum_{i=1}^n \beta_i)\varphi_q(1-\varphi_p(\sum_{i=1}^n \alpha_i))}.$$
Then for each λ satisfying
$$\frac{1}{\xi_1 M^{\frac{1}{q-1}}} \le \lambda \le \frac{1}{\frac{1}{m+1}2^{\frac{1}{q-1}}lM^{\frac{1}{q-1}}}, \tag{3.1}$$
and $a \in (0,b)$, Equation (1.1) *has at least three solutions $x_1(t), x_2(t), x_3(t)$ satisfying*

(i) $\|x_i\| \le d,\ i = 1,2,3$;
(ii) $b < \min\{|x_1(t)||t \in [0,\tau]\}$;
(iii) $\|x_2\| > a,\ \min\{x_2(t)|t \in [0,\tau]\} < b$;
(iv) $\|x_3\| < a$.

Proof. We divide the proof of this theorem in four steps.

Step 1. Let us show $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$. In fact, for any $u \in \overline{P(\gamma,d)}$, it is not difficult to see
$$\|Tu\| = \max\{(Tu)(0), -(Tu)'(1)\}. \tag{3.2}$$
From (2.3), (3.1), and (H3), we obtain
$$(Tu)(0) = -\frac{\sum_{i=1}^n \beta_i \int_{\xi_i}^1 \varphi_q(Au - \int_0^s \lambda f(r,u(r),u'(r))dr)ds}{1-\sum_{i=1}^n \beta_i}$$

$$
\begin{aligned}
&\quad - \int_0^1 \varphi_q(Au - \int_0^s \lambda f(r,u(r),u'(r))dr)ds \\
&\le -\frac{\sum_{i=1}^n \beta_i \int_{\xi_i}^1 \varphi_q(-k\int_0^1 \lambda f(r,u(r),u'(r))dr - \int_0^s \lambda f(r,u(r),u'(r))dr)ds}{1-\sum_{i=1}^n \beta_i} \\
&\quad - \int_0^1 \varphi_q(-k\int_0^1 \lambda f(r,u(r),u'(r))dr - \int_0^s \lambda f(r,u(r),u'(r))dr)ds \\
&\le \lambda^{q-1} \frac{1-\sum_{i=1}^n \beta_i\xi_i}{(1-\sum_{i=1}^n \beta_i)\varphi_q(1-\varphi_p(\sum_{i=1}^n \alpha_i))} \varphi_q\Big(\int_0^1 f(r,u(r),u'(r))dr\Big) \\
&< (\frac{1}{m+1})^{q-1} 2\lambda^{q-1} l^{q-1} M\|u\| \\
&\le (\frac{1}{m+1})^{q-1} 2\lambda^{q-1} l^{q-1} Md \le d
\end{aligned}
$$

and

$$
\begin{aligned}
-(Tu)'(1) &= -\varphi_q(Au - \int_0^1 \lambda f(r,u(r),u'(r))dr) \\
&\le \varphi_q(\int_0^1 \lambda f(r,u(r),u'(r))dr + \int_0^s \lambda f(r,u(r),u'(r))dr) \\
&\le \lambda^{q-1} \frac{1}{\varphi_q(1-\varphi_p(\sum_{i=1}^n \alpha_i))} \varphi_q(\int_0^1 f(r,u(r),u'(r))dr) \\
&< (\frac{1}{m+1})^{q-1} 2\lambda^{q-1} l^{q-1} \frac{1}{\varphi_q(1-\varphi_p(\sum_{i=1}^n \alpha_i))} \|u\| \le d.
\end{aligned}
$$

Thus $\|Tu\| = \max\{(Tu)(0), -(Tu)'(1)\} \le d$. Hence $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$.

Step 2. Check condition (S1) of Lemma 2.1. Choose an integer $w > 0$ such that $\max\{\frac{1}{1-\xi_1}, \frac{1}{N}\} < w \le \frac{d}{b}$. Set $u(t) = wb(1-t)$. Then

$$
b < \theta(u) = wb,\ \gamma(u) = wb \le d,\ b < \alpha(u) = wb(1-\xi_1) < wb.
$$

Therefore, $u(t) = wb(1-t) \in P(\gamma,\theta,\alpha,b,wb,d)$, and $\alpha(u) > b$. This guarantees that $\{u \in P(\gamma,\theta,\alpha,b,wb,d) | \alpha(u) > b\} \neq \emptyset$. For any $u \in P(\gamma,\theta,\alpha,b,wb,d)$, it is easy to see

$$
b \le u(t) \le d, \quad -d \le u'(t) \le 0, \quad \forall t \in [\tau,\xi_1].
$$

Thus by (H4), $f(t,u(t),u'(t)) > \varphi_p(b)$.

By Lemma 2.2 and Lemma 2.3, it is not difficult to see

$$
\begin{aligned}
&\alpha(Tu) = (Tu)(\xi_1) \\
&= -\frac{\sum_{i=1}^n \beta_i \int_{\xi_i}^1 \varphi_q(Au - \int_0^s \lambda f(r,u(r),u'(r))dr)ds}{1-\sum_{i=1}^n \beta_i} \\
&\quad - \int_{\xi_1}^1 \varphi_q(Au - \int_0^s \lambda f(r,u(r),u'(r))dr)ds \\
&\ge \frac{\sum_{i=1}^n \beta_i \int_{\xi_i}^1 \varphi_q(k\int_0^{\xi_1} \lambda f(r,u(r),u'(r))dr + \int_0^s \lambda f(r,u(r),u'(r))dr)ds}{1-\sum_{i=1}^n \beta_i} \\
&\quad + \int_{\xi_1}^1 \varphi_q(k\int_0^{\xi_1} \lambda f(r,u(r),u'(r))dr + \int_0^s \lambda f(r,u(r),u'(r))dr)ds
\end{aligned}
$$

$$
\begin{aligned}
&\ge \lambda^{q-1}\frac{1-\sum_{i=1}^n\beta_i\xi_i}{(1-\sum_{i=1}^n\beta_i)\varphi_q(1-\varphi_p(\sum_{i=1}^n\alpha_i))}\varphi_q(\int_\tau^{\xi_1} f(r,u(r),u'(r))dr)\\
&> \lambda^{q-1}\xi_1^{q-1}Mb \ge b.
\end{aligned}
$$

This shows that condition (S1) of Lemma (2.1) is satisfied.

Step 3. Examine (S2) of Lemma 2.1. For any $u \in P(\gamma,\alpha,b,d)$ with $\theta(Tu) > wb$, we know

$$
\theta(Tu) = -(Tu)'(\tau) = \varphi_q\Big(\int_0^\tau \lambda f(r,u(r),u'(r))dr - Au\Big) > wb. \tag{3.3}
$$

Therefore by (2.3) and (3.3),

$$
\begin{aligned}
&\alpha(Tu) = (Tu)(\xi_1)\\
&= -\frac{\sum_{i=1}^n\beta_i\int_{\xi_i}^1\varphi_q(Au-\int_0^s\lambda f(r,u(r),u'(r))dr)ds}{1-\sum_{i=1}^n\beta_i}\\
&\quad -\int_{\xi_1}^1\varphi_q(Au-\int_0^s\lambda f(r,u(r),u'(r))dr)ds\\
&\ge \frac{\sum_{i=1}^n\beta_i\int_{\xi_i}^1\varphi_q\Big(k\int_0^\tau\lambda f(r,u(r),u'(r))dr-Au\Big)ds}{1-\sum_{i=1}^n\beta_i}\\
&\quad +\int_{\xi_1}^1\varphi_q(k\int_0^\tau\lambda f(r,u(r),u'(r))dr-Au)ds\\
&= \frac{\sum_{i=1}^n\beta_i(1-\xi_i)+(1-\sum_{i=1}^n\beta_i)(1-\xi_1)}{1-\sum_{i=1}^n\beta_i}\varphi_q(\int_0^\tau\lambda f(r,u(r),u'(r))dr-Au)\\
&> Nwb > b.
\end{aligned}
$$

Thus, condition (S2) of Lemma (2.1) is satisfied.

Step 4. Finally we show (S3) of Lemma 2.1 holds. Since $\psi(0) = 0 < a$, we know $0 \notin R(\gamma,\psi,a,d)$. For each $u \in R(\gamma,\psi,a,d)$, $\psi(u) = \|u\| = a$, by (2.3), (3.1), and (H3), we obtain

$$
\begin{aligned}
(Tu)(0) &= -\frac{\sum_{i=1}^n\beta_i\int_{\xi_i}^1\varphi_q(Au-\int_0^s\lambda f(r,u(r),u'(r))dr)ds}{1-\sum_{i=1}^n\beta_i}\\
&\quad -\int_0^1\varphi_q(Au-\int_0^s\lambda f(r,u(r),u'(r))dr)ds\\
&\le \lambda^{q-1}\frac{1-\sum_{i=1}^n\beta_i\xi_i}{(1-\sum_{i=1}^n\beta_i)\varphi_q(1-\varphi_p(\sum_{i=1}^n\alpha_i))}\varphi_q\Big(\int_0^1 f(r,u(r),u'(r))dr\Big)\\
&< (\frac{1}{m+1})^{q-1}2\lambda^{q-1}l^{q-1}M\|u\|\\
&= (\frac{1}{m+1})^{q-1}2\lambda^{q-1}l^{q-1}Ma \le a
\end{aligned}
$$

and

$$
\begin{aligned}
-(Tu)'(1) &= -\varphi_q(Au-\int_0^1\lambda f(r,u(r),u'(r))dr)\\
&\le \varphi_q(k\int_0^1\lambda f(r,u(r),u'(r))dr+\int_0^s\lambda f(r,u(r),u'(r))dr)
\end{aligned}
$$

$$
\begin{aligned}
&\le \lambda^{q-1} \frac{1}{\varphi_q(1-\varphi_p(\sum_{i=1}^n \alpha_i))} \varphi_q\Big(\int_0^1 f(r,u(r),u'(r))dr\Big) \\
&< (\frac{1}{m+1})^{q-1} 2\lambda^{q-1} l^{q-1} \frac{1}{\varphi_q(1-\varphi_p(\sum_{i=1}^n \alpha_i))} \|u\| \le a.
\end{aligned}
$$

Therefore,

$$
\psi(u) = \|u\| = \max\{(Tu)(0), -(Tu)'(1)\} < a.
$$

So condition (S3) of Lemma (2.1) is satisfied. Thus an application of Lemma 2.1 implies that the boundary value problem (1.1) has at least three solutions $x_1(t), x_2(t),\ x_3(t)$ satisfying (i)–(iv). □

We remark that in Theorem 3.1, the two solutions $x_1(t)$ and $x_2(t)$ are positive, while $x_3(t)$ may be the trivial solution.

3.1. **Example.** Consider the differential equation

$$
\begin{gathered}
(\varphi_p(u'(t)))' + \lambda f(t,u(t),u'(t)) = 0, \quad t \in (0,1); \\
u'(0) = \sum_{i=1}^{2} \alpha_i u'(\xi_i), \quad u(1) = \sum_{i=1}^{2} \beta_i u(\xi_i),
\end{gathered} \tag{3.4}
$$

where $p = 3/2$, $q = 3$, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1/4$, $\xi_1 = 0.9$, $\xi_2 = 0.95$,

$$
f(t,u,v) = 1.8t^{10(\sqrt{2}+1)/9}\sqrt{u+|v|}, \quad (t,u,v) \in [0,1] \times [0,+\infty) \times (-\infty,0].
$$

Choose $l = 1.835055448$, $m = 10(\sqrt{2}+1)/9$, $H = 20000$, $d = 10000$, $b = 100$, $a = 50$, $\tau = 0.88$, then by simple calculations, it is easy to show (H1)-(H4) are satisfied. Therefore, by Theorem 3.1, for $9\sqrt{430}/430 \le \lambda \le 1.461370837$, Equation (3.4) has at least three solutions.

Acknowledgements. The authors want to thank the anonymous referee for the suggestions on the paper.

References

[1] R. I. Avery, A. C. Peterson; *Three positive fixed points of nonlinear operators on ordered Banach spaces*, Comput. Math. Appl., **42** (2001), 313-322.

[2] W. Ge; *Boundary value problems for nonlinear differential equations*, Science Press, Beijing, 2007 (in Chinese).

[3] W. Ge, J. Ren; *An extension of Mawhin's continuation theorem and its applications to boundary value problems with a p-Laplacian*, Nonl. Anal., **58** (2004), 477-488.

[4] D. Guo, V. Lakshmikantham; *Nonlinear problems in abstract cones*, Academic Press, New York, 1988.

[5] X. He, W. Ge; *Existence of positive solution for a one-dimensional p-Laplacian boundary value problem*, Acta. Mathematica. Sinica., **46**(4)(2003), 805-810 (in Chinese).

[6] X. He, W. Ge; *Twin positive solutions for the one-dimensional p-Laplacian boundary value problems*, Nonl. Anal., **56** (7) (2004), 975-984.

[7] S. Hong; *Triple positive solutions of three-point boundary value problems for p-Laplacian dynamic equations on time scales*, Comput. Math. Appl., **206**(2007), 967-976.

[8] D. Ji, Y. Tian, W. Ge; *The existence of positive solution of multi-point boundary value problem with a p-Laplace operator*, Acta. Mathematica. Sinica., **52** (1)(2009), 1-8 (in Chinese).

[9] H. Lv, D. O'Regan, C. Zhang; *Multiple positive solutions for the one dimensional singular p-Laplacian*, Appl. Math. Comput., **133** (2002), 407-422.

[10] D. Ma, Z. Du, W. Ge; *Existence and iteration of monotone positive solutions for multi-point boundary value problem with p-Laplacian operator*, Comput. Math. Appl., **50**(2005), 729-739.

[11] J. Wang; *The existence of positive solutions for the one-dimensional p-Laplacian*, Proc. Amer. Appl., **125** (8)(1997), 2275-2283.

[12] Y. Wang, W. Ge; *M-point boundary value problem for second order nonlinear differential equation*, J. Anal. Anal., **85** (2006), 659-667.
[13] Y. Wang, W. Ge; *Existence of triple positive solutions for multi-point boundary value problems with a one dimensional p-Laplacian*, Comput. Math. Appl., **54** (2007), 793-807.
[14] Y. Wang, C. Hou; *Existence of multiple positive solutions for one-dimensional p-Laplace*, J. Math. Anal. Appl., **315** (2006), 144-153.

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