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A REMARK ON THE REGULARITY FOR THE 3D NAVIER-STOKES EQUATIONS IN TERMS OF THE TWO COMPONENTS OF THE VELOCITY

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Abstract. In this note, we study the regularity of Leray-Hopf weak solutions to the Navier-Stokes equation, with the condition

$$\nabla(u_1,u_2,0)\in L^{\frac{2}{1-r}}(0,T;\dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)),$$

where $\dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)$ is the Morrey-Campanato space for $0<r<1$. Since

$$L^{1/3}(\mathbb{R}^3)\subset \dot{X}_r(\mathbb{R}^3)\subset \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3),$$

the above regularity condition allows us to improve the results obtained by Fan and Gao [6].

1. Introduction

Consider the Navier-Stokes equation, in $\mathbb{R}^3$,

$$\begin{gathered}\partial_t u+(u\cdot\nabla)u-\Delta u+\nabla p=0,\quad (x,t)\in\mathbb{R}^3\times(0,T),\\ \operatorname{div} u=0,\quad (x,t)\in\mathbb{R}^3\times(0,T),\\ u(x,0)=u_0(x),\quad x\in\mathbb{R}^3,\end{gathered}\tag{1.1}$$

where $u=u(x,t)$ is the velocity field, $p=p(x,t)$ is the scalar pressure and $u_0(x)$ with $\operatorname{div} u_0=0$ in the sense of distribution is the initial velocity field. For simplicity, we assume that the external force has a scalar potential and is included in the pressure gradient.

In their classical article, Leray [12] and Hopf [9] independently constructed a weak solution u of (1.1) for arbitrary $u_0\in L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0=0$. The solution is called the Leray-Hopf weak solution. Regularity of such Leray-Hopf weak solutions is one of the most significant open problems in mathematical fluid mechanics.

By a weak solution we mean a function $u\in L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;\dot{H}^1(\mathbb{R}^3))$ satisfying (1.1) in sense of distributions. See e.g. [17] for an exposition of the theory of weak solutions.

Introducing the class $L^\alpha(0,T;L^q(\mathbb{R}^3))$, it is shown that if we have a Leray-Hopf weak solution u belonging to $L^\alpha((0,T);L^q(\mathbb{R}^3))$ with the exponents α and

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q satisfying $\frac{2}{\alpha}+\frac{3}{q}\le 1$, $2\le\alpha<\infty$, $3<q\le\infty$, then the solution $u(x,t)\in C^\infty(\mathbb{R}^3\times(0,T))$ [16, 14, 15, 5, 7, 18, 19]. The limit case $\alpha=\infty$, $q=3$ was covered much later Escauriaza, Seregin and Sverak in [4]. Bae and Choe [2] proved that u is strong if $\widetilde{u}\in L^\alpha(0,T;L^q(\mathbb{R}^3))$ with $\frac{2}{\alpha}+\frac{3}{q}=1$ and $q>3$. Later, Chae-Choe [3] obtained an improved regularity criterion of [1] imposing condition on only two components of the velocity, namely if

$$\nabla\tilde{u}\in L^\alpha(0,T;L^q(\mathbb{R}^3))\quad\text{with } \frac{2}{\alpha}+\frac{3}{q}\le 2,\ 1\le\alpha<\infty,$$
$$\tilde{u}=(u_1,u_2,0)$$

then the weak solution becomes smooth. See also [20, 21] for recent improvements of these criteria, via one velocity component. Recently, Fan and Gao [6] improved the regularity criterion in [3], under the condition

$$\nabla\tilde{u}\in L^{\frac{2}{2-r}}(0,T;\dot{X}_r(\mathbb{R}^3))\quad\text{for some } 0<r<1,$$

where $\dot{X}_r$ is the multiplier space (see definition below).

The purpose of this note is to imporve the results in [3] and [6], by proving that if $\nabla\tilde{u}\in L^{\frac{2}{2-r}}(0,T;\dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3))$ with $0<r<1$, then the weak solution becomes smooth. Here $\dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)$ is the Morrey-Campanato space, which is strictly larger than $L^{1/3}(\mathbb{R}^3)$ and $\dot{X}_r(\mathbb{R}^3)$ (see the next section for the related embedding relations).

2. Preliminaries and the main result

Now, we recall the definition and some properties of the spaces to be used later. These spaces play an important role in studying the regularity of solutions to partial differential equations; see e.g. [8] and the references therein.

Definition 2.1. For $0\le r<3/2$, the space $\dot{X}_r(\mathbb{R}^3)$ is defined as the space of functions $f(x)\in L^2_{\rm loc}(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{X}_r}=\sup_{\|g\|_{\dot{H}^r}\le 1}\|fg\|_{L^2}<\infty.$$

where we denote by $\dot{H}^r(\mathbb{R}^3)$ the completion of the space $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_{\dot{H}^r}=\|(-\Delta)^{r/2}u\|_{L^2}$.

We have the following homogeneity properties: For all $x_0\in\mathbb{R}^3$,

$$\|f(\cdot+x_0)\|_{\dot{X}_r}=\|f\|_{\dot{X}_r}$$
$$\|f(\lambda\cdot)\|_{\dot{X}_r}=\frac{1}{\lambda^r}\|f\|_{\dot{X}_r},\quad \lambda>0.$$

Also we have the imbedding

$$L^{1/3}(\mathbb{R}^3)\hookrightarrow\dot{X}_r(\mathbb{R}^3)\quad\text{for } 0\le r<\frac{3}{2}\,.$$

Now we recall the definition of the Morrey-Campanato spaces.

Definition 2.2. For $1<p\le q\le+\infty$, the Morrey-Campanato space $\dot{\mathcal{M}}_{p,q}(\mathbb{R}^3)$ is defined by

$$\dot{\mathcal{M}}_{p,q}(\mathbb{R}^3)=\big\{f\in L^p_{\rm loc}(\mathbb{R}^3):\|f\|_{\dot{\mathcal{M}}_{p,q}}=\sup_{x\in\mathbb{R}^3}\sup_{R>0}R^{3/q-3/p}\|f\|_{L^p(B(x,R))}<\infty\big\} \tag{2.1}$$

It is easy to check the equality

$$\|f(\lambda\cdot)\|_{\dot{\mathcal{M}}_{p,q}} = \frac{1}{\lambda^{3/q}}\|f\|_{\dot{\mathcal{M}}_{p,q}}, \quad \lambda > 0.$$

For $2 < p \le 3/r$ and $0 < r < 3/2$ we have the following embeddings:

$$L^{1/3}(\mathbb{R}^3) \hookrightarrow L^{3/r,\infty}(\mathbb{R}^3) \hookrightarrow \dot{\mathcal{M}}_{p,3/r}(\mathbb{R}^3) \hookrightarrow \dot{X}_r(\mathbb{R}^3) \hookrightarrow \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3).$$

The relation

$$L^{3/r,\infty}(\mathbb{R}^3) \hookrightarrow \dot{\mathcal{M}}_{p,3/r}(\mathbb{R}^3)$$

is shown as follows.

$$\begin{aligned}
\|f\|_{\dot{\mathcal{M}}_{p,\frac{3}{r}}} &\le \sup_E |E|^{\frac{r}{3}-\frac{1}{2}}\Big(\int_E |f(y)|^p dy\Big)^{1/p} \quad (f \in L^{3/r,\infty}(\mathbb{R}^3))\\
&= \Big(\sup_E |E|^{\frac{pr}{3}-1}\int_E |f(y)|^p dy\Big)^{1/p}\\
&\cong \Big(\sup_{R>0} R|\{x\in\mathbb{R}^3 : |f(y)|^p > R\}|^{pr/3}\Big)^{1/p}\\
&= \sup_{R>0} R|\{x\in\mathbb{R}^p : |f(y)| > R\}|^{r/3}\\
&\cong \|f\|_{L^{3/r,\infty}}.
\end{aligned}$$

For $0 < r < 1$, we use the fact that

$$L^2 \cap \dot{H}^1 \subset \dot{B}^r_{2,1} \subset \dot{H}^r.$$

Thus we can replace the space $\dot{X}_r$ by the pointwise multipliers from Besov space $\dot{B}^r_{2,1}$ to L^2. Then we have the following lemma given in [11].

Lemma 2.3. *For $0 \le r < 3/2$, the space $\dot{Z}_r(\mathbb{R}^3)$ is defined as the space of functions $f(x) \in L^2_{\rm loc}(\mathbb{R}^3)$ such that*

$$\|f\|_{\dot{Z}_r} = \sup_{\|g\|_{\dot{B}^r_{2,1}}\le 1} \|fg\|_{L^2} < \infty.$$

Then $f \in \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)$ if and only if $f \in \dot{Z}_r(\mathbb{R}^3)$ with equivalence of norms.

Additionally, for $2 < p \le \frac{3}{r}$ and $0 \le r < \frac{3}{2}$, we have the following inclusions [10, 11]:

$$\dot{\mathcal{M}}_{p,3/r}(\mathbb{R}^3) \hookrightarrow \dot{X}_r(\mathbb{R}^3) \hookrightarrow \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3) = \dot{Z}_r(\mathbb{R}^3).$$

The relation

$$\dot{X}_r(\mathbb{R}^3) \hookrightarrow \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)$$

is shown as follows: Let $f \in \dot{X}_r(\mathbb{R}^3)$, $0 < R \le 1$, $x_0 \in \mathbb{R}^3$ and $\phi \in C_0^\infty(\mathbb{R}^3)$, $\phi \equiv 1$ on $B(\frac{x_0}{R}, 1)$. We have

$$\begin{aligned}
R^{r-\frac{3}{2}}\Big(\int_{|x-x_0|\le R} |f(x)|^2 dx\Big)^{1/2} &= R^r\Big(\int_{|y-\frac{x_0}{R}|\le 1} |f(Ry)|^2 dy\Big)^{1/2}\\
&\le R^r\Big(\int_{y\in\mathbb{R}^3} |f(Ry)\phi(y)|^2 dy\Big)^{1/2}\\
&\le R^r\|f(R.)\|_{\dot{X}_r}\|\phi\|_{H^r}\\
&\le \|f\|_{\dot{X}_r}\|\phi\|_{H^r}\\
&\le C\|f\|_{\dot{X}_r}.
\end{aligned}$$

The following result well be used in the proof of Theorem 2.5.

Lemma 2.4. *For $0 < r < 1$, we have*

$$\|f\|_{\dot{B}^r_{2,1}} \leq C\|f\|_{L^2}^{1-r}\|\nabla f\|_{L^2}^r.$$

Proof. The idea comes from [13] (see also [22]). According to the definition of Besov spaces, one has

$$\begin{aligned}
\|f\|_{\dot{B}^r_{2,1}} &= \sum_{j\in\mathbb{Z}} 2^{jr}\|\Delta_j f\|_{L^2} \\
&\leq \sum_{j\leq k} 2^{jr}\|\Delta_j f\|_{L^2} + \sum_{j>k} 2^{j(r-1)}2^j\|\Delta_j f\|_{L^2} \\
&\leq (\sum_{j\leq k} 2^{2jr})^{1/2}(\sum_{j\leq k}\|\Delta_j f\|_{L^2}^2)^{1/2} + (\sum_{j>k} 2^{2j(r-1)})^{\frac{1}{2}}(\sum_{j>k} 2^{2j}\|\Delta_j f\|_{L^2}^2)^{1/2} \\
&\leq C\Big(2^{rk}\|f\|_{L^2} + 2^{k(r-1)}\|f\|_{\dot{H}^1}\Big) \\
&= C(2^{rk}A^{-r} + 2^{k(r-1)}A^{1-r})\|f\|_{L^2}^{1-r}\|f\|_{\dot{H}^1}^r,
\end{aligned}$$

where $A = \|f\|_{\dot{H}^1}/\|f\|_{L^2}$.

Choose k such that $2^{rk}A^{-r} \leq 1$; that is, $k \leq [\log A^r]$. Then

$$\begin{aligned}
\|f\|_{\dot{B}^r_{2,1}} &\leq C(1 + 2^{k(r-1)}A^{1-r})\|f\|_{L^2}^{1-r}\|f\|_{\dot{H}^1}^r \\
&\leq C\|f\|_{L^2}^{1-r}\|\nabla f\|_{L^2}^r,
\end{aligned}$$

and so the proof is complete. □

Since $L^{1/3}(\mathbb{R}^3) \subset \dot{X}_r(\mathbb{R}^3)\subset\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$, the above regularity criterion alloy us to improve the results obtained by Fan and Gao [6]. Our main result on (1.1) reads as follows.

Theorem 2.5. *Let $\tilde{u} = u_1e_1 + u_2e_2$ be the first two components of a Leray-Hopf weak solution of the Navier-Stokes equation corresponding to $u_0 \in H^1(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$. Suppose that $\nabla\tilde{u} \in L^{\frac{2}{1-r}}(0, T, \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3))$ with $0 < r < 1$, then u becomes the classical solution on $(0, T]$.*

Proof. We follow the ideas of the proof in [6]. By differentiating the equations (1.1) with respect to x_k, we take the scalar product with $\partial_k u$, and integrate over $\mathbb{R}^3$. A suitable integration by parts yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla u(t,.)\|_{L^2}^2 + \|\nabla^2 u(t,.)\|_{L^2}^2 &= -\int_{\mathbb{R}^3}\nabla[(u.\nabla)u].\nabla u\,dx \\
&= \sum_{i,j,k}\int_{\mathbb{R}^3}\partial_k u_i.\partial_i u_j.\partial_k u_j dx.
\end{aligned} \tag{2.2}$$

Following [6], we only need to deal with the case $i = j = 3$. Since $\partial_1u_1 + \partial_2u_2 + \partial_3u_3 = 0$, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^3}\partial_k u_i.\partial_i u_j.\partial_k u_j dx &= -\int_{\mathbb{R}^3}\partial_k u_3.(\partial_1u_1 + \partial_2u_2).\partial_k u_3 dx \\
&\leq \int_{\mathbb{R}^3}|\nabla\widetilde{u}||\nabla u|^2 dx.
\end{aligned}$$

Using Hölder's inequality and Lemma 2.3, we have

$$\begin{aligned}\int_{\mathbb{R}^3}|\nabla\widetilde{u}||\nabla u|^2dx &\le \|\nabla u\|_{L^2}\|\nabla u\cdot\nabla\tilde{u}\|_{L^2}\\ &\le C\|\nabla\tilde{u}\|_{\dot{\mathcal{M}}_{2,3/r}}\|\nabla u\|_{L^2}\|\nabla u\|_{\dot{B}^r_{2,1}}\\ &\le C\|\nabla\tilde{u}\|_{\dot{\mathcal{M}}_{2,3/r}}\|\nabla u\|_{L^2}\|\nabla u\|_{L^2}^{1-r}\|\nabla^2 u\|_{L^2}^r\\ &= C\Big(\|\nabla\tilde{u}\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{2-r}}\|\nabla u\|_{L^2}^2\Big)^{\frac{2-r}{2}}\|\nabla^2 u\|_{L^2}^r\\ &\le \frac{1}{2}\|\nabla^2 u\|_{L^2}^2 + C\|\nabla\tilde{u}\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{2-r}}\|\nabla u\|_{L^2}^2.\end{aligned}$$

This estimates combined with (2.2), yield

$$\frac{d}{dt}\|\nabla u(t,.)\|_{L^2}^2 + \|\nabla^2 u(t,.)\|_{L^2}^2 \le C\|\nabla\tilde{u}\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{2-r}}\|\nabla u\|_{L^2}^2.$$

By Gronwall' s inequality we have

$$\|\nabla u(t,.)\|_{L^2}^2 \le \|\nabla u(0,.)\|_{L^2}^2\exp\Big(C\int_0^T\|\nabla\tilde{u}(\cdot,\tau)\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{2}{2-r}}\,d\tau\Big).$$

This completes the proof. □

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