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BOUNDEDNESS AND EXPONENTIAL STABILITY OF HIGHLY NONLINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this article we consider nonlinear stochastic differential systems and use Lyapunov functions to study the boundedness and exponential asymptotic stability of solutions. We provide several examples in which we consider stochastic systems with unbounded terms.

1. Introduction

The method of Lyapunov functions has been an important tool in the development of stability theory for nonlinear deterministic dynamical systems. There is also large body of literature on the boundedness and stability using Lyapunov functions or functionals, see for example [1]–[5] and [7]. Erhart [11] used Lyapunov functions to obtained sufficient conditions for the existence of solution of a dynamical system on time scales. Since Itô introduced the stochastic integral, Lyapunov functions have been employed to study various qualitative and quantitative properties including stochastic stability and stochastic boundedness of stochastic differential equations (SDE). See for example, Kushner [8], Mao [9, 10], Hasminskii [6] and the references therein. The report in [13] presents an interesting survey of Lyapunov functions techniques in stochastic differential equations.

In a recent work Raffoul [12] considered the deterministic nonlinear system

$$\begin{gathered} \dot{x} = f(t,x), \quad t \ge 0, \\ x(t_0) = x_0, \quad t_0 \ge 0,\ x_0 \in \mathbb{R}^n \end{gathered}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a given nonlinear continuous function in t and x, where $t \in \mathbb{R}^+$. There non-negative Lyapunov functions are used for establishing sufficient conditions for the boundedness and exponential asymptotic stability of deterministic nonlinear differential dynamic systems with unbounded forcing terms. It makes sense to ask that if the solutions of the deterministic systems are bounded or exponentially stable then whether or not the same would hold for the solutions of the stochastic system, created from the deterministic one by adding a nonlinear stochastic term to it. To answer this question we will have to generalize the definitions and results in [12] to a class of nonlinear stochastic

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differential equations and prove several theorems that can be used to verify the stochastic boundedness and exponential asymptotic stability. As an application to results obtained, we furnish several examples.

As a motivational example, consider the one-dimensional stochastic differential equation

$$dx(t) = -a(t)x(t)dt + g(x(t),t)dB(t), \quad t \geq t_0 \geq 0, \tag{1.1}$$

with initial condition $x(t_0) = x_0 \in \mathbb{R}$, where $B(t)$ is a one-dimensional standard Brownian motion. We assume that the function $a(t)$ and $g(x,t)$ satisfy $g^2(x,t) \leq \lambda^2(t)x^2 + h^2(t)$ and $2a(t) \geq \lambda^2(t)$ for some function $\lambda(t)$ and $h(t)$. Let $\alpha(t) = 2a(t) - \lambda^2(t)$ and choose the Lyapunov function $V(x,t) = x^2$. Then along the solutions $x := x(t)$ of (1.1), we have

$$dV(x,t) \leq -\alpha(t)V(x,t)dt + h^2(t)dt + 2xg(x,t)dB(t). \tag{1.2}$$

Multiply both sides of (1.2) with the integrating factor $e^{\int_{t_0}^t \alpha(s)ds}$ and then integrate from t_0 to t to obtain

$$\begin{aligned} V(x,t) \leq e^{-\int_{t_0}^t \alpha(s)ds} V(x_0,t_0) + \int_{t_0}^t e^{-\int_u^t \alpha(s)ds} h^2(u)\,du \\ + 2\int_{t_0}^t e^{-\int_u^t \alpha(s)ds} x(u)g(x(u),u)dB(u). \end{aligned} \tag{1.3}$$

We note that inequality (1.3) was readily available due to the form (1.2). However, in general to find a Lyapunov function that satisfies (1.2) is extremely difficult and may requires strong conditions on the known coefficients. This brings us to the following question: What can be said about the boundedness of solutions when along the solutions of (1.1), V satisfies

$$dV(x) \leq -\alpha(t)W(x)dt + h^2(t)dt + 2xg(x,t)dB(t), W(x) \neq V(x) \quad \text{for } x \neq 0? \tag{1.4}$$

Here $\alpha : [0,\infty) \to [0,\infty)$ is continuous and $W : [0,\infty) \to [0,\infty)$ is continuous with $W(0) = 0$, $W(x)$ is strictly increasing, and $W(x) \to \infty$ as $x \to \infty$. Such a function is called a wedge.

In the next section we present our main results and in Section 3 we give examples as application of the theory.

2. Main Results

Let $B(t) = (B_1(t), B_2(t), \dots, B_m(t))^T$ be an m-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We consider the n-dimensional stochastic differential equation

$$dx(t) = f(x(t),t)dt + g(x(t),t)dB(t), \quad t \geq 0, \tag{2.1}$$

with initial condition $x(t_0) = x_0 \in \mathbb{R}^n$. Here $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in \mathbb{R}^n$, $t_0 \geq 0$, $f : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^{n\times m}$ are given nonlinear continuous functions such that (2.1) has a solution on some small interval. In what follows we use $x(t,t_0,x_0)$ or simply $x(t)$ for a solution of (2.1). We also use $\|x\|$ to denote the Euclidean norm for vector $x \in \mathbb{R}^n$.

Definition 2.1. A solution $x(t,t_0,x_0)$ to (2.1) is said to be stochastically bounded, or bounded in probability, if it satisfies

$$E^{x_0}\|x(t,t_0,x_0)\| \leq C(\|x_0\|,t_0), \quad \text{for all } t \geq t_0, \tag{2.2}$$

where E^{x_0} denotes the expectation operator with respect to the probability law associated with x_0, $C : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a constant depending on t_0 and x_0. We say that solutions of (2.1) are uniformly stochastically bounded if C is independent of t_0.

Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$ denote the family of non-negative functions $V(x,t)$ defined on $\mathbb{R}^n \times \mathbb{R}^+$ that are twice continuously differentiable in x and continuously differentiable in t. Define an operator $\mathcal{L}$ on functions in $C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$ by

$$\mathcal{L}V(x,t) = V_t(x,t) + \sum_{i=1}^{n} V_{x_i}(x,t) f_i(x,t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m} V_{x_i x_j}(x,t) g_{ik}(x,t) g_{jk}(x,t), \tag{2.3}$$

where f_i is the ith component of vector f and g_{ij} is the ij-entry of matrix g. Let $C(\mathbb{R}^+; \mathbb{R}^+)$ denote the family of continuous functions with non-negative domain and non-negative range.

In this article we assume that the function $V(x,t)$ chosen from $C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$ satisfies the following assumption.

Assumption 2.2. We assume that for any solution $x(t)$ of (2.1) and for any fixed $0 \le t_0 \le T < \infty$, the following conditions hold:

$$E^{x_0}\Big\{ \int_{t_0}^{T} V_{x_i}^2(x(t),t) g_{ik}^2(x(t),t)\, dt \Big\} < \infty, \quad 1 \le i \le n,\ 1 \le k \le m. \tag{2.4}$$

Remark 2.3. A special case of the general condition (2.4) is the following condition. Assume that there exists a function $\sigma(t)$ such that

$$|V_{x_i}(x,t) g_{ik}(x,t)| \le \sigma(t), \quad x \in \mathbb{R}^n,\ 1 \le i \le n,\ \ 1 \le k \le m, \tag{2.5}$$

and for any fixed $0 \le t_0 \le T < \infty$,

$$\int_{t_0}^{T} \sigma^2(t)\, dt < \infty. \tag{2.6}$$

We will present examples in which conditions (2.5) and (2.6) are satisfied.

Theorem 2.4. *Assume there exists a function $V(x,t)$ in $C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$ satisfying Assumption 2.2, such that for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$,*

$$W(\|x\|) \le V(x,t) \le \phi(\|x\|), \tag{2.7}$$

$$\mathcal{L}V(x,t) \le -\alpha(t)\psi(\|x\|) + \beta(t), \tag{2.8}$$

$$V(x,t) - \psi(\phi^{-1}(V(x,t))) \le \gamma, \tag{2.9}$$

where $W, \phi, \psi, \alpha, \beta \in C(\mathbb{R}^+; \mathbb{R}^+)$, W, ϕ, ψ are strictly increasing, W is convex, and γ is a non-negative constant. Then all solutions of (2.1) satisfy

$$E^{x_0}\|x(t,t_0,x_0)\| \le W^{-1}\big\{ V(x_0,t_0) e^{-\int_{t_0}^{t} \alpha(s)\, ds} + \int_{t_0}^{t} \big(\gamma\alpha(u) + \beta(u)\big) e^{-\int_{u}^{t} \alpha(s)\, ds}\, du \big\}, \tag{2.10}$$

for all $t \ge t_0$.

Proof. Let $x(t) := x(t,t_0,x_0)$ be any solution of (2.1). For notational brevity, let

$$dB^x(t) = \sum_{i=1}^{n} V_{x_i}(x(t),t) \sum_{k=1}^{m} g_{ik}(x(t),t) dB_k(t).$$

Applying Itô's formula to $e^{\int_{t_0}^t \alpha(s)\,ds}V(x(t),t)$ give

$$
\begin{aligned}
&d\Big(e^{\int_{t_0}^t \alpha(s)\,ds}V(x(t),t)\Big)\\
&= e^{\int_{t_0}^t \alpha(s)\,ds}\Big(\alpha(t)V(x(t),t)+\mathcal{L}V(x(t),t)\Big)dt + e^{\int_{t_0}^t \alpha(s)\,ds}dB^x(t)\\
&\le e^{\int_{t_0}^t \alpha(s)\,ds}\Big(\alpha(t)V(x(t),t)-\alpha(t)\psi(\|x(t)\|)+\beta(t)\Big)dt + e^{\int_{t_0}^t \alpha(s)\,ds}dB^x(t)\\
&\quad \text{by (2.8)}\\
&\le e^{\int_{t_0}^t \alpha(s)\,ds}\Big(\alpha(t)[V(x(t),t)-\psi(\phi^{-1}(V(x(t),t)))]+\beta(t)\Big)dt + e^{\int_{t_0}^t \alpha(s)\,ds}dB^x(t)\\
&\quad \text{by (2.7)}\\
&\le e^{\int_{t_0}^t \alpha(s)\,ds}\Big(\gamma\alpha(t)+\beta(t)\Big)dt + e^{\int_{t_0}^t \alpha(s)\,ds}dB^x(t) \quad \text{by (2.9)}.
\end{aligned}
$$

Integrating both sides from t_0 to t, we have

$$
\begin{aligned}
&e^{\int_{t_0}^t \alpha(s)\,ds}V(x(t),t)\\
&\le V(x_0,t_0)+\int_{t_0}^t e^{\int_{t_0}^u \alpha(s)\,ds}\Big(\gamma\alpha(u)+\beta(u)\Big)\,du + \int_{t_0}^t e^{\int_{t_0}^u \alpha(s)\,ds}dB^x(u).
\end{aligned}
$$

Dividing both sides by $e^{\int_{t_0}^t \alpha(s)\,ds}$,

$$
\begin{aligned}
V(x(t),t) \le e^{-\int_{t_0}^t \alpha(s)\,ds}V(x_0,t_0)&+\int_{t_0}^t e^{-\int_u^t \alpha(s)\,ds}\Big(\gamma\alpha(u)+\beta(u)\Big)\,du\\
&+\int_{t_0}^t e^{-\int_u^t \alpha(s)\,ds}dB^x(u).
\end{aligned}
\tag{2.11}
$$

Taking expectation of both sides and noting that $E^{x_0}\big\{\int_{t_0}^t e^{-\int_u^t \alpha(s)\,ds}dB^x(u)\big\}=0$, we have in view of (2.4) that

$$
E^{x_0}[V(x(t),t)] \le e^{-\int_{t_0}^t \alpha(s)\,ds}V(x_0,t_0)+\int_{t_0}^t e^{-\int_u^t \alpha(s)\,ds}\Big(\gamma\alpha(u)+\beta(u)\Big)\,du. \tag{2.12}
$$

Finally, since W is convex, by Jensen's Inequality for expectation, we have,

$$
W(E^{x_0}\|x(t)\|) \le E^{x_0}[W(\|x(t)\|)] \le E^{x_0}[V(x(t),t)],
$$

which, when combined with (2.12), yields (2.10). □

Theorem 2.5. *Assume there exists a function $V(x,t)$ in $C^{2,1}(\mathbb{R}^n\times\mathbb{R}^+;\mathbb{R}^+)$ satisfying Assumption 2.2, such that for all $(x,t)\in\mathbb{R}^n\times\mathbb{R}^+$,*

$$
\|x\|^p \le V(x,t) \le \|x\|^q, \tag{2.13}
$$

$$
\mathcal{L}V(x,t) \le -\alpha(t)\|x\|^r+\beta(t), \tag{2.14}
$$

$$
V(x,t)-V^{r/q}(x,t) \le \gamma, \tag{2.15}
$$

where $\alpha,\beta\in C(\mathbb{R}^+;\mathbb{R}^+)$, p, q, r are positive constants, $p\ge 1$, and γ is a non-negative constant. Then all solutions of (2.1) satisfy

$$
E^{x_0}\|x(t,t_0,x_0)\| \le \big\{V(x_0,t_0)e^{-\int_{t_0}^t \alpha(s)\,ds}+\int_{t_0}^t(\gamma\alpha(u)+\beta(u))\,e^{-\int_u^t \alpha(s)\,ds}\,du\big\}^{1/p}, \tag{2.16}
$$

for all $t\ge t_0$.

Proof. If we chose $W(x) = x^p$, $\phi(x) = x^q$ and $\psi(x) = x^r$, then the conditions (2.7)-(2.9) reduce to (2.13)-(2.15). Also note that x^p is convex for $p \geq 1$. It then follows from Theorem 2.4 that (2.16) holds. $\square$

The proof of the next theorem is similar to that of Theorem 2.4 and hence omitted.

Theorem 2.6. *Assume there exists a function $V(x,t)$ in $C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$ satisfying Assumption 2.2, such that for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$,*

$$\|x\|^p \leq V(x,t), \tag{2.17}$$

$$\mathcal{L}V(x,t) \leq -\alpha(t)V^q(x,t) + \beta(t), \tag{2.18}$$

$$V(x,t) - V^q(x,t) \leq \gamma, \tag{2.19}$$

where $\alpha, \beta \in C(\mathbb{R}^+; \mathbb{R}^+)$, p, q are positive constants, $p \geq 1$, and γ is a non-negative constant. Then all solutions of (2.1) *satisfy*

$$E^{x_0}\|x(t,t_0,x_0)\| \leq \left\{V(x_0,t_0)e^{-\int_{t_0}^t \alpha(s)\,ds} + \int_{t_0}^t (\gamma\alpha(u) + \beta(u))\, e^{-\int_u^t \alpha(s)\,ds}\,du\right\}^{1/p}, \tag{2.20}$$

for all $t \geq t_0$.

Definition 2.7. Suppose $f(0,t) = 0$ and $g(0,t) = 0$. We say that the zero solution of (2.1) is α-exponentially asymptotically stable in probability, if there exists a positive continuous function $\alpha(t)$ such that $\int_{t_0}^t \alpha(s)\,ds \to \infty$ as $t \to \infty$ and constants d, $C \in \mathbb{R}^+$ such that for any solution $x(t,t_0,x_0)$ of (2.1),

$$E^{x_0}\|x(t,t_0,x_0)\| \leq C(\|x_0\|, t_0)\Big(e^{-\int_{t_0}^t \alpha(s)\,ds}\Big)^d, \quad \text{for all } t \geq t_0, \tag{2.21}$$

where the constant C may depend on t_0 and x_0. The zero solution of (2.1) is said to be α-uniformly exponentially asymptotically stable in probability if C is independent of t_0.

Corollary 2.8. *(1) Assume either of the hypothesis Theorem 2.4 or Theorem 2.5 hold. In addition,*

$$\int_{t_0}^t (\gamma\alpha(u) + \beta(u))\, e^{-\int_u^t \alpha(s)\,ds}\,du \leq M, \quad \forall t \geq t_0 \geq 0, \tag{2.22}$$

for some positive constant M, then all solutions of (2.1) *are uniformly stochastically bounded.*

(2) Assume the hypothesis of Theorem 2.6 hold. If condition (2.22) is satisfied, then all solutions of (2.1) *are stochastically bounded.*

Proof. If the hypothesis of Theorem 2.4 hold, then by (2.10) and (2.22), we have

$$E^{x_0}\|x(t)\| \leq W^{-1}\left\{V(x_0,t_0)e^{-\int_{t_0}^t \alpha(s)\,ds} + M\right\} \leq W^{-1}\{\phi(\|x_0\|) + M\} := C(\|x_0\|),$$

by (2.7). If the hypothesis of Theorem 2.5 hold, then by (2.16) and (2.22), we have

$$E^{x_0}\|x(t)\| \leq \left\{V(x_0,t_0)e^{-\int_{t_0}^t \alpha(s)\,ds} + M\right\}^{1/p} \leq \{\|x_0\|^q + M\}^{1/p} := C(\|x_0\|),$$

by (2.13). By Definition 2.1, all solutions of (2.1) are uniformly stochastically bounded in either case.

If the hypothesis of Theorem 2.6 hold, then by (2.20) and (2.22), we have

$$E^{x_0}\|x(t)\| \le \big\{V(x_0,t_0)e^{-\int_{t_0}^t \alpha(s)\,ds} + M\big\}^{1/p} \le \{V(x_0,t_0)+M\}^{1/p} := C(\|x_0\|,t_0),$$

which implies that all solutions of (2.1) are stochastically bounded. □

Corollary 2.9. *Assume $f(0,t)=0$ and $g(0,t)=0$. Assume*

$$\int_{t_0}^t \big(\gamma\alpha(u)+\beta(u)\big)e^{\int_{t_0}^u \alpha(s)\,ds}\,du \le M, \quad \forall t \ge t_0 \ge 0, \tag{2.23}$$

for some positive constant M, and

$$\int_{t_0}^t \alpha(s)\,ds \to \infty, \quad \text{as } t\to\infty. \tag{2.24}$$

(1) If the hypothesis of Theorem 2.5 hold, then the zero solution of (2.1) is α-uniformly exponentially asymptotically stable in probability with $d=1/p$.

(2) If the hypothesis of Theorem 2.6 hold, then the zero solution of (2.1) is α-exponentially asymptotically stable in probability with $d=1/p$.

Proof. From (2.12) we have

$$E^{x_0}[V(x(t),t)] \le \big\{V(x_0,t_0) + \int_{t_0}^t e^{\int_{t_0}^u \alpha(s)\,ds}\big(\gamma\alpha(u)+\beta(u)\big)\,du\big\}e^{-\int_{t_0}^t \alpha(s)\,ds}.$$

Then by (2.23),

$$E^{x_0}[V(x(t),t)] \le \big\{V(x_0,t_0)+M\big\}e^{-\int_{t_0}^t \alpha(s)\,ds}. \tag{2.25}$$

Note that x^p is convex for $p\ge 1$. By Jensen's Inequality, we have $(E^{x_0}\|x(t)\|)^p \le E^{x_0}(\|x(t)\|^p) \le E^{x_0}[V(x(t),t)]$. It follows that

$$E^{x_0}\|x(t)\| \le \big\{V(x_0,t_0)+M\big\}^{1/p}\Big(e^{-\int_{t_0}^t \alpha(s)\,ds}\Big)^{1/p} := C(\|x_0\|,t_0)\Big(e^{-\int_{t_0}^t \alpha(s)\,ds}\Big)^{1/p}, \tag{2.26}$$

which implies that the zero solution of (2.1) is α-exponentially asymptotically stable in probability with $d=1/p$.

If the hypothesis of Theorem 2.5 hold, then using (2.13) in (2.26), we can further obtain

$$E^{x_0}\|x(t)\| \le \big\{\|x_0\|^q + M\big\}^{1/p}\Big(e^{-\int_{t_0}^t \alpha(s)\,ds}\Big)^{1/p} := C(\|x_0\|)\Big(e^{-\int_{t_0}^t \alpha(s)\,ds}\Big)^{1/p}.$$

In this case, the zero solution of (2.1) is α-uniformly exponentially asymptotically stable in probability with $d=\frac{1}{p}$. □

3. Examples

In this section we provide three examples to illustrate the application of the results we obtained in the previous section.

Example 1. For $a(t), b(t), h(t) \geq 0$, consider the scalar stochastic differential equation

$$dx(t) = \big[-\big(a(t)+\frac{7}{6}\big)x(t)+b(t)x^{\frac{1}{3}}(t)+h(t)\big]dt+g(x(t),t)dB(t), \quad t \geq t_0 \geq 0, \tag{3.1}$$

with initial condition $x(t_0) = x_0$, where $B(t)$ is a one-dimensional standard Brownian motion.

Let $V(x,t) = x^2$. Then along any solution $x := x(t)$ of (3.1), we have

$$\begin{aligned} \mathcal{L}V(x,t) &= V_x f + \frac{1}{2}V_{xx}g^2(x,t) \\ &= -2\big(a(t)+\frac{7}{6}\big)x^2 + 2b(t)x^{4/3} + 2xh(t) + g^2(x,t) \\ &\leq -2\big(a(t)+\frac{7}{6}\big)x^2 + 2b(t)x^{4/3} + x^2 + h^2(t) + g^2(x,t). \end{aligned}$$

To further simplify $\mathcal{L}V(x,t)$, we make use of Young's inequality, which says for any two nonnegative real numbers w and z, we have

$$wz \leq \frac{w^e}{e} + \frac{z^f}{f}, \quad \text{with } \frac{1}{e} + \frac{1}{f} = 1.$$

Thus, for $e = 3/2$ and $f = 3$, we obtain

$$2|b(t)|x^{4/3} \leq 2\Big[\frac{1}{3}|b(t)|^3 + \frac{(x^{4/3})^{3/2}}{3/2}\Big] = \frac{4}{3}x^2 + \frac{2}{3}|b(t)|^3.$$

As a result, we have

$$\mathcal{L}V(x,t) \leq -2a(t)x^2 + \frac{2}{3}b^3(t) + h^2(t) + g^2(x,t). \tag{3.2}$$

Let $g(x,t) = t^{1/2}x/(x^2+1)$. Then

$$V_x g(x,t) = 2t^{1/2}\frac{x^2}{x^2+1} \leq 2t^{1/2} := \sigma(t).$$

Hence conditions (2.5) and (2.6) are satisfied. We also have $g^2(x,t) = tx^2/(x^2+1)^2 \leq t/4$. It follows that

$$\mathcal{L}V(x,t) \leq -\alpha(t)x^2 + \beta(t), \tag{3.3}$$

where $\alpha(t) = 2a(t)$, $\beta(t) = \frac{2}{3}b^3(t) + h^2(t) + \frac{t}{4}$. We can easily check that conditions (2.13)-(2.15) of Theorem 2 are satisfied with $p = q = r = 2$ and $\gamma = 0$. Specifically, let $a(t) = \frac{t}{2}$, $b(t) = t^{\frac{1}{3}}$, and $h(t) = t^{1/2}$. Then, $\alpha(t) = t$, $\beta(t) = \frac{23}{12}t$ and (3.3) becomes

$$\mathcal{L}V(x,t) \leq -tx^2 + \frac{23}{12}t.$$

We note that

$$\int_{t_0}^t \big(\gamma\alpha(u) + \beta(u)\big)e^{-\int_u^t \alpha(s)ds}\,du = \frac{23}{12}\int_{t_0}^t ue^{-\int_u^t s ds}\,du \leq \frac{23}{12},$$

for all $t \geq t_0 \geq 0$. Thus condition (2.22) holds. By Corollary 2.8 all solutions of

$$\begin{gathered} dx(t) = \big[-\big(\frac{t}{2}+\frac{7}{6}\big)x(t) + t^{\frac{1}{3}}x^{\frac{1}{3}}(t) + t^{1/2}\big]dt + \frac{t^{1/2}x(t)}{x^2(t)+1}dB(t), \\ x(t_0) = x_0,\ t \geq t_0 \geq 0, \end{gathered} \tag{3.4}$$

are uniformly stochastically bounded and satisfy

$$E^{x_0}\|x(t,t_0,x_0)\| \le \big\{x_0^2 + \frac{23}{12}\big\}^{1/2}, \quad \forall t \ge t_0 \ge 0.$$

Next, if we take $a(t) = t/2$, $b(t) = e^{-\kappa_1 t/3}$, $h(t) = 0$, and $g(x,t) = e^{-\kappa_2 T/2}x/(x^2+1)$, $\kappa_1, \kappa_2 > 1$, then conditions (2.5) and (2.6) are satisfied. We have $\alpha(t) = t$ and $\beta(t) = \frac{2}{3}e^{-\kappa_1 t} + \frac{1}{4}e^{-\kappa_2 t}$ for inequality (3.3). Using them in (2.23), we have

$$\begin{aligned}\int_{t_0}^{t}\big(\gamma\alpha(u)+\beta(u)\big)e^{\int_{t_0}^{u}\alpha(s)\,ds}\,du &= \int_{t_0}^{t}\big(\frac{2}{3}e^{-\kappa_1 t}+\frac{1}{4}e^{-\kappa_2 t}\big)e^{\int_{t_0}^{u}ds}\,du\\ &\le \frac{2}{3(\kappa_1-1)}+\frac{1}{4(\kappa_2-1)},\end{aligned}$$

for all $t \ge t_0 \ge 0$. Hence condition (2.23) is satisfied. We can easily see that condition (2.24) is also satisfied. By Corollary 2.9 we know that the zero solution of

$$\begin{gathered} dx(t) = \big[-(\frac{t}{2}+\frac{7}{6})x(t) + e^{-\frac{\kappa_1}{3}t}x^{\frac{1}{3}}(t)\big]dt + e^{-\frac{\kappa_2}{2}t}\frac{x(t)}{x^2(t)+1}dB(t),\\ x(t_0) = x_0,\ t \ge t_0 \ge 0, \end{gathered} \tag{3.5}$$

is α-uniformly exponentially asymptotically stable in probability with $d = 1/2$.

We note that in Example 1 the condition (2.15) did not come into play due to the fact that $r = q = 2$. In the next example, we consider a non-linear stochastic differential equation in which the condition (2.15) naturally comes into play.

Example 2. Consider the stochastic differential equation

$$dx(t) = \big[-a(t)x^3(t) + b(t)x^{\frac{1}{3}}(t) + h(t)\big]dt + g(x(t),t)dB(t), \quad t \ge t_0 \ge 0, \tag{3.6}$$

with initial condition $x(t_0) = x_0$, where function $a(t)$, $b(t)$, $h(t)$ and $g(x,t)$ will be specified later. Use the same function $V(x,t) = x^2$ as in Example 1. Then along any solution $x := x(t)$ of (3.6), we have

$$\mathcal{L}V(x,t) = -2a(t)x^4 + 2b(t)x^{4/3} + 2xh(t) + g^2(x,t).$$

Using Young's inequality for term $b(t)x^{4/3}$ with $e = 3$ and $f = \frac{3}{2}$, we have

$$|b(t)||x|^{4/3} \le \frac{x^4}{3} + \frac{2}{3}|b(t)|^{3/2}.$$

Using Young's inequality for term $xh(t)$ with $e = 4$ and $f = 4/3$, we have

$$|x||h(t)| \le \frac{x^4}{4} + \frac{3}{4}|h(t)|^{4/3}.$$

It follows that

$$\mathcal{L}V(x,t) \le \big(-2a(t) + \frac{7}{6}\big)x^4 + \frac{4}{3}|b(t)|^{3/2} + \frac{3}{2}|h(t)|^{4/3} + g^2(x,t). \tag{3.7}$$

Let $g(x,t) = t^{1/2}x/(x^2+1)$. Then

$$\mathcal{L}V(x,t) \le -\alpha(t)x^4 + \beta(t), \tag{3.8}$$

where $\alpha(t) = 2a(t) - \frac{7}{6}$, $\beta(t) = \frac{4}{3}|b(t)|^{3/2} + \frac{3}{2}|h(t)|^{4/3} + \frac{1}{4}t$. Hence we have $p = q = 2$ and $r = 4$.

Note that

$$V(x,t) - V^{r/q}(x,t) = x^2(1-x^2) \le \frac{1}{4}, \quad \text{for } x \in \mathbb{R}.$$

Hence condition (2.15) is satisfied with constant $\gamma = \frac{1}{4}$. To ensure that $\alpha(t) \geq 0$ we need $a(t) \geq \frac{7}{12}$. Specifically, let $a(t) = \frac{t}{2} + \frac{7}{12}$, $b(t) = t^{\frac{2}{3}}$, and $h(t) = t^{\frac{3}{4}}$. Then we have $\alpha(t) = t$, $\beta(t) = \frac{37}{12}t$ and (3.8) becomes

$$\mathcal{L}V(x,t) \leq -tx^4 + \frac{37}{12}t.$$

Consequently, (2.16) implies

$$E^{x_0}\|x(t,t_0,x_0)\| \leq \big\{x_0^2 e^{-\int_{t_0}^t s\,ds} + \int_{t_0}^t \big(\frac{1}{4}u + \frac{37}{12}u\big)e^{-\int_u^t s\,ds}\,du\big\}^{1/2}, \tag{3.9}$$

which yields

$$E^{x_0}\|x(t,t_0,x_0)\| \leq \big\{x_0^2 + \frac{10}{3}\big\}^{1/2}, \quad \forall t \geq t_0 \geq 0. \tag{3.10}$$

Therefore, all solutions of

$$\begin{gathered} dx(t) = \big[-\big(\frac{t}{2} + \frac{7}{12}\big)x^3(t) + t^{\frac{2}{3}}x^{\frac{1}{3}}(t) + t^{\frac{3}{4}}\big]dt + \frac{t^{1/2}x(t)}{x^2(t)+1}dB(t), \\ x(t_0) = x_0,\ t \geq t_0 \geq 0, \end{gathered} \tag{3.11}$$

are uniformly stochastically bounded.
As an application of Theorem 2.6, we consider a two-dimensional system in the next example.

Example 3. Let $x(t) = (x_1(t), x_2(t))^T \in \mathbb{R}^2$. For $a(t) > 0$, consider:

$$\begin{aligned} dx_1(t) &= \big[x_2(t) - a(t)x_1(t)|x_1(t)| + \frac{x_1(t)h_1(t)}{1+x_1^2(t)}\big]dt \\ &\quad + g_{11}(x(t),t)dB_1(t) + g_{12}(x(t),t)dB_2(t), \\ dx_2(t) &= \big[-x_1(t) - a(t)x_2(t)|x_2(t)| + \frac{x_2(t)h_2(t)}{1+x_2^2(t)}\big]dt \\ &\quad + g_{21}(x(t),t)dB_1(t) + g_{22}(x(t),t)dB_2(t), \end{aligned} \tag{3.12}$$

with initial condition $x(t_0) = x_0 \in \mathbb{R}^2$. Let $V(x,t) = x_1^2 + x_2^2$. We assume that the functions g_{ij}, $i,j = 1,2$ satisfy the conditions (2.5) and (2.6), and

$$g^2(x,t) := g_{11}^2(x,t) + g_{12}^2(x,t) + g_{21}^2(x,t) + g_{22}^2(x,t) \leq M(t).$$

Along the solution $(x_1,x_2) := (x_1(t),x_2(t))$ of (3.12), we have

$$\begin{aligned} \mathcal{L}V(x,t) &= -2a(t)(|x_1|^3 + |x_2|^3) + \frac{2x_1^2h_1(t)}{1+x_1^2} + \frac{2x_2^2h_2(t)}{1+x_2^2} + g^2(x,t) \\ &\leq -4a(t)\big(\frac{|x_1|^3}{2} + \frac{|x_2|^3}{2}\big) + 2(|h_1(t)| + |h_2(t)|) + M(t). \end{aligned}$$

Using the inequality

$$\big(\frac{a+b}{2}\big)^l \leq \frac{a^l}{2} + \frac{b^l}{2}, \quad a,b > 0,\ l > 1,$$

we have

$$\begin{aligned} \mathcal{L}V(x,t) &\leq -4a(t)\frac{(x_1^2+x_2^2)^{3/2}}{2^{3/2}} + 2(|h_1(t)| + |h_2(t)|) + M(t) \\ &\leq -\sqrt{2}a(t)V^{3/2}(x,t) + 2(|h_1(t)| + |h_2(t)|) + M(t). \end{aligned}$$

Thus $p = 2$, $q = \frac{3}{2}$. Condition (2.18) is satisfied with $\alpha(t) = \sqrt{2}a(t)$ and $\beta(t) = 2(|h_1(t)|+|h_2(t)|)+M(t)$. To check condition (2.19), we note that for $(x_1, x_2) \in \mathbb{R}^2$,

$$V(x,t) - V^{3/2}(x,t) = x_1^2 + x_2^2 - (x_1^2 + x_2^2)^{3/2} \le \frac{4}{27}.$$

Thus (2.19) is satisfied with $\gamma = \frac{4}{27}$. From Theorem 2.6, we conclude that all solutions of (3.12) satisfy (2.20). Specifically,

$$\begin{aligned} E^{x_0}\|x(t,t_0,x_0)\| \le \Big\{ \|x_0\|^2 e^{-\sqrt{2}\int_{t_0}^t a(s)\,ds} + \int_{t_0}^t \Big(\frac{4\sqrt{2}}{27} a(u) + 2(|h_1(u)| + |h_2(u)|) \\ + M(u)\Big) e^{-\sqrt{2}\int_u^t a(s)\,ds}\,du \Big\}^{1/2}, \end{aligned}$$

for all $t \ge t_0 \ge 0$.

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