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PRECISE ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO DAMPED SIMPLE PENDULUM EQUATIONS

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ABSTRACT. We consider the simple pendulum equation

$$-u''(t) + \epsilon f(u'(t)) = \lambda \sin u(t), \quad t \in I := (-1, 1),$$
$$u(t) > 0, \quad t \in I, \quad u(\pm 1) = 0,$$

where $0 < \epsilon \le 1$, $\lambda > 0$, and the friction term is either $f(y) = \pm|y|$ or $f(y) = -y$. Note that when $f(y) = -y$ and $\epsilon = 1$, we have well known original damped simple pendulum equation. To understand the dependance of solutions, to the damped simple pendulum equation with $\lambda \gg 1$, upon the term $f(u'(t))$, we present asymptotic formulas for the maximum norm of the solutions. Also we present an asymptotic formula for the time at which maximum occurs, for the case $f(u) = -u$.

1. Introduction

We consider the damped simple pendulum equation

$$-u''(t) + \epsilon f(u'(t)) = \lambda \sin u(t), \quad t \in I := (-1, 1), \tag{1.1}$$
$$u(t) > 0, \quad t \in I, \tag{1.2}$$
$$u(\pm 1) = 0, \tag{1.3}$$

where $0 < \epsilon \le 1$, $\lambda > 0$, and the damping term is either $f(y) = \pm|y|$ or $f(y) = -y$. It is known that there exists a solution $u_{\epsilon,\lambda}$ to (1.1)–(1.3) for $0 < \epsilon \le 1$ and $\lambda \gg 1$, with $\|u_{\epsilon,\lambda}\|_\infty < \pi$; see for example [1].

The purpose of this paper is to study the asymptotic behavior of $u_{\epsilon,\lambda}(t)$ as $\lambda \to \infty$; this is useful for understanding the effect of the damping term on the asymptotic behavior of $u_{\epsilon,\lambda}$. First, we recall some properties of the solution $u_{0,\lambda}$ for the simple pendulum equation without friction (i.e. the case where $\epsilon = 0$):

$$-u''(t) = \lambda \sin u(t), \quad t \in I, \tag{1.4}$$
$$u(t) > 0, \quad t \in I, \tag{1.5}$$
$$u(\pm 1) = 0. \tag{1.6}$$

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It is well known that $u_{0,\lambda} \to \pi$ locally uniformly in I as $\lambda \to \infty$. Furthermore (cf. Lemma 2.1 in Section 2), as $\lambda \to \infty$,

$$\|u_{0,\lambda}\|_\infty = \pi - 8e^{-\sqrt{\lambda}} - 32\sqrt{\lambda}e^{-3\sqrt{\lambda}} + o(\sqrt{\lambda}e^{-3\sqrt{\lambda}}). \tag{1.7}$$

It should be mentioned that the asymptotic behavior of solutions to the original and perturbed simple pendulum problems have been studied in [5, 6, 7]. We also refer the reader to [3] for the basic properties of the solution to simple pendulum problems. As far as the author knows, there are only a few works concerning the precise properties of solutions to (1.1)–(1.3). In particular, an asymptotic formulas such as (1.7) for $\|u_{\epsilon,\lambda}\|_\infty$ has not been obtained yet. Therefore, it seems worth considering the precise asymptotic behavior of $\|u_{\epsilon,\lambda}\|_\infty$ as $\lambda \to \infty$, for having a better understanding of the effect of the friction term.

Now we state our main results. We denote by $u_{1,\epsilon,\lambda}$, $u_{2,\epsilon,\lambda}$ and $u_{3,\epsilon,\lambda}$ the solutions of (1.1)–(1.3) with $f(y) = -|y|$, $f(y) = |y|$ and $f(y) = -y$, respectively.

Theorem 1.1. *Let $f(y) = -|y|$ and let $0 < \epsilon \le 1$ be fixed. Then, as $\lambda \to \infty$,*

$$\|u_{1,\epsilon,\lambda}\|_\infty = \pi - 8e^{-\epsilon}e^{-\sqrt{\lambda}} + O(\lambda^{-1/2}e^{-\sqrt{\lambda}}). \tag{1.8}$$

Since $u_{1,\epsilon,\lambda}$ is a super-solution of (1.4)–(1.6), (1.8) is well understood and reasonable from a viewpoint of (1.7). Moreover, the formula (1.8) gives us the clear relationship between $\|u_{0,\lambda}\|_\infty$ and $\|u_{1,\epsilon,\lambda}\|_\infty$.

The following result can be proved by the same arguments as those used in the proof of Theorem 1.1.

Theorem 1.2. *Let $f(y) = |y|$ and $0 < \epsilon \le 1$ be fixed. Then, as $\lambda \to \infty$,*

$$\|u_{2,\epsilon,\lambda}\|_\infty = \pi - 8e^{\epsilon}e^{-\sqrt{\lambda}} + O(\lambda^{-1/4}e^{-\sqrt{\lambda}}). \tag{1.9}$$

We also note that $u_{2,\epsilon,\lambda}$ is a sub-solution of (1.4)–(1.6), (1.9) is also reasonable result.

Now we consider the case $f(y) = -y$. Let $0 < \epsilon \le 1$ be fixed. Let $t_{\epsilon,\lambda} \in I$ be the unique point satisfying $u_{3,\epsilon,\lambda}(t_{\epsilon,\lambda}) = \|u_{3,\epsilon,\lambda}\|_\infty$. Then we know from [2] that $t_{\epsilon,\lambda} < 0$ for $\lambda \gg 1$.

Theorem 1.3. *Let $f(y) = -y$. Then, as $\lambda \to \infty$,*

$$t_{\epsilon,\lambda} = -\frac{\epsilon}{\sqrt{\lambda}} + O\big(\lambda^{-3/4}\big), \tag{1.10}$$

$$\|u_{3,\epsilon,\lambda}\|_\infty = \pi - 8e^{-\sqrt{\lambda}} + O(\lambda^{-1/4}e^{-\sqrt{\lambda}}). \tag{1.11}$$

By (1.10), we obtain a precise asymptotic formula for $t_{\epsilon,\lambda}$ as $\lambda \to \infty$. Moreover, since the second term of (1.11) is the same as that of (1.7), the friction term does not have any effect on the second term of $\|u_{3,\epsilon,\lambda}\|_\infty$.

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.1 based on the crucial tool Lemma 2.2, which will be proved in Section 3. We prove Theorem 1.2 in Section 4 by almost the same argument as that to prove Theorem 1.1. We apply the modified argument for the proof of Theorem 1.1 to the proof of Theorem 1.3 in Section 5.

2. Proof of Theorem 1.1

In the following two sections, we let $f(y) = -|y|$. We fix $0 < \epsilon \le 1$. Further, we assume that $\lambda \gg 1$ and we write $u_{\epsilon,\lambda} = u_{1,\epsilon,\lambda}$ for simplicity. We consider the solution $u_{\epsilon,\lambda}(t)$ with $\|u_{\epsilon,\lambda}\|_\infty < \pi$. We know

$$u_{\epsilon,\lambda}(t) = u_{\epsilon,\lambda}(-t), \quad t \in I, \tag{2.1}$$

$$u'_{\epsilon,\lambda}(t) > 0, \quad t \in [-1, 0), \tag{2.2}$$

$$u_{\epsilon,\lambda}(0) = \|u_{\epsilon,\lambda}\|_\infty, \tag{2.3}$$

$$u_{\epsilon,\lambda}(t) \to \pi \quad \text{as } \lambda \to \infty, \quad (t \in I). \tag{2.4}$$

Note that (2.1)-(2.3) follow from [2]. (2.4) is a direct consequence of (1.7), (2.3) and (2.6) below.

By (1.1) and (2.2), for $-1 \le t \le 0$, we have

$$\{u''_{\epsilon,\lambda}(t) + \epsilon u'_{\epsilon,\lambda}(t) + \lambda \sin u_{\epsilon,\lambda}(t)\} u'_{\epsilon,\lambda}(t) = 0.$$

By this equality and (2.3), for $-1 \le t \le 0$, we have

$$\begin{aligned} &\frac{1}{2} u'_{\epsilon,\lambda}(t)^2 + \epsilon \int_{-1}^{t} |u'_{\epsilon,\lambda}(s)|^2 ds - \lambda \cos u_{\epsilon,\lambda}(t) \\ &= \epsilon \int_{-1}^{0} |u'_{\epsilon,\lambda}(s)|^2 ds - \lambda \cos \|u_{\epsilon,\lambda}\|_\infty = \text{constant}. \end{aligned} \tag{2.5}$$

For $-1 \le t \le 0$, we obtain

$$\frac{1}{2} u'_{\epsilon,\lambda}(t)^2 = \lambda(\cos u_{\epsilon,\lambda}(t) - \cos \|u_{\epsilon,\lambda}\|_\infty) + \epsilon \int_{t}^{0} |u'_{\epsilon,\lambda}(s)|^2 ds. \tag{2.6}$$

For $-1 \le t \le 0$, we put

$$A(\theta) := A_\lambda(\theta) = \lambda(\cos\theta - \cos \|u_{\epsilon,\lambda}\|_\infty), \tag{2.7}$$

$$B(t) := B_\lambda(t) = \int_{t}^{0} |u'_{\epsilon,\lambda}(s)|^2 ds. \tag{2.8}$$

Then by (2.2) and (2.6)–(2.8), for $-1 \le t \le 0$,

$$u'_{\epsilon,\lambda}(t) = \sqrt{2(A(u_{\epsilon,\lambda}(t)) + \epsilon B(t))}. \tag{2.9}$$

Then

$$1 = \int_{-1}^{0} dt = \frac{1}{\sqrt{2}} \int_{-1}^{0} \frac{u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t)) + \epsilon B(t)}} dt = \frac{1}{\sqrt{2}}(I + II), \tag{2.10}$$

where

$$I = \int_{-1}^{0} \frac{u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t))}} dt, \tag{2.11}$$

$$\begin{aligned} II &= \int_{-1}^{0} \frac{u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t)) + \epsilon B(t)}} dt - \int_{-1}^{0} \frac{u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t))}} dt \\ &= \int_{-1}^{0} \frac{-\epsilon B(t) u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t))}\sqrt{A(u_{\epsilon,\lambda}(t)) + \epsilon B(t)}} \\ &\quad \times \frac{1}{\left(\sqrt{A(u_{\epsilon,\lambda}(t))} + \sqrt{A(u_{\epsilon,\lambda}(t)) + \epsilon B(t)}\right)} dt. \end{aligned} \tag{2.12}$$

Lemma 2.1. *Let $d_\lambda := \pi - \|u_{\epsilon,\lambda}\|_\infty$. Then, as $\lambda \to \infty$,*

$$I = \sqrt{\frac{2}{\lambda}}\Big(\log\frac{4}{\sin(d_\lambda/2)} + \frac{1}{4}(1+o(1))\big(\log\frac{4}{\sin(d_\lambda/2)}\big)\sin^2\frac{d_\lambda}{2}\Big). \tag{2.13}$$

Proof. Put $\theta = u_{\epsilon,\lambda}(t)$. Then

$$\begin{aligned} I &= \frac{1}{\sqrt{\lambda}}\int_0^{\|u_{\epsilon,\lambda}\|_\infty}\frac{1}{\sqrt{\cos\theta - \cos\|u_{\epsilon,\lambda}\|_\infty}}d\theta \\ &= \frac{\sqrt{2}}{\sqrt{\lambda}\sin(\|u_{\epsilon,\lambda}\|_\infty/2)}\int_0^{\|u_{\epsilon,\lambda}\|_\infty/2}\frac{1}{\sqrt{1-\sin^2\varphi/\sin^2(\|u_{\epsilon,\lambda}\|_\infty/2)}}d\varphi \\ &= \sqrt{\frac{2}{\lambda}}\int_0^{\pi/2}\frac{1}{\sqrt{1-\sin^2(\|u_{\epsilon,\lambda}\|_\infty/2)\sin^2\phi}}d\phi \\ &= \sqrt{\frac{2}{\lambda}}K(k), \end{aligned} \tag{2.14}$$

where K is the complete elliptic integral of the first kind and $k = \sin(\|u_{\epsilon,\lambda}\|_\infty/2)$. Then by [4], we have

$$K(k) = \log\frac{4}{k'} + \frac{1}{4}\big(\log\frac{4}{k'}\big)k'^2(1+o(1)), \tag{2.15}$$

where $k' = \sqrt{1-k^2} = \cos(\|u_{\epsilon,\lambda}\|_\infty/2) = \cos((\pi - d_\lambda)/2) = \sin(d_\lambda/2)$. By this and (2.14), we obtain (2.13). Thus the proof is complete. $\square$

Since $II < 0$, by (2.10), (2.15) and Lemma 2.1, we have

$$1 < \frac{1}{\sqrt{2}}I \le \frac{1}{\sqrt{\lambda}}\big(1 + C\sin^2\frac{d_\lambda}{2}\big)\log\frac{4}{\sin(d_\lambda/2)}. \tag{2.16}$$

Then

$$\sin\frac{d_\lambda}{2} \le 4(1+o(1))e^{-\sqrt{\lambda}}, \quad \frac{d_\lambda}{2} \le 4(1+o(1))e^{-\sqrt{\lambda}}, \quad \sin\|u_{\epsilon,\lambda}\|_\infty \le 8(1+o(1))e^{-\sqrt{\lambda}}. \tag{2.17}$$

Lemma 2.2. *As $\lambda \to \infty$,*

$$II = \frac{\sqrt{2}\epsilon}{\lambda}\log\big(\sin\frac{d_\lambda}{2}\big) + O(\lambda^{-1}). \tag{2.18}$$

The proof of the above lemma will be given in Section 3. We accept Lemma 2.2 tentatively to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemmas 2.1 and 2.2 and (2.17), we have

$$\begin{aligned} 1 = \frac{1}{\sqrt{2}}(I + II) &= \frac{1}{\sqrt{\lambda}}\Big(\log\frac{4}{\sin(d_\lambda/2)} + \frac{1}{4}(1+o(1))\sin^2\frac{d_\lambda}{2}\log\frac{4}{\sin(d_\lambda/2)}\Big) \\ &\quad + \frac{\epsilon}{\lambda}\log\sin\frac{d_\lambda}{2} + O(\lambda^{-1}) \\ &= \frac{1}{\sqrt{\lambda}}\big(\log 4 - \log\sin\frac{d_\lambda}{2}\big) + \frac{\epsilon}{\lambda}\log\sin\frac{d_\lambda}{2} + O(\lambda^{-1}). \end{aligned} \tag{2.19}$$

This implies

$$\big(1-\frac{\epsilon}{\sqrt{\lambda}}\big)\log\sin\frac{d_\lambda}{2}=\log 4-\sqrt{\lambda}+O(\lambda^{-1/2}). \tag{2.20}$$

By this,

$$\begin{aligned}\log\sin\frac{d_\lambda}{2}&=\big(1+\frac{\epsilon}{\sqrt{\lambda}}+O(\lambda^{-1})\big)\big(\log 4-\sqrt{\lambda}+O(\lambda^{-1/2})\big)\\&=-\sqrt{\lambda}+\log 4-\epsilon+O(\lambda^{-1/2}).\end{aligned} \tag{2.21}$$

By this and Taylor expansion,

$$\sin\frac{d_\lambda}{2}=\frac{d_\lambda}{2}\big(1-\frac{d_\lambda^2}{3}+o(d_\lambda^2)\big)=4e^{-\epsilon}e^{-\sqrt{\lambda}}(1+O(\lambda^{-1/2})).$$

By this and (2.17), we obtain Theorem 1.1. $\square$

3. Proof of Lemma 2.2

In this section, we focus our attention on the proof of Lemma 2.2. Let $0<\delta\ll 1$ be fixed. We define $t_\delta:=t_{\lambda,\delta}<0$ by $u_{\epsilon,\lambda}(t_\delta)=\|u_{\epsilon,\lambda}\|_\infty-\delta$. We set

$$\begin{aligned}&II=II_1+II_2\\&:=\int_{t_\delta}^0\frac{-\epsilon B(t)u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t))}\sqrt{A(u_{\epsilon,\lambda}(t))+\epsilon B(t)}(\sqrt{A(u_{\epsilon,\lambda}(t))}+\sqrt{A(u_{\epsilon,\lambda}(t))+\epsilon B(t)})}dt\\&+\int_{-1}^{t_\delta}\frac{-\epsilon B(t)u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t))}\sqrt{A(u_{\epsilon,\lambda}(t))+\epsilon B(t)}\big(\sqrt{A(u_{\epsilon,\lambda}(t))}+\sqrt{A(u_{\epsilon,\lambda}(t))+\epsilon B(t)}\big)}dt.\end{aligned} \tag{3.1}$$

To obtain Lemma 2.2, we estimate II_1 and II_2 by series of lemmas.

Lemma 3.1. *For $-1\le t\le 0$,*

$$B(t)\le\sqrt{2A(u_{\epsilon,\lambda}(t))}(\|u_{\epsilon,\lambda}\|_\infty-u_{\epsilon,\lambda}(t))+2\epsilon(\|u_{\epsilon,\lambda}\|_\infty-u_{\epsilon,\lambda}(t))^2. \tag{3.2}$$

Proof. Since $\|u_{\epsilon,\lambda}\|_\infty<\pi$, we see from (1.1) that $u''_{\epsilon,\lambda}(t)\le 0$ for $t\in I$. This along with (2.8) and (2.9) implies that for $-1\le t\le 0$,

$$\begin{aligned}0&<B(t)\\&\le\max_{t\le s\le 0}|u'_{\epsilon,\lambda}(s)|\int_t^0 u'_{\epsilon,\lambda}(s)ds\\&=u'_{\epsilon,\lambda}(t)(\|u_{\epsilon,\lambda}\|_\infty-u_{\epsilon,\lambda}(t))\\&=\sqrt{2A(u_{\epsilon,\lambda}(t))+2\epsilon B(t)}(\|u_{\epsilon,\lambda}\|_\infty-u_{\epsilon,\lambda}(t)).\end{aligned} \tag{3.3}$$

By (3.3),

$$B(t)^2-2\epsilon B(t)(\|u_{\epsilon,\lambda}\|_\infty-u_{\epsilon,\lambda}(t))^2-2A(u_{\epsilon,\lambda}(t))(\|u_{\epsilon,\lambda}\|_\infty-u_{\epsilon,\lambda}(t))^2\le 0.$$

Since $\sqrt{a+b}\le\sqrt{a}+\sqrt{b}$ for $a,b\ge 0$, by this, we obtain

$$\begin{aligned}B(t)&\le\epsilon(\|u_{\epsilon,\lambda}\|_\infty-u_{\epsilon,\lambda}(t))^2\\&\quad+\sqrt{\epsilon^2(\|u_{\epsilon,\lambda}\|_\infty-u_{\epsilon,\lambda}(t))^4+2A(u_{\epsilon,\lambda}(t))(\|u_{\epsilon,\lambda}\|_\infty-u_{\epsilon,\lambda}(t))^2}\\&\le 2\epsilon(\|u_{\epsilon,\lambda}\|_\infty-u_{\epsilon,\lambda}(t))^2+\sqrt{2A(u_{\epsilon,\lambda}(t))}(\|u_{\epsilon,\lambda}\|_\infty-u_{\epsilon,\lambda}(t)).\end{aligned} \tag{3.4}$$

The proof is complete. $\square$

By Taylor expansion, for $t_\delta \le t \le 0$ and $0 < \kappa \ll 1$,

$$
\begin{aligned}
\cos u_{\epsilon,\lambda}(t) - \cos \|u_{\epsilon,\lambda}\|_\infty &\le \sin \|u_{\epsilon,\lambda}\|_\infty (\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t)) \\
&\quad + \frac{1}{2}(\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t))^2,
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
\cos u_{\epsilon,\lambda}(t) - \cos \|u_{\epsilon,\lambda}\|_\infty &\ge \sin \|u_{\epsilon,\lambda}\|_\infty (\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t)) \\
&\quad + \frac{1}{2}(1-\kappa)(\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t))^2.
\end{aligned} \tag{3.6}
$$

Lemma 3.2. *For $\lambda \gg 1$,*

$$
|II_1| \le -\frac{\sqrt{2}\epsilon}{\lambda} \log \sin\big(\frac{d_\lambda}{2}\big) + O(\lambda^{-1}). \tag{3.7}
$$

Proof. By (3.1) and Lemma 3.1,

$$
\begin{aligned}
|II_1| &\le \epsilon \int_{t_\delta}^0 \frac{B(t)u'_{\epsilon,\lambda}(t)}{2A(u_{\epsilon,\lambda}(t))^{3/2}} dt = X_1 + X_2 \\
&:= \frac{\epsilon}{\sqrt{2}\lambda} \int_{t_\delta}^0 \frac{\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t)}{\cos u_{\epsilon,\lambda}(t) - \cos \|u_{\epsilon,\lambda}\|_\infty} u'_{\epsilon,\lambda}(t) dt \\
&\quad + \epsilon^2 \int_{t_\delta}^0 \frac{(\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t))^2}{(\lambda(\cos u_{\epsilon,\lambda}(t) - \cos \|u_{\epsilon,\lambda}\|_\infty)^{3/2}} u'_{\epsilon,\lambda}(t) dt \\
&= \frac{\epsilon}{\sqrt{2}\lambda} \int_{\|u_{\epsilon,\lambda}\|_\infty - \delta}^{\|u_{\epsilon,\lambda}\|_\infty} \frac{\|u_{\epsilon,\lambda}\|_\infty - \theta}{\cos\theta - \cos \|u_{\epsilon,\lambda}\|_\infty} d\theta \\
&\quad + \frac{\epsilon^2}{\lambda^{3/2}} \int_{\|u_{\epsilon,\lambda}\|_\infty - \delta}^{\|u_{\epsilon,\lambda}\|_\infty} \frac{(\|u_{\epsilon,\lambda}\|_\infty - \theta)^2}{(\cos\theta - \cos \|u_{\epsilon,\lambda}\|_\infty)^{3/2}} d\theta.
\end{aligned} \tag{3.8}
$$

We first calculate X_1. We put

$$
\begin{aligned}
X_1 &= Q_1 + Q_2 \\
&:= \frac{\epsilon}{\sqrt{2}\lambda} \int_{\|u_{\epsilon,\lambda}\|_\infty - \delta}^{\|u_{\epsilon,\lambda}\|_\infty} \frac{\|u_{\epsilon,\lambda}\|_\infty - \theta}{\cos\theta - \cos\pi} d\theta \\
&\quad + \frac{\epsilon}{\sqrt{2}\lambda} \int_{\|u_{\epsilon,\lambda}\|_\infty - \delta}^{\|u_{\epsilon,\lambda}\|_\infty} \Big(\frac{\|u_{\epsilon,\lambda}\|_\infty - \theta}{\cos\theta - \cos \|u_{\epsilon,\lambda}\|_\infty} - \frac{\|u_{\epsilon,\lambda}\|_\infty - \theta}{\cos\theta - \cos\pi} \Big) d\theta.
\end{aligned} \tag{3.9}
$$

We see that

$$
\begin{aligned}
Q_1 &= Q_{11} + Q_{12} \\
&:= \frac{\epsilon}{\sqrt{2}\lambda} \int_{\|u_{\epsilon,\lambda}\|_\infty - \delta}^{\|u_{\epsilon,\lambda}\|_\infty} \frac{\|u_{\epsilon,\lambda}\|_\infty - \pi}{\cos\theta + 1} d\theta + \frac{\epsilon}{\sqrt{2}\lambda} \int_{\|u_{\epsilon,\lambda}\|_\infty - \delta}^{\|u_{\epsilon,\lambda}\|_\infty} \frac{\pi - \theta}{\cos\theta + 1} d\theta.
\end{aligned} \tag{3.10}
$$

Then by (2.17),

$$
\begin{aligned}
Q_{11} &= \frac{-d_\lambda \epsilon}{\sqrt{2}\lambda} \int_{\|u_{\epsilon,\lambda}\|_\infty - \delta}^{\|u_{\epsilon,\lambda}\|_\infty} \frac{1}{\cos\theta + 1} d\theta \\
&= \frac{-d_\lambda \epsilon}{\sqrt{2}\lambda} \big[\tan\frac{\theta}{2}\big]_{\|u_{\epsilon,\lambda}\|_\infty - \delta}^{\|u_{\epsilon,\lambda}\|_\infty} \\
&= \frac{-d_\lambda \epsilon}{\sqrt{2}\lambda} \big[\frac{\cos(d_\lambda/2)}{\sin(d_\lambda/2)} - \frac{\sin(\pi - d_\lambda - \delta)/2)}{\cos(\pi - d_\lambda - \delta)/2)}\big] = O(\lambda^{-1}).
\end{aligned} \tag{3.11}
$$

Next,

$$
\begin{aligned}
Q_{12} &= \frac{\epsilon}{\sqrt{2}\lambda}\int_{\|u_{\epsilon,\lambda}\|_\infty-\delta}^{\|u_{\epsilon,\lambda}\|_\infty} \frac{\pi-\theta}{\cos\theta+1}d\theta \\
&= \frac{\epsilon}{\sqrt{2}\lambda}\int_{d_\lambda}^{d_\lambda+\delta} \frac{y}{1-\cos y}dy \\
&= \frac{\epsilon}{\sqrt{2}\lambda}\int_{d_\lambda}^{d_\lambda+\delta} y\Big(-\cot\frac{y}{2}\Big)'\,dy \\
&= \frac{\epsilon}{\sqrt{2}\lambda}\Big(-(d_\lambda+\delta)\cot\frac{d_\lambda+\delta}{2} + d_\lambda\cot\frac{d_\lambda}{2} \\
&\quad + 2\log\sin\big(\frac{d_\lambda+\delta}{2}\big) - 2\log\sin\big(\frac{d_\lambda}{2}\big)\Big) \\
&= -\frac{\sqrt{2}\epsilon}{\lambda}\log\sin\frac{d_\lambda}{2} + O(\lambda^{-1}).
\end{aligned}
\tag{3.12}
$$

Now, we calculate Q_2.

$$
\begin{aligned}
Q_2 &= \frac{\epsilon}{\sqrt{2}\lambda}(1+\cos\|u_{\epsilon,\lambda}\|_\infty)\int_{\|u_{\epsilon,\lambda}\|_\infty-\delta}^{\|u_{\epsilon,\lambda}\|_\infty} \frac{1}{\cos\theta+1}\cdot\frac{\|u_{\epsilon,\lambda}\|_\infty-\theta}{\cos\theta-\cos\|u_{\epsilon,\lambda}\|_\infty}d\theta \\
&\le \frac{\epsilon}{\sqrt{2}\lambda}(1-\cos d_\lambda)\frac{1}{\sin\|u_{\epsilon,\lambda}\|_\infty}\int_{\|u_{\epsilon,\lambda}\|_\infty-\delta}^{\|u_{\epsilon,\lambda}\|_\infty}\frac{1}{\cos\theta+1}d\theta \\
&\le C\epsilon d_\lambda\lambda^{-1}\big[\tan\frac{\theta}{2}\big]_{\|u_{\epsilon,\lambda}\|_\infty-\delta}^{\|u_{\epsilon,\lambda}\|_\infty} \\
&= C\epsilon\lambda^{-1}d_\lambda\left(\frac{\cos(d_\lambda/2)}{\sin(d_\lambda/2)} - \tan\frac{\pi-d_\lambda-\delta}{2}\right) = O(\lambda^{-1}).
\end{aligned}
\tag{3.13}
$$

By (3.9)–(3.13), we obtain

$$
X_1 \le -\frac{\sqrt{2}\epsilon}{\lambda}\log\sin\frac{d_\lambda}{2} + O(\lambda^{-1}). \tag{3.14}
$$

Finally, we calculate X_2. By (2.17) and (3.6),

$$
\begin{aligned}
X_2 &= \epsilon^2\lambda^{-3/2} \\
&\quad\times\int_{\|u_{\epsilon,\lambda}\|_\infty-\delta}^{\|u_{\epsilon,\lambda}\|_\infty}\frac{(\|u_{\epsilon,\lambda}\|_\infty-\theta)^2}{(\sin\|u_{\epsilon,\lambda}\|_\infty+(1/2)(1-\kappa)(\|u_{\epsilon,\lambda}\|_\infty-\theta))^{3/2}(\|u_{\epsilon,\lambda}\|_\infty-\theta)^{3/2}}d\theta \\
&\le C\epsilon^2\lambda^{-3/2}\int_{\|u_{\epsilon,\lambda}\|_\infty-\delta}^{\|u_{\epsilon,\lambda}\|_\infty}\frac{(\|u_{\epsilon,\lambda}\|_\infty-\theta)^{1/2}}{(\sin\|u_{\epsilon,\lambda}\|_\infty+(\|u_{\epsilon,\lambda}\|_\infty-\theta))^{3/2}}d\theta \\
&= C\epsilon^2\lambda^{-3/2}\int_0^\delta\frac{y^{1/2}}{(\sin\|u_{\epsilon,\lambda}\|_\infty+y)^{3/2}}dy \\
&\le C\epsilon^2\lambda^{-3/2}\int_0^\delta\frac{1}{\sin\|u_{\epsilon,\lambda}\|_\infty+y}dy \\
&\le C\epsilon^2\lambda^{-3/2}|\log\sin\|u_{\epsilon,\lambda}\|_\infty| = O(\lambda^{-1}).
\end{aligned}
$$

By this and (3.14), we obtain (3.7). Thus the proof is complete. $\square$

We estimate II_1 from below. To do this, we need the following lemma.

Lemma 3.3. *For $\lambda \gg 1$ and $t_\delta < t < 0$,*

$$\begin{aligned} B(t) \geq \frac{\sqrt{\lambda}}{2}&\sqrt{-\cos u_{\epsilon,\lambda}(t)}(\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t))^2 \\ &- \frac{1}{2}\sqrt{\lambda}\frac{\sin\|u_{\epsilon,\lambda}\|_\infty}{\sqrt{-\cos u_{\epsilon,\lambda}(t)}}(\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t)). \end{aligned} \tag{3.15}$$

Proof. We recall that for constants $a, b > 0$,

$$\begin{aligned} &\int \sqrt{ax^2+bx}dx \\ &= \frac{2ax+b}{4a}\sqrt{ax^2+bx} - \frac{b^2}{8a}\frac{1}{\sqrt{a}}\log\left|2ax+b+2\sqrt{a(ax^2+bx)}\right|. \end{aligned} \tag{3.16}$$

By Taylor expansion, for $\|u_{\epsilon,\lambda}\|_\infty - \delta \leq u_{\epsilon,\lambda}(t) \leq \theta \leq \|u_{\epsilon,\lambda}\|_\infty$, we have

$$\cos\theta - \cos\|u_{\epsilon,\lambda}\|_\infty \geq \sin\|u_{\epsilon,\lambda}\|_\infty(\|u_{\epsilon,\lambda}\|_\infty - \theta) - \frac{1}{2}\cos u_{\epsilon,\lambda}(t)(\|u_{\epsilon,\lambda}\|_\infty - \theta)^2.$$

By this, (2.7)–(2.9) and (3.16), for $t_\delta \leq t \leq 0$,

$$\begin{aligned} B(t) &= \int_t^0 \sqrt{2A(u_{\epsilon,\lambda}(t)) + 2\epsilon B(t)}u'_{\epsilon,\lambda}(t)dt \\ &\geq \sqrt{2\lambda}\int_{u_{\epsilon,\lambda}(t)}^{\|u_{\epsilon,\lambda}\|_\infty} \sqrt{\cos\theta - \cos\|u_{\epsilon,\lambda}\|_\infty}d\theta \\ &\geq \sqrt{\lambda}\int_0^{\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t)} \sqrt{-x^2\cos u_{\epsilon,\lambda}(t) + 2x\sin\|u_{\epsilon,\lambda}\|_\infty}dx \\ &= \sqrt{\lambda}\Big(\frac{1}{2}z - \frac{\sin\|u_{\epsilon,\lambda}\|_\infty}{2\cos u_{\epsilon,\lambda}(t)}\Big)\sqrt{-z^2\cos u_{\epsilon,\lambda}(t) + 2z\sin\|u_{\epsilon,\lambda}\|_\infty} \\ &\quad - \frac{\sqrt{\lambda}}{2}\frac{\sin^2\|u_{\epsilon,\lambda}\|_\infty}{(-\cos u_{\epsilon,\lambda}(t))^{3/2}}\big\{\log(R_\lambda(u_{\epsilon,\lambda}(t)) \\ &\quad + 2\sin\|u_{\epsilon,\lambda}\|_\infty) - \log(2\sin\|u_{\epsilon,\lambda}\|_\infty)\big\}, \end{aligned} \tag{3.17}$$

where

$$R_\lambda(u_{\epsilon,\lambda}(t)) = -2z\cos u_{\epsilon,\lambda}(t) + 2\sqrt{z^2\cos^2 u_{\epsilon,\lambda}(t) - 2z\cos u_{\epsilon,\lambda}(t)\sin\|u_{\epsilon,\lambda}\|_\infty} \tag{3.18}$$

and $z := \|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t)$. We know that $\log(1+x) \leq x$ for $x \geq 0$. By this,

$$\begin{aligned} &\log(R_\lambda(u_{\epsilon,\lambda}(t)) + 2\sin\|u_{\epsilon,\lambda}\|_\infty) - \log(2\sin\|u_{\epsilon,\lambda}\|_\infty) \\ &= \log\Big(1 + \frac{-2z\cos u_{\epsilon,\lambda}(t) + 2\sqrt{z^2\cos^2 u_{\epsilon,\lambda}(t) - 2z\cos u_{\epsilon,\lambda}(t)\sin\|u_{\epsilon,\lambda}\|_\infty}}{2\sin\|u_{\epsilon,\lambda}\|_\infty}\Big) \\ &\leq \frac{-z\cos u_{\epsilon,\lambda}(t) + \sqrt{-\cos u_{\epsilon,\lambda}(t)}\sqrt{-z^2\cos u_{\epsilon,\lambda}(t) + 2z\sin\|u_{\epsilon,\lambda}\|_\infty}}{\sin\|u_{\epsilon,\lambda}\|_\infty}. \end{aligned}$$

By this and (3.17), we obtain (3.15). Thus the proof is complete. □

Lemma 3.4. *For $\lambda \gg 1$,*

$$|II_1| \geq -\frac{\sqrt{2}\epsilon}{\lambda}\log\sin\frac{d_\lambda}{2} + O(\lambda^{-1}). \tag{3.19}$$

Proof. By (3.1), we have

$$
\begin{aligned}
|II_1| &\geq \epsilon\int_{t_\delta}^{0} \frac{B(t)u'_{\epsilon,\lambda}(t)}{2\sqrt{A(u_{\epsilon,\lambda}(t)}(A(u_{\epsilon,\lambda}(t))+\epsilon B(t))}dt := II_{1,1} - II_{1,2}\\
&= \epsilon\int_{t_\delta}^{0} \frac{B(t)u'_{\epsilon,\lambda}(t)}{2A(u_{\epsilon,\lambda}(t))^{3/2}}dt\\
&\quad + \epsilon\Big(\int_{t_\delta}^{0} \frac{B(t)u'_{\epsilon,\lambda}(t)}{2\sqrt{A(u_{\epsilon,\lambda}(t))}(A(u_{\epsilon,\lambda}(t))+\epsilon B(t))}dt - \int_{t_\delta}^{0}\frac{B(t)u'_{\epsilon,\lambda}(t)}{2A(u_{\epsilon,\lambda}(t))^{3/2}}dt\Big).
\end{aligned}
\tag{3.20}
$$

By Lemma 3.3, we put

$$
\begin{aligned}
&II_{1,1} = II_{1,2,1} - II_{1,2,2}\\
&= \frac{\epsilon}{2}\sqrt{\lambda}\int_{t_\delta}^{0}\frac{\sqrt{-\cos u_{\epsilon,\lambda}(t)}(\|u_{\epsilon,\lambda}\|_\infty - u_\lambda(t))^2 u'_{\epsilon,\lambda}(t)}{2\lambda^{3/2}(\cos u_{\epsilon,\lambda}(t) - \cos\|u_{\epsilon,\lambda}\|_\infty)^{3/2}}dt\\
&\quad - \frac{\sqrt{\lambda}\epsilon}{2}\sin\|u_{\epsilon,\lambda}\|_\infty\int_{t_\delta}^{0}\frac{(\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t))u'_{\epsilon,\lambda}(t)}{2\sqrt{-\cos u_{\epsilon,\lambda}(t)}\lambda^{3/2}(\cos u_{\epsilon,\lambda}(t) - \cos\|u_{\epsilon,\lambda}\|_\infty)^{3/2}}dt.
\end{aligned}
\tag{3.21}
$$

By Taylor expansion, for $t_\delta \leq t \leq 0$ and $0 < \eta \ll 1$,

$$
\sqrt{-\cos\|u_{\epsilon,\lambda}\|_\infty} - \sqrt{-\cos u_{\epsilon,\lambda}(t)} \leq \frac{1+\eta}{2}\sin u_{\epsilon,\lambda}(t)(\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t)), \tag{3.22}
$$

$$
\sqrt{-\cos\|u_{\epsilon,\lambda}\|_\infty} = (\cos d_\lambda)^{1/2} = 1 - \frac{1}{4}(1+o(1))d_\lambda^2. \tag{3.23}
$$

By (3.5), (3.21) and (3.22),

$$
\begin{aligned}
II_{1,2,1} &\geq \frac{\epsilon}{4\lambda}\int_{\|u_{\epsilon,\lambda}\|_\infty-\delta}^{\|u_{\epsilon,\lambda}\|_\infty}\frac{\sqrt{-\cos\theta}(\|u_{\epsilon,\lambda}\|_\infty-\theta)^{1/2}}{(\sin\|u_{\epsilon,\lambda}\|_\infty + (1/2)(\|u_{\epsilon,\lambda}\|_\infty-\theta))^{3/2}}d\theta\\
&= \frac{\epsilon}{4\lambda}\int_{\|u_{\epsilon,\lambda}\|_\infty-\delta}^{\|u_{\epsilon,\lambda}\|_\infty}\frac{\sqrt{-\cos\|u_{\epsilon,\lambda}\|_\infty}(\|u_{\epsilon,\lambda}\|_\infty-\theta)^{1/2}}{(\sin\|u_{\epsilon,\lambda}\|_\infty + (1/2)(\|u_{\epsilon,\lambda}\|_\infty-\theta))^{3/2}}d\theta\\
&\quad - \frac{\epsilon}{4\lambda}\int_{\|u_{\epsilon,\lambda}\|_\infty-\delta}^{\|u_{\epsilon,\lambda}\|_\infty}\frac{(\sqrt{-\cos\|u_{\epsilon,\lambda}\|_\infty} - \sqrt{-\cos\theta})(\|u_{\epsilon,\lambda}\|_\infty-\theta)^{1/2}}{(\sin\|u_{\epsilon,\lambda}\|_\infty + (1/2)(\|u_{\epsilon,\lambda}\|_\infty-\theta))^{3/2}}d\theta\\
&\geq \frac{\epsilon}{4\lambda}\sqrt{-\cos\|u_{\epsilon,\lambda}\|_\infty}\int_0^\delta\frac{y^{1/2}}{(\sin\|u_{\epsilon,\lambda}\|_\infty + (1/2)y)^{3/2}}dy\\
&\quad - \frac{1+\eta}{8\lambda}\epsilon\sin(\|u_{\epsilon,\lambda}\|_\infty - \delta)\int_0^\delta\frac{y^{3/2}}{(\sin\|u_{\epsilon,\lambda}\|_\infty + (1/2)y)^{3/2}}dy.
\end{aligned}
\tag{3.24}
$$

Then

$$
\begin{aligned}
\frac{\epsilon}{4\lambda}\int_0^\delta\frac{y^{1/2}}{(\sin\|u_{\epsilon,\lambda}\|_\infty + (1/2)y)^{3/2}}dy &= \frac{\epsilon}{\sqrt{2}\lambda}\int_0^\delta\frac{y^{1/2}}{(2\sin\|u_{\epsilon,\lambda}\|_\infty + y)^{3/2}}dy\\
&= \frac{\sqrt{2}\epsilon}{\lambda}\int_0^{\sqrt{\delta/(2\sin\|u_{\epsilon,\lambda}\|_\infty)}}\frac{z^2}{(1+z^2)^{3/2}}dz\\
&= -\frac{\sqrt{2}\epsilon}{\lambda}\log\sin\|u_{\epsilon,\lambda}\|_\infty + O(\lambda^{-1}).
\end{aligned}
\tag{3.25}
$$

Further, by (2.17),

$$\frac{1+\eta}{8\lambda}\epsilon\sin(\|u_{\epsilon,\lambda}\|_\infty-\delta)\int_0^\delta\frac{y^{3/2}}{(\sin\|u_{\epsilon,\lambda}\|_\infty+(1/2)y)^{3/2}}dy\le C\epsilon\lambda^{-1}. \tag{3.26}$$

By (3.24)–(3.26),

$$\begin{aligned}
II_{1,2,1}&\ge-\frac{\sqrt{2}\epsilon}{\lambda}\log\sin\|u_{\epsilon,\lambda}\|_\infty+O(\lambda^{-1})\\
&=-\frac{\sqrt{2}\epsilon}{\lambda}\log\sin d_\lambda+O(\lambda^{-1})\\
&=-\frac{\sqrt{2}\epsilon}{\lambda}\Big(\log 2+\sin\frac{d_\lambda}{2}+\cos\frac{d_\lambda}{2}\Big)+O(\lambda^{-1})\\
&=-\frac{\sqrt{2}\epsilon}{\lambda}\log\sin\frac{d_\lambda}{2}+O(\lambda^{-1}).
\end{aligned}\tag{3.27}$$

Next, by (3.6),

$$\begin{aligned}
II_{1,2,2}&\le C\epsilon\lambda^{-1}\int_{\|u_{\epsilon,\lambda}\|_\infty-\delta}^{\|u_{\epsilon,\lambda}\|_\infty}\frac{\sin\|u_{\epsilon,\lambda}\|_\infty}{(\sin\|u_{\epsilon,\lambda}\|_\infty+(\|u_{\epsilon,\lambda}\|_\infty-\theta))^{3/2}(\|u_{\epsilon,\lambda}\|_\infty-\theta)^{1/2}}d\theta\\
&\le C\epsilon\lambda^{-1}\int_0^{\sqrt{\delta/(\sin\|u_{\epsilon,\lambda}\|_\infty)}}\frac{1}{(1+z^2)^{3/2}}dz\le C\epsilon\lambda^{-1}.
\end{aligned}\tag{3.28}$$

Finally, by Lemma 3.1 and (3.20), for $z_\lambda(u_{\epsilon,\lambda}(t)):=\|u_{\epsilon,\lambda}\|_\infty-u_{\epsilon,\lambda}(t)$,

$$\begin{aligned}
|II_{1,2}|&=\epsilon^2\int_{t_\delta}^0\frac{B(t)^2u'_{\epsilon,\lambda}(t)}{A(u_{\epsilon,\lambda}(t))^{5/2}}dt\\
&\le C\epsilon^2\int_{t_\delta}^0\frac{A(u_{\epsilon,\lambda}(t))z_\lambda(u_{\epsilon,\lambda}(t))^2+\epsilon^2z_\lambda(u_{\epsilon,\lambda}(t))^4}{A(u_{\epsilon,\lambda}(t))^{5/2}}u'_{\epsilon,\lambda}(t)dt\\
&=C\epsilon^2\int_{\|u_{\epsilon,\lambda}\|_\infty-\delta}^{\|u_{\epsilon,\lambda}\|_\infty}\frac{z_\lambda(\theta)^2}{A(\theta)^{3/2}}d\theta+C\epsilon^4\int_{\|u_{\epsilon,\lambda}\|_\infty-\delta}^{\|u_{\epsilon,\lambda}\|_\infty}\frac{z_\lambda(\theta)^4}{A(\theta)^{5/2}}d\theta\\
&:=Y_1+Y_2.
\end{aligned}\tag{3.29}$$

Then by (2.17), (3.6) and (3.25),

$$\begin{aligned}
Y_1&=C\epsilon^2\lambda^{-3/2}\int_{\|u_{\epsilon,\lambda}\|_\infty-\delta}^{\|u_{\epsilon,\lambda}\|_\infty}\frac{(\|u_{\epsilon,\lambda}\|_\infty-u_{\epsilon,\lambda}(t))^2}{(\cos\theta-\cos|u_{\epsilon,\lambda}\|_\infty)^{3/2}}d\theta\\
&\le C\epsilon^2\lambda^{-3/2}\int_{\|u_{\epsilon,\lambda}\|_\infty-\delta}^{\|u_{\epsilon,\lambda}\|_\infty}\frac{(\|u_{\epsilon,\lambda}\|_\infty-u_{\epsilon,\lambda}(t))^{1/2}}{(\sin\|u_{\epsilon,\lambda}\|_\infty+(1/2)(1-\kappa)(\|u_{\epsilon,\lambda}\|_\infty-u_{\epsilon,\lambda}(t)))^{3/2}}d\theta\\
&\le C\epsilon^2\lambda^{-3/2}\int_0^\delta\frac{y^{1/2}}{(\sin\|u_{\epsilon,\lambda}\|_\infty+y)^{3/2}}dy\\
&\le\epsilon^2\lambda^{-3/2}(C+\log\sin\|u_{\epsilon,\lambda}\|_\infty)=O(\lambda^{-1}).
\end{aligned}\tag{3.30}$$

By the same argument as that just above, by (2.17) and (3.6), we obtain

$$Y_2=C\epsilon^4\lambda^{-5/2}\int_{\|u_{\epsilon,\lambda}\|_\infty-\delta}^{\|u_{\epsilon,\lambda}\|_\infty}\frac{(\|u_{\epsilon,\lambda}\|_\infty-u_{\epsilon,\lambda}(t))^4}{(\cos\theta-\cos|u_{\epsilon,\lambda}\|_\infty)^{5/2}}d\theta$$

$$
\begin{aligned}
&\le C\epsilon^4\lambda^{-5/2}\Big(\int_0^\delta \frac{1}{\sin\|u_{\epsilon,\lambda}\|_\infty + y}dy + C\Big)\\
&\le C\epsilon^4\lambda^{-5/2}\left(|\log\sin\|u_{\epsilon,\lambda}\|_\infty| + C\right) = O(\lambda^{-3/2}).
\end{aligned}
$$

Thus the proof is complete. □

Lemma 3.5. *For $\lambda \gg 1$, we have $|II_2| \le C\lambda^{-1}$.*

Proof. By (3.1) and Lemma 3.1,

$$
\begin{aligned}
&|II_2|\\
&= \int_{-1}^{t_\delta} \frac{\epsilon B(t)u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t)) + \epsilon B(t)}\sqrt{A(u_{\epsilon,\lambda}(t))}(\sqrt{A(u_{\epsilon,\lambda}(t)) + \epsilon B(t)} + \sqrt{A(u_{\epsilon,\lambda}(t))})}dt\\
&\le C\epsilon\int_{-1}^{t_\delta}\frac{B(t)u'_{\epsilon,\lambda}(t)}{2A(u_{\epsilon,\lambda}(t))^{3/2}}dt\\
&\le C\epsilon\int_0^{\|u_{\epsilon,\lambda}\|_\infty-\delta}\frac{2\epsilon(\|u_{\epsilon,\lambda}\|_\infty-\theta)^2 + \sqrt{2\lambda(\cos\theta - \cos\|u_{\epsilon,\lambda}\|_\infty)}(\|u_{\epsilon,\lambda}\|_\infty-\theta)}{(\lambda(\cos\theta-\cos\|u_{\epsilon,\lambda}\|_\infty))^{3/2}}d\theta\\
&\le C\epsilon(\lambda^{-3/2}+\lambda^{-1}).
\end{aligned}
$$

The proof is complete. □

Now Lemma 2.2 follows from Lemmas 3.2–3.5. The proof is complete.

4. Proof of Theorem 1.2

Let $f(y) = |y|$ in this section. We fix $0 < \epsilon \le 1$. Further, we assume that $\lambda \gg 1$ and we write $u_{\epsilon,\lambda} = u_{2,\epsilon,\lambda}$ for simplicity. We consider the solution $u_{\epsilon,\lambda}(t)$ with $\|u_{\epsilon,\lambda}\|_\infty < \pi$. We know that the properties (2.1)–(2.4) are also valid. By the same argument as that to obtain (2.5), we have

$$
\frac{1}{2}u'_{\epsilon,\lambda}(t)^2 = \lambda(\cos u_{\epsilon,\lambda}(t) - \cos\|u_{\epsilon,\lambda}\|_\infty) - \epsilon\int_t^0 |u'_{\epsilon,\lambda}(s)|^2 ds. \tag{4.1}
$$

Then by (2.6)–(2.8), for $-1 \le t \le 0$,

$$
u'_{\epsilon,\lambda}(t) = \sqrt{2(A(u_{\epsilon,\lambda}(t)) - \epsilon B(t))}. \tag{4.2}
$$

By this,

$$
1 = \int_{-1}^0 dt = \frac{1}{\sqrt{2}}\int_{-1}^0 \frac{u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t)) - \epsilon B(t)}}dt = \frac{1}{\sqrt{2}}(I + III), \tag{4.3}
$$

where

$$
I = \int_{-1}^0 \frac{u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t))}}dt, \tag{4.4}
$$

$$
\begin{aligned}
III &= \int_{-1}^0 \frac{u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t)) - \epsilon B(t)}}dt - \int_{-1}^0 \frac{u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t))}}dt\\
&= \int_{-1}^0 \frac{\epsilon B(t)u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t))}\sqrt{A(u_{\epsilon,\lambda}(t)) - \epsilon B(t)}}\\
&\quad\times\frac{1}{\left(\sqrt{A(u_{\epsilon,\lambda}(t))} + \sqrt{A(u_{\epsilon,\lambda}(t)) - \epsilon B(t)}\right)}dt.
\end{aligned} \tag{4.5}
$$

Let $0 < \delta \ll 1$ be fixed. Further, let $-1 < t_\delta < 0$ satisfy $u_{\epsilon,\lambda}(t_\delta) = \|u_{\epsilon,\lambda}\|_\infty$. We put

$$
\begin{aligned}
&III = III_1 + III_2 \\
&:= \int_{t_\delta}^0 \frac{\epsilon B(t) u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t))}\sqrt{A(u_{\epsilon,\lambda}(t)) - \epsilon B(t)}(\sqrt{A(u_{\epsilon,\lambda}(t))} + \sqrt{A(u_{\epsilon,\lambda}(t)) - \epsilon B(t)})} dt \\
&+ \int_{-1}^{t_\delta} \frac{\epsilon B(t) u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t))}\sqrt{A(u_{\epsilon,\lambda}(t)) - \epsilon B(t)}(\sqrt{A(u_{\epsilon,\lambda}(t))} + \sqrt{A(u_{\epsilon,\lambda}(t)) - \epsilon B(t)})} dt.
\end{aligned} \tag{4.6}
$$

Lemma 4.1. *For $\lambda \gg 1$ and $-1 < t < 0$,*

$$
B(t) \le \sqrt{2A(u_{\epsilon,\lambda}(t))}(\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t)). \tag{4.7}
$$

Proof. By (2.2), (2.8) and (4.2),

$$
\begin{aligned}
B(t) &= \int_t^0 u'_{\epsilon,\lambda}(s)^2 ds \\
&\le \int_t^0 \sqrt{2A(u_{\epsilon,\lambda}(s))} u'_{\epsilon,\lambda}(s) ds \\
&\le \sqrt{2A(u_{\epsilon,\lambda}(t))} \int_t^0 u'_{\epsilon,\lambda}(s) ds \\
&= \sqrt{2A(u_{\epsilon,\lambda}(t))}(\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t)).
\end{aligned}
$$

Thus the proof if complete. □

By (3.6) and Lemma 4.1, we obtain the estimate of III_1 from above as follows. Indeed, for $t_\delta \le t \le 0$,

$$
\begin{aligned}
A(u_{\epsilon,\lambda}(t)) - \epsilon B(t) &\ge A(u_{\epsilon,\lambda}(t)) - \epsilon \sqrt{2A(u_{\epsilon,\lambda}(t)}(\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t)) \\
&= A(u_{\epsilon,\lambda}(t)) \Big(1 - \frac{\sqrt{2}(\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t))}{\sqrt{A(u_{\epsilon,\lambda}(t))}}\Big) \\
&\ge A(u_{\epsilon,\lambda}(t)) \Big(1 - \frac{2(\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t))}{\sqrt{(\lambda(1-\kappa)/2)}(\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t))}\Big) \\
&\ge A(u_{\epsilon,\lambda}(t)) \Big(1 - \frac{C}{\sqrt{\lambda}}\Big).
\end{aligned} \tag{4.8}
$$

By this and (4.6),

$$
\begin{aligned}
III_1 &\le \int_{t_\delta}^0 \frac{\epsilon B(t) u'_{\epsilon,\lambda}(t)}{2\sqrt{A(u_{\epsilon,\lambda}(t))}(A(u_{\epsilon,\lambda}(t)) - \epsilon B(t))} \\
&\le \int_{t_\delta}^0 \frac{\epsilon B(t) u'_{\epsilon,\lambda}(t)}{2\sqrt{A(u_{\epsilon,\lambda}(t))} A(u_{\epsilon,\lambda}(t))(1 - C/\sqrt{\lambda})} dt \\
&\le \Big(1 + \frac{C}{\sqrt{\lambda}}\Big) \int_{t_\delta}^0 \frac{\epsilon B(t) u'_{\epsilon,\lambda}(t)}{2\sqrt{A(u_{\epsilon,\lambda}(t))} A(u_{\epsilon,\lambda}(t))} dt.
\end{aligned}
$$

By this, Lemma 4.1 and the same argument as that used in Lemma 3.2, for $\lambda \gg 1$, we obtain

$$III_1 \le -\frac{\epsilon\sqrt{2}}{\lambda}\log\sin\frac{d_\lambda}{2} + O(\lambda^{-1}). \tag{4.9}$$

Furthermore, by (4.3) and (4.8),

$$\begin{aligned}
1 &\le \frac{1}{\sqrt{2}}\int_{-1}^{0}\frac{u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t))}(1 - C/\sqrt{\lambda})}dt \\
&\le \frac{1}{\sqrt{2}}\Big(1 + \frac{C}{\sqrt{\lambda}}\Big)\int_{-1}^{0}\frac{u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t))}}dt.
\end{aligned}$$

By this and (2.14)–(2.16), we obtain

$$\sin\frac{d_\lambda}{2} \le Ce^{-\sqrt{\lambda}}, \quad \sin\|u_{\epsilon,\lambda}\|_\infty \le Ce^{-\sqrt{\lambda}}. \tag{4.10}$$

Lemma 4.2. *For $\lambda \gg 1$ and $t_\delta < t < 0$,*

$$\begin{aligned}
B(t) \ge \sqrt{2}&\int_{u_{\epsilon,\lambda}(t)}^{\|u_{\epsilon,\lambda}\|_\infty}\sqrt{\lambda(\cos\theta - \cos\|u_{\epsilon,\lambda}\|_\infty)}d\theta \\
&- 2\sqrt{\epsilon}A(u_{\epsilon,\lambda}(t))^{1/4}(\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t))^{3/2}.
\end{aligned} \tag{4.11}$$

Proof. By (2.8), (4.2) and (4.7),

$$\begin{aligned}
B(t) &= \int_t^0 u'_{\epsilon,\lambda}(s)^2 ds \\
&\ge \int_t^0 \sqrt{2A(u_{\epsilon,\lambda}(s))}u'_{\epsilon,\lambda}(s)ds - \int_t^0 \sqrt{2\epsilon B(s)}u'_{\epsilon,\lambda}(s)ds \\
&\ge \int_t^0 \sqrt{2A(u_{\epsilon,\lambda}(s))}u'_{\epsilon,\lambda}(s)ds - \sqrt{2\epsilon B(t)}\int_t^0 u'_{\epsilon,\lambda}(s)ds \\
&\ge \int_t^0 \sqrt{2A(u_{\epsilon,\lambda}(s))}u'_{\epsilon,\lambda}(s)ds - \sqrt{2\epsilon B(t)}(\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t)) \\
&\ge \int_t^0 \sqrt{2A(u_{\epsilon,\lambda}(s))}u'_{\epsilon,\lambda}(s)ds - 2\sqrt{\epsilon}A(u_{\epsilon,\lambda}(t))^{1/4}(\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t))^{3/2} \\
&:= W_1 - W_2.
\end{aligned} \tag{4.12}$$

The proof is complete. □

Proof of Theorem 1.2. We calculate the estimate of III_1 from below. By (3.6), (4.6), (4.8), (4.10) and (4.12)

$$
\begin{aligned}
&\epsilon \int_{t_\delta}^{0} \frac{W_2 u'_{\epsilon,\lambda}(t)}{2\sqrt{A(u_{\epsilon,\lambda}(t))}(A(u_{\epsilon,\lambda}(t)) - \epsilon B(t))} dt \\
&\le C \int_{t_\delta}^{0} \frac{A(u_{\epsilon,\lambda}(t))^{1/4}(\|u_{\epsilon,\lambda}\|_\infty - u_{\epsilon,\lambda}(t))^{3/2}}{\sqrt{A(u_{\epsilon,\lambda}(t))}(A(u_{\epsilon,\lambda}(t))(1 - C/\sqrt{\lambda})} u'_{\epsilon,\lambda}(t) dt \\
&\le C\Big(1 + \frac{C}{\sqrt{\lambda}}\Big)\lambda^{-5/4} \int_{\|u_{\epsilon,\lambda}\|_\infty - \delta}^{\|u_{\epsilon,\lambda}\|_\infty} \frac{(\|u_{\epsilon,\lambda}\|_\infty - \theta)^{3/2}}{(\cos\theta - \cos\|u_{\epsilon,\lambda}\|_\infty)^{5/4}} d\theta \\
&\le C\Big(1 + \frac{C}{\sqrt{\lambda}}\Big)\lambda^{-5/4} \int_0^{\delta} \frac{y^{1/4}}{(\sin\|u_{\epsilon,\lambda}\|_\infty + y)^{5/4}} dy \\
&\le C\Big(1 + \frac{C}{\sqrt{\lambda}}\Big)\lambda^{-5/4} \int_0^{\delta/\sin\|u_{\epsilon,\lambda}\|_\infty} \frac{z^{1/4}}{(1+z)^{5/4}} dz \\
&\le C\Big(1 + \frac{C}{\sqrt{\lambda}}\Big)\lambda^{-5/4} |\log\sin\|u_{\epsilon,\lambda}\|_\infty| \\
&\le C\lambda^{-3/4}.
\end{aligned}
\tag{4.13}
$$

By (4.8), (4.12) and Lemmas 3.3 and 3.4, for $\lambda \gg 1$,

$$
\epsilon \int_{t_\delta}^{0} \frac{W_1 u'_{\epsilon,\lambda}(t)}{2\sqrt{A(u_{\epsilon,\lambda}(t))}(A(u_{\epsilon,\lambda}(t)) - \epsilon B(t))} dt \le \frac{-\epsilon\sqrt{2}}{\lambda} \log\sin\frac{d_\lambda}{2} + O(\lambda^{-1}). \tag{4.14}
$$

Further, by (4.6), (4.8) and Lemma 3.5, for $\lambda \gg 1$, we obtain

$$
III_2 \le C\lambda^{-1}. \tag{4.15}
$$

By (4.6), (4.9) and (4.13)–(4.15), we obtain

$$
III = \frac{-\epsilon\sqrt{2}}{\lambda} \log\sin\frac{d_\lambda}{2} + O(\lambda^{-3/4}). \tag{4.16}
$$

By this, (2.14) and the same argument as (2.18)–(2.20), we obtain (1.9). The proof is complete. □

5. Proof of Theorem 1.3

In this section, let $f(y) = -y$. We write $t_\lambda = t_{\epsilon,\lambda} < 0$ for simplicity. The proof of the Theorem 1.3 is almost the same as those of Theorems 1.1 and 1.2. We begin with the fundamental properties of $u_{\epsilon,\lambda}$.

Lemma 5.1. (i) $u'_{\epsilon,\lambda}(t) > 0$ *for* $-1 \le t < t_\lambda$ *and* $u'_{\epsilon,\lambda}(t) > 0$ *for* $t_\lambda < t < 1$.
(ii) $u_{\epsilon,\lambda}(t) \to \pi$ *locally uniformly in* I *as* $\lambda \to \infty$.
(iii) $t_\lambda < 0$ *and* $t_\lambda \to 0$ *as* $\lambda \to \infty$.

Since the proof of Lemma 5.1 is quite easy, we omit it. To prove Theorem 1.3, we repeat the same arguments as those in Sections 3 and 4. We see that

$$
1 + t_\lambda = \int_{-1}^{t_\lambda} dt = \frac{1}{\sqrt{2}} \int_{-1}^{t_\lambda} \frac{u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t) + \epsilon B(t)}} dt = \frac{1}{\sqrt{2}}(P + Q), \tag{5.1}
$$

where

$$
P = \int_{-1}^{t_\lambda} \frac{u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t)}} dt, \tag{5.2}
$$

$$\begin{aligned} Q &= \int_{-1}^{t_\lambda} \frac{u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t) + \epsilon B(t)}} dt - \int_{-1}^{t_\lambda} \frac{u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t)}} dt \\ &= \int_{-1}^{t_\lambda} \frac{-\epsilon B(t) u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t)}\sqrt{A(u_{\epsilon,\lambda}(t) + \epsilon B(t)}(\sqrt{A(u_{\epsilon,\lambda}(t)} + \sqrt{A(u_{\epsilon,\lambda}(t) + \epsilon B(t)})} dt. \end{aligned} \tag{5.3}$$

Then it is clear that $P = I$ in (2.11). Furthermore, by using the same argument as that in Section 3, it is easy to show that $Q = II + O(\lambda^{-1})$. We also find that

$$1 - t_\lambda = \int_{t_\lambda}^{1} dt = \frac{1}{\sqrt{2}} \int_{t_\lambda}^{1} \frac{-u'_{\epsilon,\lambda}(t)}{\sqrt{A(u_{\epsilon,\lambda}(t) - \epsilon B(t)}} dt = \frac{1}{\sqrt{2}}(R + S), \tag{5.4}$$

where $R = I$ and $S = III + O(\lambda^{-3/4})$. By (5.1) and (5.4), we obtain

$$2t_\lambda = \sqrt{2}Q + O(\lambda^{-3/4}) = \sqrt{2}II + O(\lambda^{-3/4}) = \frac{2\epsilon}{\lambda} \log\big(\sin\frac{d_\lambda}{2}\big) + O(\lambda^{-3/4}), \tag{5.5}$$

which implies

$$t_\lambda = \frac{\epsilon}{\lambda} \log\big(\sin\frac{d_\lambda}{2}\big) + O(\lambda^{-3/4}). \tag{5.6}$$

By this and (5.1), we obtain

$$1 = \frac{1}{\sqrt{2}} I + O(\lambda^{-3/4}). \tag{5.7}$$

By this and Lemma 2.1, we obtain (1.11). By (1.11) and (5.6), we obtain (1.10). The proof is complete.

References

[1] H. Amann; *Existence and multiplicity theorem for semi-linear elliptic boundary value problems*, Math. Z. **150** (1976), 281–295.

[2] H. Berestycki and L. Nirenberg; *Monotonicity, symmetry and antisymmetry of solutions of semilinear elliptic equations*, J. Geom. Phys. **5** (1988), 237–275.

[3] D. G. Figueiredo; *On the uniqueness of positive solutions of the Dirichlet problem* $-\triangle u = \lambda \sin u$, Pitman Res. Notes in Math. **122** (1985), 80–83.

[4] I. S. Gradshteyn and I. M. Ryzhik; *Table of integrals, series and products* (fifth edition), Academic Press, San Diego, 1994.

[5] T. Shibata, *Precise spectral asymptotics for the Dirichlet problem* $-u''(t) + g(u(t)) = \lambda \sin u(t)$, J. Math. Anal. Appl. **267** (2002), 576–598.

[6] T. Shibata; *Asymptotic shapes of solutions to nonlinear eigenvalue problems*, Electronic Journal of Differential Equations **2005**, No. 37 (2005), 1–16.

[7] T. Shibata; *Asymptotic expansion of solutions to nonlinear elliptic eigenvalue problems*, Proc. Amer. Math. Soc. **133** (2005), 2597–2604.

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