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PERIODIC SOLUTIONS FOR A LIÉNARD EQUATION WITH TWO DEVIATING ARGUMENTS

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Abstract. In this work, we prove the existence and uniqueness of periodic solutions for a Liénard equation with two deviating arguments. Our main tools are the Mawhin's continuation theorem and the Schwarz inequality. We obtain our results under weaker conditions than those in [14], as shown by an example in the last section of this artticle.

1. Introduction

The Liénard equation can be derived from many fields, such as physics, mechanics and engineering technique fields. An important question is whether this equation can support periodic solutions. In the past several years, the existence of periodic solutions to Liénard equation has been widely discussed, notably by Liénard [6] and by Levinson and Smith [5]. Recently, Zhou and Long [14] studied the existence and uniqueness of periodic solutions of the following Liénard equation with two deviating arguments

$$x''(t) + f(x(t))x'(t) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) = e(t), \tag{1.1}$$

where $f, \tau_1, \tau_2, e \in C(\mathbb{R}, \mathbb{R})$, $g_1, g_2 \in C(\mathbb{R}^2, \mathbb{R})$, $\tau_1(t), \tau_2(t), g_1(t, x), g_2(t, x), e(t)$ are periodic functions with period T, with respect to t, T-periodic for short.

In recent years, there have been many publications on the existence of periodic solutions of the Liénard equation of the type (1.1); see for example [2, 7, 8, 1, 9, 10, 11, 12, 13]. However, as far as we know, there are fewer results on the existence and uniqueness of periodic solutions to (1.1). Applying Mawhin's continuation theorem and some analysis techniques, Zhou and Long [14] provided a sufficient condition for the existence and uniqueness of periodic solutions to (1.1), but their results can be improved.

The main purpose of this paper is to provide a new sufficient condition for guaranteeing the existence and uniqueness of T-periodic solutions to (1.1), by using Mawhin's continuation theorem and Schwarz inequality. Our results hold under

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weaker conditions than those in [14], and that are verifiable as shown by an example in the last section.

2. Preliminaries

For convenience, we define

$$|x|_\infty = \max_{t\in[0,T]} |x(t)|, \quad |x'|_\infty = \max_{t\in[0,T]} |x'(t)|,$$

$$|x|_k = \Big(\int_0^T |x(t)|^k dt \Big)^{1/k}, \quad \bar{e} = \frac{1}{T}\int_0^T e(t)dt$$

Let

$$C_T^1 := \{x \in C^1(\mathbb{R},\mathbb{R}) : x \text{ is } T\text{-periodic}\}, \quad C_T := \{x \in C(\mathbb{R},\mathbb{R}) : x \text{ is } T\text{-periodic}\},$$

which are Banach spaces with the norms

$$\|x\|_{C_T^1} = \max\{|x|_\infty, |x'|_\infty\}, \quad \|x\|_{C_T} = |x|_\infty.$$

The following conditions will be used in this paper:

- (H0) There exist $C_1 \ge 0$, $C_2 \ge 0$, $b_1 \ge 0$ and $b_2 \ge 0$ such that $|f(x_1) - f(x_2)| \le C_1|x_1 - x_2|$, $|f(x)| \le C_2$ and $|g_i(t,u) - g_i(t,v)| \le b_i|u - v|$, for all $x_1, x_2, x, t, u, v \in \mathbb{R}$, $i = 1, 2$.

The following Mawhin's continuation theorem is useful in obtaining the existence of T-periodic solutions of (1.1).

Lemma 2.1 ([3, p. 40])**.** *Let X and Y be two Banach spaces. Suppose that $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero and $N : X \to Y$ is L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X. Moreover, assume that all the following conditions are satisfied:*

- (i) *$Lx \ne \lambda Nx$, for all $x \in \partial\Omega \cap D(L), \lambda \in (0, 1)$;*
- (ii) *$Nx \notin \operatorname{Im} L$, for all $x \in \partial\Omega \cap \ker L$;*
- (iii) *the Brouwer degree $\deg\{JQN, \Omega \cap \ker L, 0\} \ne 0$, where $J : \operatorname{Im} Q \to \ker L$ is an isomorphism.*

Then equation $Lx = Nx$ has at least one solution on $\overline{\Omega} \cap D(L)$.

Lemma 2.2. *If $x \in C^2(\mathbb{R},\mathbb{R})$ with $x(t + T) = x(t)$, then*

$$|x'|_2^2 \le \big(\frac{T}{2\pi}\big)^2 |x''|_2^2.$$

The proof of the above lemma is a direct consequence of the Wirtinger inequality; see for example [4]. Consider the homotopic equation of (1.1), for $\lambda \in (0, 1)$,

$$x''(t) + \lambda f(x(t))x'(t) + \lambda g_1(t, x(t - \tau_1(t))) + \lambda g_2(t, x(t - \tau_2(t))) = \lambda e(t). \quad (2.1)$$

We have the following lemma.

Lemma 2.3. *Suppose that the following conditions are satisfied:*

- (H1) *one of the following conditions holds:*
  - (1) *$(g_i(t,u) - g_i(t,v))(u - v) > 0$ for all $t, u, v \in \mathbb{R}$, $u \ne v$, $i = 1, 2$,*
  - (2) *$(g_i(t,u) - g_i(t,v))(u - v) < 0$ for all $t, u, v \in \mathbb{R}$, $u \ne v$, $i = 1, 2$;*
- (H2) *there exists $d \ge 0$ such that one of the following conditions holds:*
  - (1) *$x(g_1(t,x) + g_2(t,x) - \bar{e}) > 0$, for all $t \in \mathbb{R}$, $|x| > d$,*
  - (2) *$x(g_1(t,x) + g_2(t,x) - \bar{e}) < 0$, for all $t \in \mathbb{R}$, $|x| > d$;*

If $x(t)$ is a T-periodic solution of (2.1), *then*

$$|x|_\infty \le d + \frac{\sqrt{T}}{2}|x'|_2. \tag{2.2}$$

Proof. Let $x(t)$ be an arbitrary T-periodic solution of (2.1). Then, integrating (2.1) from 0 to T, we have

$$\int_0^T [g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t))) - e(t)]dt = 0, \tag{2.3}$$

which implies that there exists $t_1 \in \mathbb{R}$ such that

$$g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) - \bar{e} = 0. \tag{2.4}$$

Now we show the following statement.

Claim. If $x(t)$ is a T-periodic solution of (2.1), then there exists $t_2 \in \mathbb{R}$ such that

$$|x(t_2)| \le d. \tag{2.5}$$

Assume, by way of contradiction, that (2.5) does not hold. Then

$$|x(t)| > d \quad \text{for all } t \in \mathbb{R}, \tag{2.6}$$

which, together with (H1), (H2) and (2.4), imply that one of the following four relations holds:

$$x(t_1-\tau_1(t_1)) > x(t_1-\tau_2(t_1)) > d, \tag{2.7}$$

$$x(t_1-\tau_2(t_1)) > x(t_1-\tau_1(t_1)) > d, \tag{2.8}$$

$$x(t_1-\tau_1(t_1)) < x(t_1-\tau_2(t_1)) < -d, \tag{2.9}$$

$$x(t_1-\tau_2(t_1)) < x(t_1-\tau_1(t_1)) < -d. \tag{2.10}$$

Suppose that (2.7) holds, in view of (H1)(1), (H1)(2), (H2)(1) and (H2)(2), we consider fours cases as follows:

Case (i): If (H1)(1) and (H2)(1) hold, according to (2.7), we have

$$\begin{aligned} 0 &< g_1(t_1, x(t_1-\tau_2(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) - \bar{e} \\ &< g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) - \bar{e}, \end{aligned}$$

which contradicts (2.4). Thus (2.5) is true.

Case (ii): If (H1)(2) and (H2)(1) hold, according to (2.7), we have

$$\begin{aligned} 0 &< g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_1(t_1))) - \bar{e} \\ &< g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) - \bar{e}, \end{aligned}$$

which contradicts (2.4). Thus (2.5) is true.

Case (iii): If (H1)(1) and (H2)(2) hold, according to (2.7), we have

$$\begin{aligned} 0 &> g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_1(t_1))) - \bar{e} \\ &> g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) - \bar{e}, \end{aligned}$$

which contradicts (2.4). Thus (2.5) is true.

Case (iv): If (H1)(2) and (H2)(2) hold, according to (2.7), we have

$$\begin{aligned} 0 &> g_1(t_1, x(t_1-\tau_2(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) - \bar{e} \\ &> g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) - \bar{e}, \end{aligned}$$

which contradicts (2.4). Thus (2.5) is true.

Suppose that (2.8)(or (2.9), or (2.10)) holds; using methods similar to those used in Case (i)–(iv), we can show that (2.5) is true. This completes the proof of the above claim.

Let $t_2 = kT + \widetilde{t}_2$, where $\widetilde{t}_2 \in [0, T]$ and k is an integer. Then noticing $x(t) = x(t+T)$ and (2.5), for any $t \in [\widetilde{t}_2, \widetilde{t}_2 + T]$, we obtain

$$|x(t)| = \Big|x(\widetilde{t}_2) + \int_{\widetilde{t}_2}^{t} x'(s)ds\Big| \le d + \int_{\widetilde{t}_2}^{t} |x'(s)|ds$$

and

$$|x(t)| = \Big|x(\widetilde{t}_2 + T) + \int_{\widetilde{t}_2+T}^{t} x'(s)ds\Big| \le d + \Big| - \int_{t}^{\widetilde{t}_2+T} x'(s)ds\Big| \le d + \int_{t}^{\widetilde{t}_2+T} |x'(s)|ds.$$

Combining the two inequalities above, we obtain

$$|x(t)| \le d + \frac{1}{2}\int_0^T |x'(s)|ds.$$

Using Schwarz inequality yields

$$|x|_\infty = \max_{t\in[\widetilde{t}_2, \widetilde{t}_2+T]} |x(t)| \le d + \frac{1}{2}\int_0^T |x'(s)|ds \le d + \frac{1}{2}|1|_2|x'|_2 = d + \frac{1}{2}\sqrt{T}|x'|_2. \tag{2.11}$$

This completes the proof. □

Lemma 2.4. *Suppose* (H0)–(H2) *hold. Also suppose the following condition holds*

(H3) $C_2\frac{T}{2\pi} + (b_1 + b_2)\frac{T^2}{4\pi} < 1$.

If $x(t)$ is a T-periodic solution of (1.1)*, then $|x'|_\infty \le D$, where*

$$D = \frac{[(b_1 + b_2)d + \max\{|g_1(t, 0)| + |g_2(t, 0)| : 0 \le t \le T\} + |e|_\infty]T}{2\big(1 - C_2\frac{T}{2\pi} - b_1\frac{T^2}{4\pi} - b_2\frac{T^2}{4\pi}\big)}.$$

Proof. Let $x(t)$ be a T-periodic solution of (1.1). From (H1) and (H2), we can easily show that (2.2) also holds. Multiplying $x''(t)$ and (1.1) and then integrating it from 0 to T, by Lemma 2, (H0), (2.2) and Schwarz inequality, we have

$$\begin{aligned}
&|x''|_2^2\\
&= -\int_0^T f(x(t))x'(t)x''(t)dt - \int_0^T g_1(t, x(t-\tau_1(t)))x''(t)dt\\
&\quad - \int_0^T g_2(t, x(t-\tau_2(t)))x''(t)dt + \int_0^T e(t)x''(t)dt\\
&\le \int_0^T |f(x(t))||x'(t)||x''(t)|dt + \int_0^T |g_1(t, x(t-\tau_1(t)))||x''(t)|dt\\
&\quad + \int_0^T |g_2(t, x(t-\tau_2(t)))||x''(t)|dt + \int_0^T |e(t)||x''(t)|dt\\
&\le C_2\int_0^T |x'(t)||x''(t)|dt + \int_0^T [|g_1(t, x(t-\tau_1(t))) - g_1(t, 0)| + |g_1(t, 0)|]|x''(t)|dt\\
&\quad + \int_0^T [|g_2(t, x(t-\tau_2(t))) - g_2(t, 0)| + |g_2(t, 0)|]|x''(t)|dt + \int_0^T |e(t)||x''(t)|dt
\end{aligned}$$

$$\begin{aligned}
&\le C_2|x'|_2|x''|_2 + b_1\int_0^T |x(t-\tau_1(t))||x''(t)|dt + \int_0^T |g_1(t,0)||x''(t)|dt\\
&\quad + b_2\int_0^T |x(t-\tau_2(t))||x''(t)|dt + \int_0^T |g_2(t,0)||x''(t)|dt + \int_0^T |e(t)||x''(t)|dt\\
&\le C_2\frac{T}{2\pi}|x''|_2^2 + (b_1+b_2)\sqrt{T}|x|_\infty|x''|_2\\
&\quad + [\max\{|g_1(t,0)|+|g_2(t,0)| : 0\le t\le T\} + |e|_\infty]\sqrt{T}|x''|_2\\
&\le \big[C_2\frac{T}{2\pi} + (b_1+b_2)\frac{T^2}{4\pi}\big]|x''|_2^2\\
&\quad + \big[(b_1+b_2)d + \max\{|g_1(t,0)|+|g_2(t,0)| : 0\le t\le T\} + |e|_\infty\big]\sqrt{T}|x''|_2,
\end{aligned}$$

which, together with (H3), implies

$$|x''|_2 \le \frac{[(b_1+b_2)d + \max\{|g_1(t,0)|+|g_2(t,0)| : 0\le t\le T\} + |e|_\infty]\sqrt{T}}{1 - C_2\frac{T}{2\pi} - b_1\frac{T^2}{4\pi} - b_2\frac{T^2}{4\pi}}. \tag{2.12}$$

Since $x(0) = x(T)$, there exists $t_0 \in [0,T]$ such that $x'(t_0) = 0$, for any $t \in [t_0, t_0+T]$, we obtain

$$|x'(t)| = \Big|x'(t_0) + \int_{t_0}^t x''(s)ds\Big| \le \int_{t_0}^t |x''(s)|ds,$$

$$|x'(t)| = \Big|x'(t_0+T) + \int_{t_0+T}^t x''(s)ds\Big| \le \Big| -\int_t^{t_0+T} x''(s)ds\Big| \le \int_t^{t_0+T} |x''(s)|ds.$$

Combining these two inequalities, we obtain

$$|x'(t)| \le \frac{1}{2}\int_0^T |x''(s)|ds.$$

Using Schwarz inequality yields

$$|x'|_\infty = \max_{t\in[t_0,t_0+T]} |x'(t)| \le \frac{1}{2}\int_0^T |x''(s)|ds \le \frac{1}{2}|1|_2|x''|_2 = \frac{1}{2}\sqrt{T}|x''|_2. \tag{2.13}$$

By (2.12) and (2.13), we obtain

$$|x'|_\infty \le \frac{[(b_1+b_2)d + \max\{|g_1(t,0)|+|g_2(t,0)| : 0\le t\le T\} + |e|_\infty]T}{2\left(1 - C_2\frac{T}{2\pi} - b_1\frac{T^2}{4\pi} - b_2\frac{T^2}{4\pi}\right)} + : D.$$

This completes the proof. □

Lemma 2.5. *Suppose* (H0)–(H3) *hold. Also assume the condition*

(H4) $C_1 D\frac{T^2}{4\pi} + C_2\frac{T}{2\pi} + (b_1+b_2)\frac{T^2}{4\pi} < 1$.

Then (1.1) *has at most one T-periodic solution.*

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of (1.1). Set $Z(t) = x_1(t) - x_2(t)$. Then, we have

$$\begin{aligned}
&Z''(t) + [f(x_1(t))x_1'(t) - f(x_2(t))x_2'(t)] + [g_1(t,x_1(t-\tau_1(t))) - g_1(t,x_2(t-\tau_1(t)))]\\
&+ [g_2(t,x_1(t-\tau_2(t))) - g_2(t,x_2(t-\tau_2(t)))] = 0.
\end{aligned} \tag{2.14}$$

Since $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of (1.1), integrating (2.14) from 0 to T, we obtain

$$\begin{aligned}\int_0^T \big[&g_1(t,x_1(t-\tau_1(t))) - g_1(t,x_2(t-\tau_1(t)))\\ &+ g_2(t,x_1(t-\tau_2(t))) - g_2(t,x_2(t-\tau_2(t)))\big]dt = 0.\end{aligned}$$

Thus, in view of Mean Value Theorem for integrals, it follows that there exists $\tilde{t} \in \mathbb{R}$ such that

$$g_1(\tilde{t},x_1(\tilde{t}-\tau_1(\tilde{t}))) - g_1(\tilde{t},x_2(\tilde{t}-\tau_1(\tilde{t}))) + g_2(\tilde{t},x_1(\tilde{t}-\tau_2(\tilde{t}))) - g_2(\tilde{t},x_2(\tilde{t}-\tau_2(\tilde{t}))) = 0. \tag{2.15}$$

By (H1), (2.15) implies

$$Z(\tilde{t}-\tau_1(\tilde{t}))Z(\tilde{t}-\tau_2(\tilde{t})) = (x_1(\tilde{t}-\tau_1(\tilde{t}))-x_2(\tilde{t}-\tau_1(\tilde{t})))(x_1(\tilde{t}-\tau_2(\tilde{t}))-x_2(\tilde{t}-\tau_2(\tilde{t}))) \le 0.$$

Since $Z(t) = x_1(t) - x_2(t)$ is a continuous function in $\mathbb{R}$, it follows that there exists $\hat{t} \in \mathbb{R}$ such that

$$Z(\hat{t}) = 0. \tag{2.16}$$

Set $\hat{t} = nT + \bar{t}$, where $\bar{t} \in [0,T]$ and n is an integer. Noticing $Z(t+T) = Z(t)$, we get

$$Z(\bar{t}) = Z(nT + \bar{t}) = Z(\hat{t}) = 0. \tag{2.17}$$

Hence, for any $t \in [\bar{t}, \bar{t}+T]$, we obtain

$$|Z(t)| = \Big|Z(\bar{t}) + \int_{\bar{t}}^t Z'(s)ds\Big| \le \int_{\bar{t}}^t |Z'(s)|ds$$

and

$$|Z(t)| = \Big|Z(\bar{t}+T) + \int_{\bar{t}+T}^t Z'(s)ds\Big| = \Big| -\int_t^{\bar{t}+T} Z'(s)ds\Big| \le \int_t^{\bar{t}+T} |Z'(s)|ds.$$

Combining these two inequalities, we obtain

$$|Z(t)| \le \frac{1}{2}\int_0^T |Z'(s)|ds.$$

Using Schwarz inequality yields

$$|Z|_\infty = \max_{t\in[\bar{t},\bar{t}+T]} |Z(t)| \le \frac{1}{2}\int_0^T |Z'(s)|ds \le \frac{1}{2}|1|_2|Z'|_2 = \frac{1}{2}\sqrt{T}|Z'|_2. \tag{2.18}$$

Multiplying $Z''(t)$ and (2.14) and then integrating it from 0 to T, by Lemma 2, Lemma 4, (H0), (2.18) and Schwarz inequality, we have

$$\begin{aligned}|Z''|_2^2 = &-\int_0^T [f(x_1(t))x_1'(t) - f(x_2(t))x_2'(t)]Z''(t)dt\\ &-\int_0^T [g_1(t,x_1(t-\tau_1(t))) - g_1(t,x_2(t-\tau_1(t)))]Z''(t)dt\\ &-\int_0^T [g_2(t,x_1(t-\tau_2(t))) - g_2(t,x_2(t-\tau_2(t)))]Z''(t)dt\\ \le &\int_0^T |f(x_1(t))||x_1'(t) - x_2'(t)||Z''(t)|dt\\ &+\int_0^T |f(x_1(t)) - f(x_2(t))||x_2'(t)||Z''(t)|dt\end{aligned}$$

$$
\begin{aligned}
&+ b_1 \int_0^T |x_1(t-\tau_1(t)) - x_2(t-\tau_1(t))||Z''(t)|dt \\
&+ b_2 \int_0^T |x_1(t-\tau_2(t)) - x_2(t-\tau_2(t))||Z''(t)|dt \\
&\le \int_0^T C_2|Z'(t)||Z''(t)|dt + \int_0^T C_1 D|Z(t)||Z''(t)|dt \\
&+ b_1 \int_0^T |Z(t-\tau_1(t))||Z''(t)|dt + b_2 \int_0^T |Z(t-\tau_2(t))||Z''(t)|dt \\
&\le C_2|Z'|_2|Z''|_2 + C_1 D\sqrt{T}|Z|_\infty|Z''|_2 + (b_1+b_2)\sqrt{T}|Z|_\infty|Z''|_2 \\
&\le \big[C_1 D\frac{T^2}{4\pi} + C_2\frac{T}{2\pi} + (b_1+b_2)\frac{T^2}{4\pi}\big]|Z''|_2^2.
\end{aligned}
$$

Since $Z(t), Z'(t)$, $Z''(t)$ are continuous T-periodic functions, by (H4), (2.18) and the above inequality, we obtain

$$
Z(t) = Z'(t) = Z''(t) = 0 \quad \text{for all } t \in \mathbb{R}.
$$

Thus, $x_1(t) \equiv x_2(t)$, for all $t \in \mathbb{R}$. Hence, (1.1) has at most one T-periodic solution. This completes the proof. □

Lemma 2.6. *Suppose* (H0)–(H3) *hold. Then the set of T-periodic solutions of* (2.1) *are bounded in* C_T^1.

Proof. Let $S \subset C_T^1$ be the set of T-periodic solutions of (2.1). If $S = \emptyset$, the proof is complete. Suppose $S \ne \emptyset$, and let $x \in S$. Multiplying $x''(t)$ and (2.1) and then integrating it from 0 to T, by Lemma 2, (H0), (2.2) and Schwarz inequality, we have

$$
\begin{aligned}
&|x''|_2^2 \\
&= -\lambda \int_0^T f(x(t))x'(t)x''(t)dt - \lambda \int_0^T g_1(t, x(t-\tau_1(t)))x''(t)dt \\
&\quad - \lambda \int_0^T g_2(t, x(t-\tau_2(t)))x''(t)dt + \lambda \int_0^T e(t)x''(t)dt \\
&\le \int_0^T |f(x(t))||x'(t)||x''(t)|dt + \int_0^T |g_1(t, x(t-\tau_1(t)))||x''(t)|dt \\
&\quad + \int_0^T |g_2(t, x(t-\tau_2(t)))||x''(t)|dt + \int_0^T |e(t)||x''(t)|dt \\
&\le C_2 \int_0^T |x'(t)||x''(t)|dt + \int_0^T [|g_1(t, x(t-\tau_1(t))) - g_1(t,0)| + |g_1(t,0)|]|x''(t)|dt \\
&\quad + \int_0^T [|g_2(t, x(t-\tau_2(t))) - g_2(t,0)| + |g_2(t,0)|]|x''(t)|dt + \int_0^T |e(t)||x''(t)|dt \\
&\le C_2|x'|_2|x''|_2 + b_1 \int_0^T |x(t-\tau_1(t))||x''(t)|dt + \int_0^T |g_1(t,0)||x''(t)|dt \\
&\quad + b_2 \int_0^T |x(t-\tau_2(t))||x''(t)|dt + \int_0^T |g_2(t,0)||x''(t)|dt + \int_0^T |e(t)||x''(t)|dt \\
&\le C_2\frac{T}{2\pi}|x''|_2^2 + (b_1+b_2)\sqrt{T}|x|_\infty|x''|_2
\end{aligned}
$$

$$
\begin{aligned}
&+ [\max\{|g_1(t,0)| + |g_2(t,0)| : 0 \le t \le T\} + |e|_\infty]\sqrt{T}|x''|_2 \\
&\le \big[C_2 \frac{T}{2\pi} + (b_1+b_2)\frac{T^2}{4\pi}\big]|x''|_2^2 \\
&+ \big[(b_1+b_2)d + \max\{|g_1(t,0)| + |g_2(t,0)| : 0 \le t \le T\} + |e|_\infty\big]\sqrt{T}|x''|_2,
\end{aligned}
$$

which, together with (H3), implies that there exists $M_0 > 0$ such that

$$|x''|_2 < M_0. \tag{2.19}$$

This, together with Lemma 2 and Lemma 3, leads to

$$|x|_\infty < d + \frac{\sqrt{T^3}}{4\pi} M_0. \tag{2.20}$$

On the other hand, since $x(0) = x(T)$, there exists $\bar{t}_0 \in [0,T]$ such that $x'(\bar{t}_0) = 0$. For any $t \in [\bar{t}_0, \bar{t}_0 + T]$, from (2.19), we obtain

$$|x'(t)| = \big|x'(\bar{t}_0) + \int_{\bar{t}_0}^{t} x''(s)ds\big| \le \int_0^T |x''(s)|ds \le |1|_2 |x''|_2 < \sqrt{T} M_0,$$

which implies

$$|x'|_\infty = \max_{t \in [\bar{t}_0, \bar{t}_0 + T]} |x'(t)| < \sqrt{T} M_0. \tag{2.21}$$

Let $M = \max\big\{d + \frac{\sqrt{T^3}}{4\pi} M_0, \sqrt{T} M_0\big\}$, by (2.20) and (2.21), we have $\|x\| < M$. This completes the proof. $\square$

3. Main results

Now we are in the position to give our main results.

Theorem 3.1. *Suppose* (H0)–(H2), (H4) *hold. Then* (1.1) *has a unique T-periodic solution.*

Proof. Lemma 5 states that (1.1) has at most one T-periodic solution. Thus, to prove Theorem 1, it suffices to show that (1.1) has at least one T-periodic solution. To do this, we apply Lemma 1.

By Lemma 6, there exists $M > d$ such that, for any T-periodic solution $x(t)$ of (2.1)

$$\|x\| < M. \tag{3.1}$$

Set

$$\Omega = \{x : x \in C_T^1, \|x\| < M\}. \tag{3.2}$$

Define a linear operator $L : D(L) \subset C_T^1 \to C_T$ by setting $D(L) = \{x : x \in C_T^1, x'' \in C(\mathbb{R}, \mathbb{R})$, for $x \in D(L)$, and

$$Lx = x''. \tag{3.3}$$

We also define a nonlinear operator $N : C_T^1 \to C_T$, by

$$Nx = -f(x(t))x'(t) - g_1(t, x(t - \tau_1(t))) - g_2(t, x(t - \tau_2(t))) + e(t). \tag{3.4}$$

Then (2.1) is equivalent to the operator equation

$$Lx = \lambda Nx, \quad \lambda \in (0,1). \tag{3.5}$$

It is easy to see that

$$\ker L = \mathbb{R} \quad \text{,and} \quad \operatorname{Im} L = \{x : x \in C_T, \int_0^T x(s)ds = 0\},$$

then, L is a Fredholm operator with index zero. Also let projectors $P : C^1_T \to \ker L$ and $Q : C_T \to C_T / \operatorname{Im} L$ defined by

$$Px = x(0) \quad \text{where } x \in C^1_T,$$
$$Qx = \frac{1}{T}\int_0^T x(s)ds \quad \text{where } x \in C_T,$$

hence, $\operatorname{Im} P = \operatorname{Im} Q = \ker L = \mathbb{R}$ and $\ker Q = \operatorname{Im} L$. Define the isomorphism as follows

$$J : \operatorname{Im} Q \to \ker L, \quad J(x) = x. \tag{3.6}$$

Let

$$L_P := L_{D(L)\cap \ker P} : D(L) \cap \ker P \to \operatorname{Im} L,$$

then we know, by [3, p. 41–42], that L_P has a continuous inverse L_P^{-1} on $\operatorname{Im} L$ defined by

$$(L_P^{-1}y)(t) = \int_0^T G(s,t)y(s)ds, \tag{3.7}$$

where

$$G(s,t) = \begin{cases} -\frac{s}{T}(T-t), & 0 \le s \le t; \\ -\frac{t}{T}(T-s), & t \le s \le T. \end{cases}$$

Using Ascoli-Arzela theorem we have, from (3.2) and (3.7), that $L_P^{-1}(I-Q)N(\overline{\Omega})$ is compact. On the other hand, $QN(\overline{\Omega})$ is bounded by the continuity of function QN. Thus N is L-compact on $\overline{\Omega}$. By (3.2) and (3.5), condition (i) of Lemma 1 is satisfied.

In view of (H2)(1) and (H2)(2), we will consider two cases:

Case(i): If (H2)(1) holds. Since

$$\begin{aligned} QNx &= -\frac{1}{T}\int_0^T [f(x(t))x'(t) + g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t))) - e(t)]dt \\ &= -\frac{1}{T}\int_0^T [f(x(t))x'(t) + g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t))) - \bar{e}]dt; \end{aligned}$$

for any $x \in \partial\Omega \cap \ker L$, $x = M$ or $x = -M$, $x' = 0$, we obtain

$$QN(M) = -\frac{1}{T}\int_0^T [g_1(t,M) + g_2(t,M) - \bar{e}]dt < 0, \tag{3.8}$$
$$QN(-M) = -\frac{1}{T}\int_0^T [g_1(t,-M) + g_2(t,-M) - \bar{e}]dt > 0 \tag{3.9}$$

which implies the condition (ii) of Lemma 1 is satisfied. Define

$$\begin{aligned} H(x,\mu) &= -\mu x + (1-\mu)QNx \\ &= -\mu x - (1-\mu)\frac{1}{T}\int_0^T \big[f(x(t))x'(t) + g_1(t, x(t-\tau_1(t))) \\ &\quad + g_2(t, x(t-\tau_2(t))) - e(t)\big]dt \\ &= -\mu x - (1-\mu)\frac{1}{T}\int_0^T \big[f(x(t))x'(t) + g_1(t, x(t-\tau_1(t))) \\ &\quad + g_2(t, x(t-\tau_2(t))) - \bar{e}\big]dt \end{aligned}$$

in view of (3.8) and (3.9), we get $xH(x,\mu) < 0$, for all $x \in \partial\Omega \cap \ker L$ and $\mu \in [0,1]$. Hence, $H(x,\mu)$ is a homotopic transformation, together with (3.6) and by using homotopic invariance theorem, we have

$$\deg\{JQN, \Omega \cap \ker L, 0\} = \deg\{QN, \Omega \cap \ker L, 0\} = \deg\{-x, \Omega \cap \ker L, 0\} \neq 0,$$

so condition (iii) of Lemma 1 is satisfied.

Case(ii): If (H2)(2) holds. Since

$$\begin{aligned} QNx &= -\frac{1}{T}\int_0^T [f(x(t))x'(t) + g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t))) - e(t)]dt \\ &= -\frac{1}{T}\int_0^T [f(x(t))x'(t) + g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t))) - \bar{e}]dt; \end{aligned}$$

for any $x \in \partial\Omega \cap \ker L$, $x = M$ or $x = -M$, $x' = 0$, we obtain

$$QN(M) = -\frac{1}{T}\int_0^T [g_1(t, M) + g_2(t, M) - \bar{e}]dt > 0, \tag{3.10}$$

$$QN(-M) = -\frac{1}{T}\int_0^T [g_1(t, -M) + g_2(t, -M) - \bar{e}]dt < 0 \tag{3.11}$$

which implies the condition (ii) of Lemma 1 is satisfied. Define

$$\begin{aligned} H(x,\mu) &= \mu x + (1-\mu)QNx \\ &= \mu x - (1-\mu)\frac{1}{T}\int_0^T \big[f(x(t))x'(t) + g_1(t, x(t-\tau_1(t))) \\ &\quad + g_2(t, x(t-\tau_2(t))) - e(t)\big]dt \\ &= \mu x - (1-\mu)\frac{1}{T}\int_0^T \big[f(x(t))x'(t) + g_1(t, x(t-\tau_1(t))) \\ &\quad + g_2(t, x(t-\tau_2(t))) - \bar{e}\big]dt, \end{aligned}$$

in view of (3.10) and (3.11), we get $xH(x,\mu) > 0$, for all $x \in \partial\Omega \cap \ker L$ and $\mu \in [0,1]$. Hence, $H(x,\mu)$ is a homotopic transformation, together with (3.6) and by using homotopic invariance theorem, we have

$$\deg\{JQN, \Omega \cap \ker L, 0\} = \deg\{QN, \Omega \cap \ker L, 0\} = \deg\{x, \Omega \cap \ker L, 0\} \neq 0,$$

so condition (iii) of Lemma 1 is satisfied. Therefore, it follows from Lemma 1 that (1.1) has at least one T-periodic solution. This completes the proof. □

In [14], Zhou and Long studied (1.1) and obtained the got the following results.

Theorem 3.2. *Assume* (H0), (H1), *and that the following conditions hold:*

(A2) *there exists $d \geq 0$ such that one of the following conditions holds:*
  (1) $x(g_1(t,x) + g_2(t,x) - e(t)) > 0$, *for all* $t \in \mathbb{R}$, $|x| > d$,
  (2) $x(g_1(t,x) + g_2(t,x) - e(t)) < 0$, *for all* $t \in \mathbb{R}$, $|x| > d$;

(A4) $C_1 D_1 \frac{T^2}{2\pi} + C_2 \frac{T}{2\pi} + (b_1 + b_2)\frac{T^2}{2\pi} < 1$, *where*

$$D_1 = \frac{[(b_1 + b_2)d + \max\{|g_1(t,0)| + |g_2(t,0)| : 0 \leq t \leq T\} + |e|_\infty]T}{1 - C_2\frac{T}{2\pi} - b_1\frac{T^2}{2\pi} - b_2\frac{T^2}{2\pi}}.$$

Then (1.1) *has a unique T-periodic solution.*

If $e(t) \neq$ constant, it is easy to verify that the condition (H2) is weaker than the condition (A2) since $\min_{t\in\mathbb{R}} e(t) < \bar{e} < \max_{t\in\mathbb{R}} e(t)$. On the other hand, noticing $\frac{1}{4\pi} < \frac{1}{2\pi}$ and $D < \frac{1}{2}D_1$, we can see that the condition (H_4) is also weaker than the condition (A_4). Therefore, our results improve those in [14].

4. Example and remark

In this section, we apply the main results obtained in previous sections to an example.

Consider the existence and uniqueness of a 2π-periodic solution to the Liénard equation

$$x''(t) + \frac{1}{10}\cos t x'(t) + g_1(t, x(t - \cos t)) + g_2(t, x(t - \sin t)) = e(t), \tag{4.1}$$

where $T = 2\pi$, $\tau_1(t) = \cos t$, $\tau_2(t) = \sin t$, $g_1(t,x) = \frac{1}{80\pi(1+\cos^2 t)}\arctan x$, $g_2(t,x) = \frac{1}{60\pi}(1 + \sin^2 t)\arctan x$ and $e(t) = \frac{1}{\pi}\sin t$.

It is obvious that the conditions (A0) and (A1) in [14, Theorem 1] hold. However, we can easily check that (A2) does not hold, which implies that (A4) does not hold. Hence, [14, Theorem 1] can not be applied. Meanwhile, Theorem 1 in this paper remains applicable, as we show now.

By (4.1), we can get $b_1 = \frac{1}{80\pi}, b_2 = \frac{1}{30\pi}$ and $C_1 = C_2 = \frac{1}{10}$. Noticing $\bar{e} = \frac{1}{T}\int_0^T e(t)dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\pi}\sin t dt = 0$, we can get $d = \frac{1}{10}$ (Actually, d can be an arbitrarily small positive constant.) and check that (H0)–(H2) hold. On the other hand, noticing that

$$\begin{aligned} D &= \frac{[(b_1+b_2)d + \max\{|g_1(t,0)| + |g_2(t,0)| : 0 \le t \le T\} + |e|_\infty]T}{2\big(1 - C_2\frac{T}{2\pi} - b_1\frac{T^2}{4\pi} - b_2\frac{T^2}{4\pi}\big)} \\ &= \frac{[\frac{1}{10}(\frac{1}{80\pi} + \frac{1}{30\pi}) + \frac{1}{\pi}]2\pi}{2(1 - \frac{1}{10} - \frac{1}{80} - \frac{1}{30})} \approx 1.176, \end{aligned}$$

it is easy to verify that (H4) holds since $C_1 D\frac{T^2}{4\pi} + C_2\frac{T}{2\pi} + (b_1+b_2)\frac{T^2}{4\pi} = \frac{1}{10}\times 1.176\times\pi + \frac{1}{10} + \frac{1}{80} + \frac{1}{30} \approx 0.515 < 1$. Thus, Theorem 1 in this study shows that (1.1) has a unique 2π-periodic solution. Hence our results improve those in [14].

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