Electronic Journal of Differential Equations, Vol. 2009(2009), No. 138, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

QUASILINEAR DIFFERENTIAL EQUATIONS IN EXTERIOR DOMAINS WITH NONLINEAR BOUNDARY CONDITIONS AND APPLICATION

DUMITRU MOTREANU, NICOLAE TARFULEA

Abstract. We investigate the existence of weak solutions to a class of quasilinear elliptic equations with nonlinear Neumann boundary conditions in exterior domains. Problems of this kind arise in various areas of science and technology. An important model case related to the initial data problem in general relativity is presented. As an application of our main result, we deduce the existence of the conformal factor for the Hamiltonian constraint in general relativity in the presence of multiple black holes. We also give a proof for uniqueness in this case.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be an exterior domain with smooth compact boundary. In this paper, we study the existence and uniqueness of solutions of the following elliptic boundary-value problem

$$-\operatorname{div}[A(x,\nabla u)] = F(x,u), \quad x \in \Omega, \tag{1.1}$$

$$A(x,\nabla u)\cdot n = f(x,u), \quad x \in \partial\Omega, \tag{1.2}$$

where n stands for the unit exterior normal to $\partial\Omega$. Here $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following conditions:

- There exist $p > 1$, $a_1(\cdot) \in L^{p'}(\Omega)$ (p' is the conjugate of p, that is $1/p + 1/p' = 1$), and $b_1 > 0$ such that $|A(x,\xi)| \le a_1(x) + b_1|\xi|^{p-1}$, for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$.
- $A(x,\xi)$ is strictly monotone in ξ, that is $[A(x,\xi_2) - A(x,\xi_1)]\cdot(\xi_2 - \xi_1) > 0$, for a.e. $x \in \Omega$ and all ξ_1, $\xi_2 \in \mathbb{R}^N$ with $\xi_1 \ne \xi_2$.
- There exist $a_2 \in L^1(\Omega)$ and $b_2 > 0$ such that the following coercivity property holds $A(x,\xi)\cdot\xi \ge b_2|\xi|^p - a_2(x)$, for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$.

Problems of this type arise in many and diverse contexts like differential geometry (e.g., in the scalar curvature problem and the Yamabe problem), nonlinear elasticity, non–Newtonian fluid mechanics, mathematical biology, general relativity, and elsewhere. In Section 3 we address one of these applications related to the

2000 *Mathematics Subject Classification.* 35J65, 83C05.
Key words and phrases. Quasilinear equation; exterior domain; general relativity; nonlinear boundary conditions; initial data problem; conformal factor.

Submitted October 5, 2009. Published October 30, 2009.

initial data problem in general relativity (more precisely, the existence of conformal factor to the Hamiltonian constraint equation in the case of multiple black holes).

Nonlinear boundary value problems related to (1.1)-(1.2) have been studied for some time by numerous authors and in variiuos frameworks. For example, in [24] Pflüger considered the problem (1.1)-(1.2) for the p-Laplacian with polynomial nonlinearities on the right hand side and in the boundary condition. In this context, Pflüger showed the existence of a nontrivial, positive weak solution. Due to the unbounded domain, the lack of compactness was overcome through the use of weighted Sobolev spaces. For more recent work on this subject, see [3, 4, 12, 17, 20, 21, 25, 28, 30], and references therein.

We denote by $W^{1,p}(\Omega)$ the weighted Sobolev space (the suitable weight function in our case is $(1+|x|^2)^{-1/2}$ for $x \in \Omega$)

$$W^{1,p}(\Omega) := \{u \in L^p_{\rm loc}(\Omega) : \frac{u}{(1+|x|^2)^{1/2}} \in L^p(\Omega) \text{ and } \nabla u \in L^p(\Omega)\}.$$

Notice that on each bounded part of the open set Ω, the space $W^{1,p}(\Omega)$ coincides with the usual Sobolev space $W^{1,p}_{\rm loc}(\Omega)$. Functions in these two spaces differ only by their behaviour at infinity. For more on these spaces, see [22, 26], and references therein.

A variational formulation for the exterior boundary-value problem (1.1) and (1.2) is

$$\int_\Omega A(x, \nabla u) \cdot \nabla v \, dx - \int_\Omega F(x, u) v \, dx - \int_{\partial\Omega} f(x, u) v \, d\sigma = 0, \quad \forall v \in W^{1,p}(\Omega).$$

A function $\underline{u}$ (resp. $\overline{u}$) in $W^{1,p}(\Omega)$ is called a (weak) *subsolution* (resp. *supersolution*) of (1.1) and (1.2) if

$$\int_\Omega A(x, \nabla \underline{u}) \cdot \nabla v \, dx - \int_\Omega F(x, \underline{u}) v \, dx - \int_{\partial\Omega} f(x, \underline{u}) v \, d\sigma \le 0, \quad (\text{resp. } \ge) \tag{1.3}$$

for each $v \in W^{1,p}(\Omega)$, $v \ge 0$ a.e. in Ω.

Under the above conditions, our main result may be stated as follows.

Theorem 1.1. *Assume there exist a pair of sub- and supersolution* $\underline{u}$ *and* $\overline{u}$ *of* (1.1)-(1.2) *and that the functions* F *and* f *satisfy the following growth conditions:*

- *There exists* $a_3 \in L^{p'}(\Omega)$ *such that* $|F(x,u)| \le a_3(x)/(1+|x|^2)^{1/2}$*, for a.e.* $x \in \Omega$ *and all* $u \in [\underline{u}(x), \overline{u}(x)]$.
- *There exist* $a_4 \in L^{p'}(\partial\Omega)$ *and* $b_3 \in L^p(\partial\Omega)$ *such that* $|f(x,u)| \le a_4(x) + b_3(x)|u|^{p-1}$*, for a.e.* $x \in \partial\Omega$ *and all* $u \in [\underline{u}(x), \overline{u}(x)]$.

Then, (1.1)-(1.2) *has at least one (weak) solution* $u \in W^{1,p}(\Omega)$ *such that* $\underline{u} \le u \le \overline{u}$.

A proof of this theorem is given in Section 2. As an application of this result, we will discuss the existence of the conformal factor for the Hamiltonian constraint in general relativity in Section 3. We also provide a proof for the uniqueness of the conformal factor in the case of multiple black holes in Subsection 3.2.

2. Proof of Theorem 1.1

Let $\rho := (1+|x|^2)^{1/2}$. For $u \in W^{1,p}(\Omega)$, define

$$b(x,u) := \begin{cases} [u(x)-\overline{u}(x)]^{p-1}/\rho^p & \text{if } u(x) > \overline{u}(x) \\ 0 & \text{if } \underline{u}(x) \le u(x) \le \overline{u}(x) \\ -[\underline{u}(x)-u(x)]^{p-1}/\rho^p & \text{if } u(x) < \underline{u}(x) \end{cases}$$

$$(Tu)(x) := \begin{cases} \overline{u}(x) & \text{if } u(x) > \overline{u}(x) \\ u(x) & \text{if } \underline{u}(x) \le u(x) \le \overline{u}(x) \\ \underline{u}(x) & \text{if } u(x) < \underline{u}(x) \end{cases}$$

Next, consider the operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{F}$, and $\mathcal{G} : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ defined by:

$$\begin{gathered} \langle \mathcal{A}(u), v\rangle := \int_\Omega A(x,\nabla u)\cdot\nabla v\,dx, \quad \langle \mathcal{B}(u), v\rangle := \int_\Omega b(x,u)v\,dx, \\ \langle \mathcal{F}(u), v\rangle := -\int_\Omega F(x,Tu)v\,dx, \quad \langle \mathcal{G}(u), v\rangle := -\int_{\partial\Omega} f(x,Tu)v\,d\sigma, \\ \Gamma : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*, \quad \Gamma(u) := \mathcal{A}(u)+\mathcal{B}(u)+\mathcal{F}(u)+\mathcal{G}(u). \end{gathered}$$

The following lemma states that solving $\Gamma(u) = 0$ in $(W^{1,p}(\Omega))^*$ produces a weak solution u to problem (1.1)–(1.2), with $\underline{u} \le u \le \overline{u}$ a.e. in Ω. Its proof relies on arguments largely similar to the ones used in [15, 16] (see also [2]) in a different context.

Lemma 2.1. *Assume that $u \in W^{1,p}(\Omega)$ is a solution to $\Gamma(u) = 0$. Then u is also a week solution to* (1.1)–(1.2)*, with $\underline{u}(x) \le u(x) \le \overline{u}(x)$ a.e. in Ω.*

Proof. Since u and $\underline{u}$ are elements of $W^{1,p}(\Omega)$, it follows that $(\underline{u}-u)^+ \in W^{1,p}(\Omega)$. Then

$$\langle \Gamma u, (\underline{u}-u)^+\rangle = \langle \mathcal{A}(u)+\mathcal{B}(u)+\mathcal{F}(u)+\mathcal{G}(u), (\underline{u}-u)^+\rangle = 0, \tag{2.1}$$

and so

$$\begin{aligned} &\int_\Omega A(x,\nabla u)\cdot\nabla(\underline{u}-u)^+\,dx + \int_\Omega b(x,u)(\underline{u}-u)^+\,dx \\ &-\int_\Omega F(x,Tu)(\underline{u}-u)^+\,dx - \int_{\partial\Omega} f(x,Tu)(\underline{u}-u)^+\,d\sigma = 0. \end{aligned} \tag{2.2}$$

Since $\underline{u}$ is a subsolution to (1.1)–(1.2), we have

$$\int_\Omega A(x,\nabla\underline{u})\cdot\nabla(\underline{u}-u)^+\,dx - \int_\Omega F(x,\underline{u})(\underline{u}-u)^+\,dx - \int_{\partial\Omega} f(x,\underline{u})(\underline{u}-u)^+\,d\sigma \le 0. \tag{2.3}$$

Subtracting (2.2) from (2.3) gives

$$\begin{aligned} &\int_\Omega [A(x,\nabla\underline{u}) - A(x,\nabla u)]\cdot\nabla(\underline{u}-u)^+\,dx - \int_\Omega [F(x,\underline{u}) - F(x,Tu)](\underline{u}-u)^+\,dx \\ &-\int_{\partial\Omega} [f(x,\underline{u}) - f(x,Tu)](\underline{u}-u)^+\,d\sigma \\ &\le \int_\Omega b(x,u)(\underline{u}-u)^+\,dx. \end{aligned} \tag{2.4}$$

Observe that (from the hypotheses and Stampachia's Theorem)

$$\begin{aligned}&\int_\Omega [A(x,\nabla \underline{u}) - A(x,\nabla u)]\cdot \nabla(\underline{u}-u)^+\,dx\\&= \int_{\{\underline{u}(x)>u(x)\}} [A(x,\nabla \underline{u}) - A(x,\nabla u)]\cdot (\nabla\underline{u}-\nabla u)\,dx \ge 0.\end{aligned} \tag{2.5}$$

Furthermore, from the definition of Tu, it follows that $Tu(x) = \underline{u}(x)$ on $\{\underline{u}(x) > u(x)\}$, and so

$$\begin{aligned}&\int_\Omega [F(x,\underline{u}) - F(x,Tu)](\underline{u}-u)^+\,dx\\&= \int_{\{\underline{u}(x)>u(x)\}} [F(x,\underline{u}) - F(x,Tu)](\underline{u}-u)\,dx = 0.\end{aligned} \tag{2.6}$$

Also,

$$\begin{aligned}&\int_{\partial\Omega} [f(x,\underline{u}) - f(x,Tu)](\underline{u}-u)^+\,d\sigma\\&= \int_{\{x\in\partial\Omega:\,\underline{u}(x)>u(x)\}} [f(x,\underline{u}) - f(x,Tu)](\underline{u}-u)\,d\sigma = 0.\end{aligned} \tag{2.7}$$

By (2.4), (2.5), (2.6), and (2.7), we obtain

$$0 \le \int_\Omega b(x,u)(\underline{u}-u)^+\,dx = -\int_{\{\underline{u}(x)>u(x)\}} (\underline{u}-u)^p/\rho^p\,dx \le 0,$$

and thus $\underline{u} = u$ a.e. in $\{\underline{u}(x) > u(x)\}$. That is, the set $\{\underline{u}(x) > u(x)\}$ has measure 0. This shows that $\underline{u}(x) \le u$ a.e. in Ω. For the inequality $u(x) \le \overline{u}$ a.e. in Ω, we proceed similarly (by considering $(u-\overline{u})^+$ this time). Since $\underline{u}(x) \le u(x) \le \overline{u}(x)$ a.e. in Ω, we have both $b(x,u(x)) = 0$ and $Tu(x) = 0$ a.e. in Ω. Thus, u is a weak solution of (1.1)–(1.2). □

Lemma 2.2. *The operator Γ is bounded.*

Proof. Let M be a bounded subset of $W^{1,p}(\Omega)$, that is, there exists a constant $C_1 \ge 0$ such that $\|u\|_{W^{1,p}(\Omega)} \le C_1$, for all $u \in M$. Our goal is to prove that there is a constant $C_2 \ge 0$ such that $\|\Gamma(u)\|_{W^{1,p}(\Omega)^*} \le C_2$, for all $u \in M$. Hereafter, the symbol $\lesssim$ between two terms means that the first term is bounded from above by the second term up to a multiplicative positive constant that may depend on M but not on the individual elements of M.

For $u \in M$ and $v \in W^{1,p}(\Omega)$, we have

$$|\langle \Gamma(u), v\rangle| \le |\langle \mathcal{A}(u), v\rangle| + |\langle \mathcal{B}(u), v\rangle| + |\langle \mathcal{F}(u), v\rangle| + |\langle \mathcal{G}(u), v\rangle|. \tag{2.8}$$

Let us place an upper bound on the terms on the right-hand side of inequality (2.8).

$$\begin{aligned}|\langle \mathcal{A}(u), v\rangle| &\le \int_\Omega |A(x,\nabla u)|\cdot|\nabla u|dx\\&\le \int_\Omega (a_1(x) + b_1|\nabla u|^{p-1})\cdot|\nabla v|dx\\&\le \|a_1\|_{L^{p'}(\Omega)}\|\nabla v\|_{L^p(\Omega)} + b_1\|\nabla u\|^{p-1}_{L^p(\Omega)}\|\nabla v\|_{L^p(\Omega)}\\&\le (\|a_1\|_{L^{p'}(\Omega)} + b_1\|u\|^{p-1}_{W^{1,p}(\Omega)})\|v\|_{W^{1,p}(\Omega)}\\&\lesssim \|v\|_{W^{1,p}(\Omega)}\end{aligned} \tag{2.9}$$

Next, observe that

$$\begin{aligned}|b(x,u)| &\le \rho^{-p}(|u(x)|+|\overline{u}(x)|+|\underline{u}(x)|)^{p-1}\\ &\lesssim \rho^{-p}(|u(x)|^{p-1}+|\overline{u}(x)|^{p-1}+|\underline{u}(x)|^{p-1})\\ &\lesssim \rho^{-1}A_1(x)+\rho^{-p}|u(x)|^{p-1},\end{aligned}$$

with $A_1(x) := |\rho^{-1}\overline{u}(x)|^{p-1}+|\rho^{-1}\underline{u}(x)|^{p-1} \in L^{p'}(\Omega)$. Thus,

$$\begin{aligned}|\langle \mathcal{B}(u),v\rangle| &\le \int_\Omega |b(x,u)|\cdot|v|dx\\ &\lesssim \int_\Omega (A_1(x)+|\rho^{-1}u|^{p-1})|\rho^{-1}v|dx\\ &\lesssim (\|A_1\|_{L^{p'}(\Omega)}+\|\rho^{-1}u\|^{p-1}_{L^p(\Omega)})\|\rho^{-1}v\|_{L^p(\Omega)}\\ &\lesssim \|v\|_{W^{1,p}(\Omega)}.\end{aligned} \tag{2.10}$$

For the third term of the right-hand side of (2.8) we obtain the following upper bound

$$\begin{aligned}|\langle \mathcal{F}(u),v\rangle| &\le \int_\Omega |F(x,Tu)|\cdot|v|dx\\ &\le \int_\Omega \rho^{-1}a_3(x)|v|dx\\ &\le \|a_3\|_{L^{p'}(\Omega)}\|\rho^{-1}v\|_{L^p(\Omega)}\\ &\lesssim \|v\|_{W^{1,p}(\Omega)}.\end{aligned} \tag{2.11}$$

Finally, for the last term of the right-hand side of (2.8) we have

$$\begin{aligned}|\langle \mathcal{G}(u),v\rangle| &\le \int_{\partial\Omega} |f(x,Tu)|\cdot|v|d\sigma\\ &\le \int_{\partial\Omega} (a_4(x)+b_3(x)|Tu|^{p-1})\cdot|v|d\sigma\\ &\le \int_{\partial\Omega} [a_4(x)+b_3(x)(|\underline{u}(x)|^{p-1}+|\overline{u}(x)|^{p-1})]\cdot|v|d\sigma\\ &\le \|a_4(x)+b_3(x)(|\underline{u}(x)|^{p-1}+|\overline{u}(x)|^{p-1})\|_{L^{p'}(\partial\Omega)}\|v\|_{L^p(\partial\Omega)}\\ &\lesssim \|v\|_{W^{1,p}(\Omega)},\end{aligned} \tag{2.12}$$

where the last inequality is a consequence of the trace theorem. Returning to inequality (2.8), and using (2.9), (2.10), (2.11), and (2.12), it follows that there exists a positive constant C_2 such that $|\langle\Gamma(u),v\rangle| \le C_2\|v\|_{W^{1,p}(\Omega)}$, for all $u\in M$ and all $v\in W^{1,p}(\Omega)$; that is, $\|\Gamma(u)\|_{W^{1,p}(\Omega)^*} \le C_2$, for all $u \in M$. $\square$

Lemma 2.3. *The operator Γ is coercive; that is,*

$$\lim_{\|u\|_{W^{1,p}(\Omega)}\to\infty} \frac{\langle\Gamma(u),u\rangle}{\|u\|_{W^{1,p}(\Omega)}} = \infty. \tag{2.13}$$

Proof. First of all, observe that

$$\langle\mathcal{A}(u),u\rangle \ge b_2\|\nabla u\|^p_{L^p(\Omega)} - \|a_2\|_{L^1(\Omega)}. \tag{2.14}$$

It is easy to prove that, for $a>b$ and $p>1$, there are positive constants C_1, C_2, C_3, and C_4 (independent of a, b) such that $(a-b)^{p-1}a \ge C_1|a|^p - C_2|b|^{p-1}|a|$ and

$(a-b)^{p-1}b \le C_3|a|^{p-1}|b| - C_4|b|^p$. Then

$$\begin{aligned}\langle \mathcal{B}(u), u\rangle &= \int_{\{u>\overline{u}\}} \rho^{-p}(u-\overline{u})^{p-1}u\,dx - \int_{\{u<\underline{u}\}} \rho^{-p}(\underline{u}-u)^{p-1}u\,dx\\ &\ge \int_{\{u>\overline{u}\}} \rho^{-p}(C_1|u|^p - C_2|\overline{u}|^{p-1}|u|)dx\\ &\quad + \int_{\{u<\underline{u}\}} \rho^{-p}(C_4|u|^p - C_3|\underline{u}|^{p-1}|u|)dx\\ &\ge \min\{C_1, C_4\}\|\rho^{-1}u\|^p_{L^p(\Omega)} - C_5\|\rho^{-1}u\|_{L^p(\Omega)},\end{aligned} \tag{2.15}$$

with $C_5 := C_2\|\rho^{-1}\overline{u}\|^{p-1}_{L^p(\Omega)} + C_3\|\rho^{-1}\underline{u}\|^{p-1}_{L^p(\Omega)}$. Also,

$$\langle \mathcal{F}(u), u\rangle \ge -\|a_3\|_{L^{p'}(\Omega)}\|\rho^{-1}u\|^p_{L^p(\Omega)} \ge -\|a_3\|_{L^{p'}(\Omega)}\|u\|^p_{W^{1,p}(\Omega)} \tag{2.16}$$

and

$$\begin{aligned}\langle \mathcal{G}(u), u\rangle &\ge -\int_{\partial\Omega}[|a_4(x)| + |b_3(x)|(|\underline{u}(x)|^{p-1} + |\overline{u}(x)|^{p-1})]|u|\,d\sigma\\ &\ge -\||a_4| + |b_3|(|\underline{u}|^{p-1} + |\overline{u}|^{p-1})\|_{L^{p'}(\partial\Omega)}\|u\|_{L^p(\partial\Omega)}\\ &\ge -C_6\||a_4| + |b_3|(|\underline{u}|^{p-1} + |\overline{u}|^{p-1})\|_{L^{p'}(\partial\Omega)}\|u\|_{W^{1,p}(\Omega)},\end{aligned} \tag{2.17}$$

where the last inequality follows from the trace theorem. Combining (2.14), (2.15), (2.16), and (2.17), we get

$$\langle \Gamma(u), u\rangle \ge C_7\|u\|^p_{W^{1,p}(\Omega)} - C_8\|u\|_{W^{1,p}(\Omega)} - C_9, \quad \forall u \in W^{1,p}(\Omega),$$

with C_7, C_8, $C_9 > 0$. Because $p > 1$, this estimate implies (2.13). $\square$

Fix an integer $n_0 > \max_{x\in\partial\Omega}|x|$. For any $n \ge n_0$, we set $\Omega_n = \{x \in \Omega : |x| < n\}$ and introduce the space $W_n := \{u \in W^{1,p}(\Omega_n) : u = 0 \text{ on } |x| = n\}$. Notice that we can consider that $W_n \subset W^{1,p}(\Omega)$ by setting, for all $w \in W_n$, $w(x) = 0$ whenever $x \in \Omega$ with $|x| > n$ (which is possible since $w = 0$ on $|x| = n$). For each $n \ge n_0$, let $i_n : W_n \to W^{1,p}(\Omega)$ denote the inclusion map and $i_n^* : W^{1,p}(\Omega)^* \to W_n^*$ its adjoint operator. Fix $n \ge n_0$ and introduce the nonlinear operator $\Gamma_n : W_n \to W_n^*$ by

$$\Gamma_n := i_n^*\Gamma i_n = i_n^*\mathcal{A}i_n + i_n^*\mathcal{B}i_n + i_n^*\mathcal{F}i_n + i_n^*\mathcal{G}i_n.$$

Lemma 2.4. *For every $n \ge n_0$, the equation $\Gamma_n(u) = 0$ has at least a solution (in W_n).*

Proof. The operator $\Gamma_n : W_n \to W_n^*$ is pseudomonotone because it is the sum of the strictly monotone operator $i_n^*\mathcal{A}i_n$ and the completely continuous operators $i_n^*\mathcal{B}i_n$, $i_n^*\mathcal{F}i_n$, $i_n^*\mathcal{G}i_n$ (which is true because the domain Ω_n is bounded). The operator Γ_n is also bounded by Lemma 2.2 and the boundedness of the operators i_n and i_n^*. Moreover, from Lemma 2.3, we see that Γ_n is coercive. The application of the abstract surjectivity result (see [34, Theorem 27.A]) completes the proof. $\square$

2.1. **Proof of Theorem 1.1.** Lemma 2.4 ensures that there exists $u_n \in W_n$ such that

$$\begin{aligned}&\int_{\Omega_n} A(x, \nabla u_n)\cdot\nabla v\,dx + \int_{\Omega_n} b(x, u_n)v\,dx\\ &- \int_{\Omega_n} F(x, Tu_n)v\,dx - \int_{\partial\Omega} f(x, Tu_n)v\,d\sigma = 0\end{aligned} \tag{2.18}$$

for all $v \in W_n$. Setting $v = u_n$ in (2.18) and using the coercivity of the operator Γ_n lead to the conclusion that the sequence $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$.

Thus, up to a subsequence, we may suppose that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, $u_n \to u$ in $L^p_{\rm loc}(\Omega)$ and a.e. in Ω and $\nabla u_n \to \nabla u$ in $L^p_{\rm loc}(\Omega, \mathbb{R}^N)$, for some $u \in W^{1,p}(\Omega)$. Let $v \in C_0^\infty(\mathbb{R}^N) \cap W^{1,p}(\Omega)$. We note that $v \in W_n$ for n sufficiently large, so we can make use of (2.18) which gives

$$\begin{aligned} &\int_{\operatorname{supp}(v)} A(x, \nabla u_n) \cdot \nabla v \, dx + \int_{\operatorname{supp}(v)} b(x, u_n) v \, dx \\ &- \int_{\operatorname{supp}(v)} F(x, Tu_n) v \, dx - \int_{\partial\Omega} f(x, Tu_n) v \, d\sigma = 0. \end{aligned} \tag{2.19}$$

We may pass to the limit in (2.19) as $n \to \infty$. Since $C_0^\infty(\mathbb{R}^N) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$, we arrive at $\Gamma u = 0$. Now it suffices to invoke Lemma 2.1 for concluding that u is a weak solution of problem (1.1)–(1.2) with $\underline{u}(x) \le u(x) \le \overline{u}(x)$ a.e. in Ω.

Remark 2.5. One can show the uniqueness of the solution by assuming additional conditions, such as $F(x, \cdot)$ and $f(x, \cdot)$ are nonincreasing on the interval $[\underline{u}(x), \overline{u}(x)]$ for a.e. $x \in \Omega$. Let $u_1, u_2 \in W^{1,p}(\Omega)$ be two weak solutions to (1.1)–(1.2) belonging to the ordered interval $[\underline{u}, \overline{u}]$. Then we can write

$$\begin{aligned} &\int_\Omega (A(x, \nabla u_1) - A(x, \nabla u_2)) \cdot \nabla (u_1 - u_2)^+ \, dx \\ &= \int_\Omega (F(x, u_1) - F(x, u_2)(u_1 - u_2)^+ \, dx + \int_{\partial\Omega} (f(x, u_1) - f(x, u_2))(u_1 - u_2)^+ \, d\sigma. \end{aligned}$$

In view of our hypothesis and since the operator $A(x, \cdot)$ is strictly monotone, we derive that $u_1 \le u_2$ (and similarly that $u_2 \le u_1$) a.e. in Ω, and so $u_1 = u_2$ a.e. in Ω.

3. Application to the Initial Data Problem in General Relativity

In this section, we indicate an example where we apply Theorem 1.1 to the existence of the conformal factor in general relativity. We mention that this section contains just an example of aplication of Theorem 1.1; it is in no way intended to give a deep or extensive analysis of the complicated initial data problem in general relativity. The interested reader can find important advances on various aspects of this subject in [5, 7, 10, 11, 8, 9, 17, 18, 19, 27, 29], among many others.

In Subsection 3.1, we briefly review York-Lichnerowicz's formalism for decomposing the constraint equations. We then discuss the existence of the conformal factor under certain assumptions. Finally, in Subsection 3.2, we present an elementary proof for the uniqueness in the case of multiple black holes.

3.1. **York-Lichnerowicz conformal decomposition method.** In general relativity, spacetime is a 4-dimensional manifold of events endowed with a pseudo-Riemannian metric $g_{\alpha\beta}$. Einstein's equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ connect the spacetime curvature represented by the Einstein tensor $G_{\alpha\beta}$ with the stress-energy tensor $T_{\alpha\beta}$. In fact, these are equations for geometries, that is, their solutions are equivalent classes under spacetime diffeomorphisms of metric tensors. To break this diffeomorphism invariance, Einstein's equations must first be transformed into a system having a well-posed Cauchy problem. That is, the spacetime is foliated and each slice Σ_t is characterized by its intrinsic geometry γ_{ij} and extrinsic curvature K_{ij}, which is essentially the "velocity" of γ_{ij} in the unit normal direction to the slice. Subsequent slices are connected via the lapse function N and shift vector β^i corresponding to the Arnowitt–Deser–Misner (ADM) 3+1 formulation [1] of the line element $ds^2 = -N^2dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$. This decomposition allows one to express six of the ten components of Einstein's equations in vacuum ($T_{\alpha\beta} = 0$) as a constrained system of evolution equations for the metric γ_{ij} and the extrinsic curvature K_{ij} (repeated subscript-superscript indices means summation):

$$\dot{\gamma}_{ij} = -2NK_{ij} + 2\nabla_{(i}\beta_{j)},$$
$$\dot{K}_{ij} = N(R_{ij} + K^l_l K_{ij} - 2K_{il}K^l_j) + \beta^l\nabla_l K_{ij} + K_{il}\nabla_j\beta^l + K_{lj}\nabla_i\beta^l - \nabla_i\nabla_j N,$$
$$R^i_i + (K^i_i)^2 - K_{ij}K^{ij} = 0, \tag{3.1}$$
$$\nabla^j K_{ij} - \nabla_i K^j_j = 0, \tag{3.2}$$

where we use a dot to denote time differentiation and ∇_j for the covariant derivative associated to γ_{ij}. The spatial Ricci tensor R_{ij} has components given by second order spatial differential operators applied to the spatial metric components γ_{ij}. Indices are raised and traces taken with respect to the spatial metric γ_{ij}, and parenthesized indices are used to denote the symmetric part of a tensor (e.g., $\nabla_{(i}\beta_{j)} := (\nabla_i\beta_j + \nabla_j\beta_i)/2$).

To evolve Einstein's equations in the standard ADM 3+1 formulation, one needs to specify the 3-metric γ_{ij} and the extrinsic curvature K_{ij} on the initial time slice Σ_0. This is a difficult task, as these quantities must satisfy the constraint equations (3.1) and (3.2). We outline here the conformal decomposition method of York-Lichnerowicz (see [6, 29, 31, 32, 33]) for the vacuum constraint equations. The base of the method consists of specifying the physical data only up to conformal equivalence, under the assumption that the trace of K_{ij}, $K^i_i := \gamma^{ij}K_{ij}$, is given and fixed. In essence, this means that we look for a metric γ_{ij} conformally related to a given metric $\hat{\gamma}_{ij}$ by $\gamma_{ij} = \psi^4\hat{\gamma}_{ij}$, where the conformal factor ψ is a strictly positive function to be determined. We will denote by $\hat{\gamma}^{ij}$, $\hat{\nabla}_j$, and $\hat{R}$ the inverse metric, covariant derivative operator, and scalar curvature associated to the metric $\hat{\gamma}_{ij}$. We now relate these to quantities based on the original metric γ_{ij}. The inverse metric $\hat{\gamma}^{ij}$ and the covariant derivative $\hat{\nabla}_j$ of scalars are easy: $\gamma^{ij} = \psi^{-4}\hat{\gamma}^{ij}$ and $\nabla_j K = \hat{\nabla}_j K$ and $\nabla^j K = \psi^{-4}\hat{\nabla}^j K$ for any scalar function K. For the covariant derivative of tensors and for the scalar curvature, we need to relate the Christoffel symbols $\hat{\Gamma}^k_{ij}$ formed with respect to $\hat{\gamma}^{ij}$ to the Christoffel symbols Γ^k_{ij} formed with respect to γ^{ij}. By direct calculation

$$\Gamma^i_{jk} = \frac{1}{2}\gamma^{il}(\frac{\partial\gamma_{lj}}{\partial x^k} + \frac{\partial\gamma_{lk}}{\partial x^j} - \frac{\partial\gamma_{jk}}{\partial x^l}) = \hat{\Gamma}^i_{jk} + 2\psi^{-1}(\frac{\partial\psi}{\partial x^k}\delta^i_j + \frac{\partial\psi}{\partial x^j}\delta^i_k - \frac{\partial\psi}{\partial x^l}\hat{\gamma}_{jk}\hat{\gamma}^{il}),$$

and so

$$\Gamma^j_{jk} = \hat{\Gamma}^j_{jk} + 6\psi^{-1}\frac{\partial\psi}{\partial x^k}.$$

Now, let us relate the extrinsic curvature K^{ij} corresponding to γ_{ij} to a given symmetric $(2,0)$ tensor $\hat{K}^{ij}$ by $K^{ij} = \psi^{-s}\hat{K}^{ij}$ for some s. Then, by direct calculation,

$$\begin{aligned}\nabla_j K^{ij} &= \frac{\partial K^{ij}}{\partial x^j} + \Gamma^j_{jl}K^{il} + \Gamma^i_{jl}K^{lj} \\ &= \psi^{-s}\hat{\nabla}_j\hat{K}^{ij} - 2\psi^{-s-1}\frac{\partial\psi}{\partial x^m}\hat{\gamma}^{im}\hat{K} + (10-s)\psi^{-s-1}\frac{\partial\psi}{\partial x^l}\hat{K}^{il},\end{aligned}$$

where $\hat{K} = \hat{\gamma}_{ij}\hat{K}^{ij}$. This motivates the choice $s = 10$. Moreover, we choose the tensor $\hat{K}^{ij}$ to be trace-free, i.e., $\hat{K} = 0$. Then, the zero trace is preserved, i.e., K^{ij} is trace-free, and $\nabla_j K^{ij} = \psi^{-10}\hat{\nabla}_j \hat{K}^{ij}$. The scalar curvatures $R = \gamma_{ij}R^{ij}$ and $\hat{R} = \hat{\gamma}_{ij}\hat{R}^{ij}$ are related by $R = \psi^{-4}\hat{R} - 8\psi^{-5}\hat{\Delta}\psi$, where $\hat{\Delta}\psi := \hat{\gamma}_{ij}\hat{\nabla}^i\hat{\nabla}^j\psi$ is the Laplacian of ψ with respect to the metric $\hat{\gamma}_{ij}$.

If we choose $\hat{\gamma}_{ij}$ to be the flat metric $\hat{\gamma}_{ij} := \delta_{ij}$, then the momentum constraints (3.2) are linear, decoupled from the Hamiltonian constraint (3.1) (as a consequence of the assumption $\hat{K} = 0$), and solutions $\hat{K}_{ij}$ to them can be determined analytically (see [13, 14, 23, 29, 31, 32, 33], among others). Moreover, the Hamiltonian constraint equation reduces to the relatively simple semilinear elliptic equation

$$-\Delta\psi = \hat{H}\psi^{-7}, \tag{3.3}$$

where $\Delta := \delta_{ij}\partial^i\partial^j$ is the usual 3D Laplacian and $\hat{H} := \frac{1}{8}\hat{K}^{ij}\hat{K}_{ij}$ is a positive function. It is also necessary to specify the domain on which this equation will be solved, and the boundary conditions that will be applied. In the case of multiple black holes, our goal is to solve equation (3.3) in the exterior domain $\Omega := \{x \in \mathbf{R}^3 : |x - O_i| > R_i,\, i = \overline{1, N}\}$, where O_i, respectively R_i, $i = 1, 2, \dots, N$, are the centers, respectively the radii, of the disjoint black holes. Because we are interested in asymptotically flat spacetimes, we would like that the conformal factor approach unity as the distance from any sources approaches infinity:

$$\psi(x) \to 1, \quad \text{as } |x| \to \infty. \tag{3.4}$$

Also, we invoke an inner boundary condition (see [6, 29, 31], and references therein)

$$\frac{\partial\psi}{\partial n} + \frac{1}{2R_i}\psi = 0 \quad \text{on } \partial B(O_i,\, R_i),\ i = 1, 2, \dots, N, \tag{3.5}$$

where the normal n to $\partial B(O_i, R_i)$ points *into* the domain Ω.

Let $u := \psi - 1$. For ψ to be a solution of (3.3)–(3.5), u must satisfy the following boundary value problem in the exterior domain Ω:

$$-\Delta u = \hat{H}(1 + u)^{-7} \quad \text{in } \Omega, \tag{3.6}$$

$$u(x) \to 0 \quad \text{as } |x| \to \infty, \tag{3.7}$$

$$\frac{\partial u}{\partial n} = -\frac{1}{2R_i}(1 + u) \quad \text{on } \partial B(O_i,\, R_i),\ i = 1, 2, \dots, N. \tag{3.8}$$

Theorem 3.1. *Suppose that $\rho\hat{H} \in L^2(\Omega)$. Then there exists at least one (weak) solution $u \in W^{1,2}(\Omega)$ to* (3.6)–(3.8).

Proof. Observe that one can now apply Theorem 1.1 to the boundary-value problem (3.6)–(3.8) if a pair of sub- and supersolution $\underline{u}$ and $\overline{u}$ can be found. It is easy to see that $\underline{u} :\equiv 0$ is a subsolution to (3.6)–(3.8). Furthermore, the solution $\overline{u} \in W^{1,2}(\Omega)$ of the following Dirichlet boundary-value problem (whose existence is guarateed by [22, Theorem 2.5.14])

$$-\Delta\overline{u}(x) = \hat{H}(x) \quad \text{in } \Omega, \quad \overline{u}(x) = 0 \quad \text{on } \partial\Omega,$$

is a supersolution to (3.6)–(3.8). Moreover, by the maximum principle one obtains $\overline{u} > 0$ in Ω. Then, by Theorem 1.1 it follows that there exists a weak solution $u \in W^{1,2}(\Omega)$, with $(\underline{u} \equiv)0 \le u \le \overline{u}$, to the boundary-value problem (3.6)–(3.8). In fact, by the maximum principle again, u is strictly positive in Ω. In addition, since $u \in W^{1,2}(\Omega)$, we also have that $u(x) \to 0$, as $|x| \to \infty$, a.e. in Ω. ∎

3.2. **The uniqueness in the general case of multiple black holes.** In 1989 York [31] proved that the solution for the boundary value problem (3.3), (3.4), and (3.5) is locally unique, that is, he proved that no other solutions lie in the neighborhood of a given solution, but this does not preclude the existence of other solutions which are "significantly different." Here the normal n to $\partial\Omega$ points into the domain Ω, and this interferes with an usual existence and uniqueness analysis for the problem. That is, even though (3.5) looks like a Robin boundary condition, it has the "wrong" sign between its two terms. Therefore, as observed in [31], one cannot give a standard uniqueness argument for the problem (3.3), (3.4), and (3.5). In what follows we give a simple proof for uniqueness in the case of multiple black-holes; it has some points in common with the proof pointed out by York in [31].

Theorem 3.2. *There exists at most one solution to the elliptic exterior boundary-value problem* (3.3)*,* (3.4)*, and* (3.5)*.*

Proof. Arguing by contradiction, suppose that we have two distinct solutions for (3.3), (3.4), and (3.5). Denote by u and v these two solutions. For each $i = 1, \dots, N$, by passing to spheric coordinates with respect to O_i, we define a related function

$$\tilde{u}_i(r,\theta,\phi) = \frac{R_i}{r} u(\overline{r},\theta,\phi),$$

where $\overline{r} = R_i^2/r$, $0 < r \le R_i$. Note that $\tilde{u}_i(R_i,\theta,\phi) = u(R_i,\theta,\phi)$. Moreover, the first derivatives of u and $\tilde{u}$ agree at $r = R_i$ (we need only check the radial derivatives)

$$\frac{\partial \tilde{u}_i}{\partial r}(r,\theta,\phi) = -\frac{R_i}{r^2} u(\overline{r},\theta,\phi) - \frac{R_i^3}{r^3}\frac{\partial u}{\partial r}(\overline{r},\theta,\phi),$$

and so

$$\frac{\partial \tilde{u}_i}{\partial r}(R_i,\theta,\phi) = -\frac{1}{R_i} u(R_i,\theta,\phi) - \frac{\partial u}{\partial r}(R_i,\theta,\phi) = \frac{\partial u}{\partial r}(R_i,\theta,\phi), \tag{3.9}$$

where the last equality in (3.9) follows from the boundary condition (3.5).

Likewise, one finds that the second derivatives of u and $\tilde{u}_i$ also match at $r = R_i$.

$$\frac{\partial^2 \tilde{u}_i}{\partial r^2}(r,\theta,\phi) = \frac{2R_i}{r^3} u(\overline{r},\theta,\phi) + \frac{4R_i^3}{r^4}\frac{\partial u}{\partial r}(\overline{r},\theta,\phi) + \frac{R_i^5}{r^5}\frac{\partial^2 u}{\partial r^2}(\overline{r},\theta,\phi),$$

and so

$$\frac{\partial^2 \tilde{u}_i}{\partial r^2}(R_i,\theta,\phi) = \frac{4}{R_i}\Big(\frac{\partial u}{\partial r}(R_i,\theta,\phi) + \frac{1}{2R_i} u(R_i,\theta,\phi)\Big) + \frac{\partial^2 u}{\partial r^2}(R_i,\theta,\phi), \tag{3.10}$$

where the first term of the right-hand side of (3.10) vanishes because of the boundary condition (3.5).

Furthermore, simple computations show that

$$\Delta \tilde{u}_i(r,\theta,\phi) = \frac{R_i^5}{r^5}\Delta u(\overline{r},\theta,\phi). \tag{3.11}$$

Hence we can extend u as follows

$$U(x) = \begin{cases} u(x) & \text{for } x \in \overline{\Omega} \\ \tilde{u}_i(x) & \text{for } x \in J_i(\Omega) \subset B(O_i,R_i),\ i = 1,2,\dots,N, \end{cases}$$

where

$$J_i(x) = \frac{R_i^2}{|x - O_i|^2}(x - O_i) + O_i,$$

for all $x \neq O_i$, $i = \overline{1,N}$. Observe that U is in $C^2(\tilde\Omega)$, $\tilde\Omega := \Omega \cup J_1(\Omega) \cup J_2(\Omega) \ldots \cup J_N(\Omega)$, and (as a consequence of (3.11)) it satisfies the following differential equation in the open set $\tilde\Omega$

$$-\Delta U = \tilde{H} U^{-7},$$

where

$$\tilde{H}(x) = \begin{cases} \hat{H}(x) & \text{for } x \in \overline{\Omega} \\ R_i^{12}|x - O_i|^{-12}\hat{H}(J_i^{-1}(x)) & \text{for } x \in J_i(\Omega),\ i = 1, 2, \ldots, N. \end{cases}$$

Doing the same for v, we get its extension V in $\tilde\Omega$. Without restricting generality, we can assume $U(x) > V(x)$ in a nonzero measure subset of $\tilde\Omega$. Let $w(x) = \ln|U(x)/V(x)|$. Since both u and v tend to 1 as $|x| \to \infty$ and by the construction of U and V, it follows that $\lim_{|x|\to\infty} |U(x)/V(x)| = 1$ and $\lim_{x\to O_i} |U(x)/V(x)| = 1$, $i = 1, 2, \ldots, N$. Therefore, there exists x_0 in the closure of the set $\tilde\Omega$ such that $w(x_0) = \sup_{x\in\tilde\Omega} w(x)$. First, let us prove that x_0 must belong to $\partial\tilde\Omega$. Arguing by contradiction, assume that x_0 belongs to the interior of $\tilde\Omega$. Then, from $\nabla w(x_0) = 0$, it follows that

$$\frac{1}{U(x_0)}\nabla U(x_0) = \frac{1}{V(x_0)}\nabla V(x_0),$$

and so

$$\begin{aligned} \Delta w(x_0) &= \frac{1}{U(x_0)}\Delta U(x_0) - \frac{1}{V(x_0)}\Delta V(x_0) \\ &\quad + \frac{1}{U(x_0)^2}|\nabla U(x_0)|^2 - \frac{1}{V(x_0)^2}|\nabla V(x_0)|^2 \\ &= -\tilde{H}(x_0)\Big(\frac{1}{U(x_0)^8} - \frac{1}{V(x_0)^8}\Big) > 0, \end{aligned} \tag{3.12}$$

which is impossible. This forces x_0 to belong to $\partial\tilde\Omega$.

Suppose that $x_0 \in J_i(\partial B(O_j, R_j))$ for some i and j, with $i \neq j$. Then, for $\tilde{x}_0 := J_i^{-1}(x_0) \in \partial B(O_j, R_j)$ we have

$$w(\tilde{x}_0) = \ln|U(\tilde{x}_0)/V(\tilde{x}_0)| = \ln|u(\tilde{x}_0)/v(\tilde{x}_0)| = \ln|\tilde{u}_i(x_0)/\tilde{v}_i(x_0)| = w(x_0),$$

and so $w(\tilde{x}_0) = \sup_{x\in\tilde\Omega} w(x)$. Since $\tilde{x}_0$ is an interior point of $\tilde\Omega$, we get the same contradiction as in (3.12). $\square$

References

[1] A. Arnowitt, S. Deser, and C. Misner; in *Gravitation: An introduction to current research*, edited by L. Witten, Wiley, New York, 1962.

[2] S. Carl, V. K. Le, and D. Motreanu; *Comparison principles for nonsmooth variational problems*, Springer, New York, 2006.

[3] F. Carstea, D. Motreanu, and V. Radulescu; *Weak solutions of quasilinear problems with nonlinear boundary condition*, Nonlinear Anal. Ser. A: Theory Methods **43** (2001), 623–636.

[4] J. Chabrowski and Z.–Q. Wang; *Exterior nonlinear Neumann problem. NoDEA Nonlinear Differential Equations Appl*, **13** (2007), 683–697.

[5] P. T. Chrusciel, J. Isenberg, and D. Pollack; *Initial data engineering*, Comm. Math. Phys. **257** (2005), 29–42.

[6] G. B. Cook and S. A. Teukolsky; *Numerical Relativity: Challenges for Computational Science*, Acta Numerica **8** (1999), 1–44.

[7] S. Dain; *Trapped surfaces as boundaries for the constraint equations*, Classical Quantum Gravity **21** (2004), 555–573.

[8] M. Holst, G. Nagy, and G. Tsogtgerel; *Rough solutions of the Einstein constraints on closed manifolds without near-CMC conditions*, Comm. Math. Phys. **288** (2009), 547–613.

[9] E. Hebey, F. Pacard, and D. Pollack; *A variational analysis of Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds*, Comm. Math. Phys. **278** (2008), 117–132.

[10] J. Isenberg; *Constant mean curvature solutions of the Einstein constraint equations on closed manifolds*, Class. Quantum Grav. **12** (1995), 2249–2274.

[11] J. Isenberg and V. Moncrief; *A set of nonconstant mean curvature solutions of the Einstein constraint equations on closed manifolds*, Class. Quantum Grav. **13** (1996), 1819–1847.

[12] D. A. Kandilakis and A. N. Lyberopoulos; *Indefinite quasilinear elliptic problems with subcritical and supercritical nonlinearities on unbounded domains*, J. Differential Equations **230** (2006), 337–361.

[13] A. D. Kulkarni; *Time-asymmetric initial data for the N black hole problem in general relativity*, J. Math. Phys. **25** (1984), 1028–1034.

[14] A. D. Kulkarni, L. C. Shepley, and J. W. York, Jr.; *Initial Data for N Black Holes*, Physics Letters **96** (1983), 228–230.

[15] V. K. Le and K. Schmitt, *On boundary value problems for degenerate quasilinear elliptic equations and Inequalities*, J. Diff. Eq. **144** (1998), 170–218.

[16] V. K. Le and K. Schmitt; *Sub-supersolution theorems for quasilinear elliptic problems: A variational approach*, E. J. Diff. Eq. **118** (2004), 1–7.

[17] D. Maxwell; *Solutions of the Einstein constraint equations with apparent horizon boundaries* Comm. Math. Phys. **253** (2005), 561–583.

[18] D. Maxwell; *Rough solutions of the Einstein constraint equations*, J. Reine Angew. Math. **590** (2006), 1–29.

[19] L. Mazzieri; *Generalized gluing for Einstein constraint equations*, Calc. Var. Partial Differential Equations **34** (2009), 453–473.

[20] I. I. Mezei and C. Varga; *Multiplicity results for a double eigenvalue quasilinear problem on unbounded domain*, Nonlinear Anal. **69** (2008), 4099–4105.

[21] E. Montefusco and V. Radulescu, *Nonlinear eigenvalue problems for quasilinear operators on unbounded domains*, NoDEA Nonlinear Differential Equations Appl. **8** (2001), 481–497.

[22] J.–C. Nedelec; *Acoustic and Electromagnetic Equations*, Springer-Verlag, 2001, 59–84.

[23] N. S. O'Murchaldha and J. W. York, Jr.; *Existence and uniqueness of solutions of the Hamiltonian constraint of general relativity on compact manifolds*, J. Math. Phys. **14** (1973), 1551–1557.

[24] K. Pflüger; *Nonlinear boundary value problems in weighted Sobolev spaces*, Nonlinear Anal. **30** (1997), 1263–1270.

[25] K. Pflüger; *Existence and multiplicity of solutions to a p-Laplacian equation with nonlinear boundary condition*, Electron. J. Differential Equations 1998, No. 10, 13 pp. (electronic).

[26] J. Su, Z.–Q. Wang, and M. Willem; *Weighted Sobolev embedding with unbounded and decaying radial potentials*, J. Differential Equations **238** (2007), 201–219.

[27] N. Tarfulea; *Positive radial symmetric solutions to an exterior elliptic Robin boundary-value problem and application*, Nonlinear Analysis: Theory, Methods & Applications, **71** (2009), 1909–1915.

[28] N. Tarfulea; *Positive solutions of some nonlinear elliptic equations involving the p-Laplacian with Neumann boundary condition*, Rev. Roumaine Math. Pures Appl. **44** (1999), 143–151.

[29] J. Thornburg; *Coordinates and boundary conditions for the general relativistic initial data problem*, Class. Quantum Grav. **4** (1987), 1119–1131.

[30] P. Volkmann and A. Chaljub-Simon; *Existence of ground states with exponential decay for semilinear elliptic equations in* $\mathbf{R}^n$, J. Differ. Equations **76** (1988), 374–390.

[31] J. W. York, Jr.; *Initial Data for Collisions of Black Holes and Other Gravitational Miscellany*, in Evans, Finn, and Hobill, Cambridge U. Press, 1989, 89–109.

[32] J. W. York, Jr., in *Sources of Gravitational Radiation*, edited by L. L. Smarr, Cambridge University Press, Cambridge, England, 1979, 83–126.

[33] J. W. York, Jr. and T. Piran; *The Initial Value Problem and Beyond*, in R. A. Matzner and L. C. Shepley, editors; *Spacetime and Geometry*, 1982, 145–176.

[34] E. Zeidler; *Nonlinear Functional Analysis and its Applications II/B, Nonlinear Monotone Operators*, 1990.

Dumitru Motreanu
Département de Mathématiques, Université de Perpignan, 66025 Perpignan, France
E-mail address: motreanu@univ-perp.fr

Nicolae Tarfulea
Department of Mathematics, Purdue University Calumet, IN 46323, USA
E-mail address: tarfulea@calumet.purdue.edu