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POSITIVITY AND STABILITY FOR A SYSTEM OF TRANSPORT EQUATIONS WITH UNBOUNDED BOUNDARY PERTURBATIONS

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Abstract. This article concerns wellposedness, positivity and spectral properties of the solution of a system of transport equations with unbounded boundary perturbations. In particular we obtain that the rescaled solution converges to the unique steady-state solution as time approaches infinity on a weighted L^1-space.

1. Introduction

Inspired from a queueing network model studied by [4], [6], [8], [10], we propose in this paper to study the qualitative and the quantitative properties of the system of partial differential equations

$$
\begin{gathered}
\frac{\partial p_0(x,t)}{\partial t} + \frac{\partial p_0(x,t)}{\partial x} = \eta \int_0^1 \mu(x) p_1(x,t)dx, \quad t \geq 0,\, x \in (0,1), \\
\frac{\partial p_1(x,t)}{\partial t} + \frac{\partial p_1(x,t)}{\partial x} = -(\alpha + \mu(x)) p_1(x,t), \quad t \geq 0,\, x \in (0,1), \\
\frac{\partial p_n(x,t)}{\partial t} + \frac{\partial p_n(x,t)}{\partial x} = -(\alpha + \mu(x)) p_n(x,t) + \alpha p_{n-1}(x,t), \\
\text{for } t \geq 0,\ x \in (0,1),\ 2 \leq n \leq N+1, \\
\frac{\partial p_{N+2}(x,t)}{\partial t} + \frac{\partial p_{N+2}(x,t)}{\partial x} = -\mu(x) p_{N+2}(x,t) + \alpha p_{N+1}(x,t), \\
\text{for } t \geq 0,\ x \in (0,1),
\end{gathered}
\tag{1.1}
$$

with the boundary conditions

$$
\begin{gathered}
p_0(0,t) = p_0(1,t), \quad t \geq 0, \\
p_1(0,t) = \alpha p_0(1,t) + q\overline{\mu} p_1(1,t) + \eta \overline{\mu} p_2(1,t), \quad t \geq 0, \\
p_n(0,t) = q\overline{\mu} p_n(1,t) + \eta \overline{\mu} p_{n+1}(1,t), \quad 2 \leq n \leq N+1,\ t \geq 0, \\
p_{N+2}(0,t) = q\overline{\mu} p_{N+2}(1,t), \quad t \geq 0,
\end{gathered}
\tag{1.2}
$$

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and the initial values

$$\begin{gathered}
p_0(x,0)=f_0(x), \quad x\in(0,1),\\
p_1(x,0)=f_1(x), \quad x\in(0,1),\\
p_n(x,0)=f_n(x), \quad 2\le n\le N+1,\ x\in(0,1),\\
p_{N+2}(x,0)=f_{N+2}(x), \quad x\in(0,1),
\end{gathered} \tag{1.3}$$

where $f_i\in L^1(0,1)$ for $i\in\{0,1,\dots,N+2\}$. Using the language of operator matrices we see that equations (1.1)-(1.2) are equivalent to

$$\partial_t\begin{pmatrix}p_0\\ p_1\\ \vdots\\ p_{N+2}\end{pmatrix}+\partial_x\begin{pmatrix}p_0\\ p_1\\ \vdots\\ p_{N+2}\end{pmatrix}=Q\begin{pmatrix}p_0\\ p_1\\ \vdots\\ p_{N+2}\end{pmatrix}+R\begin{pmatrix}p_0\\ p_1\\ \vdots\\ p_{N+2}\end{pmatrix} \tag{1.4}$$

$$\begin{pmatrix}p_0(0,t)\\ p_1(0,t)\\ \vdots\\ p_{N+2}(0,t)\end{pmatrix}=\Phi\begin{pmatrix}p_0(1,t)\\ p_1(1,t)\\ \vdots\\ p_{N+2}(1,t)\end{pmatrix}, \tag{1.5}$$

where Q is the multiplication operator

$$Q=\begin{pmatrix}
0&0&0&0&.&.&.&0&0&0\\
0&D&0&0&.&.&.&0&0&0\\
0&\alpha&D&0&.&.&.&0&0&0\\
0&0&\alpha&D&.&.&.&0&0&0\\
.&.&.&.&.&.&.&0&0&0\\
0&0&0&0&.&.&.&D&0&0\\
0&0&0&0&.&.&.&\alpha&D&0\\
0&0&0&0&.&.&.&0&\alpha&-\mu(.)
\end{pmatrix},$$

and R the integral operator

$$R=\begin{pmatrix}
0&\eta\Psi&0&0&.&.&.&0&0&0\\
0&0&0&0&.&.&.&0&0&0\\
0&0&0&0&.&.&.&0&0&0\\
0&0&0&0&.&.&.&0&0&0\\
.&.&.&.&.&.&.&0&0&0\\
0&0&0&0&.&.&.&0&0&0\\
0&0&0&0&.&.&.&0&0&0\\
0&0&0&0&.&.&.&0&0&0
\end{pmatrix}$$

with $\Psi(\varphi)=\int_0^1\varphi(x)\mu(x)dx$ and $D\varphi=-(\alpha+\mu(.))\varphi$ for $\varphi\in L^1(0,1)$. The $(N+3)\times(N+3)$-matrix Φ is

$$\Phi=\begin{pmatrix}
1&0&0&0&.&.&.&0&0&0\\
\alpha&q&\eta&0&.&.&.&0&0&0\\
0&0&q&\eta&.&.&.&0&0&0\\
0&0&0&q&.&.&.&0&0&0\\
.&.&.&.&.&.&.&0&0&0\\
0&0&0&0&.&.&.&.&0&0\\
0&0&0&0&.&.&.&0&q&\eta\\
0&0&0&0&.&.&.&0&0&q
\end{pmatrix}.$$

Here and in the sequel we suppose that $\mu \in L^\infty((0,1),\mathbb{R}_+)$, $\eta \in (0,1)$, $q := 1-\eta$, $\lambda_0 > 0$ and take without loss of generality $\int_0^1 \mu(x)dx = \overline{\mu} = 1$. Hence, equations (1.4)-(1.5) are similar to a model describing the growth of a cell population proposed by Rotenberg [11] (see also [1], [2]).

On the Banach space $X := [L^1(0,1)]^{N+3}$, $N \geq 1$, endowed with the usual norm

$$\|\varphi\| := \sum_{i=0}^{N+2} \|\varphi_i\|_{L^1(0,1)}, \quad \varphi \in X,$$

one can see that Q, $R \in \mathcal{L}(X)$. Then the problem (1.3)-(1.5) can be written as the Cauchy problem

$$\begin{gathered} P'(t) = A_m P(t) + BP(t) := L_m P(t), \quad t \geq 0, \\ \Gamma_0 P(t) = \Phi\Gamma_1 P(t) := \overline{\Phi} P(t), \\ P(0) = (f_0, \dots, f_{N+2})^T \in X, \end{gathered} \tag{1.6}$$

where $B = R + Q$, the operator A_m and the trace application Γ_0 and Γ_1 are respectively defined by

$$A_m = -\frac{\partial}{\partial x} Id_X, \quad \Gamma_0 = \gamma_0 Id_X, \quad \Gamma_1 = \gamma_1 Id_X,$$

where $\gamma_i : L^1(0,1) :\to \mathbb{C}$, $\gamma_i(\varphi) = \varphi(i)$ for $i \in \{0,1\}$ and $\varphi \in L^1(0,1)$.

In Section 2 below we construct the semigroup solution $S_\Phi(\cdot)$ of the Cauchy problem (1.6) and give the explicit expression of the unperturbed semigroup $T_\Phi(\cdot)$ corresponding to A_m (i.e. B=0).

In Section 3 we prove the irreducibility of the semigroups $S_\Phi(\cdot)$ and $T_\Phi(\cdot)$, and show that the growth bound of $T_\Phi(\cdot)$ is $\omega_0(T_\Phi) = 0$.

In the last section we investigate the spectrum of the generator L_Φ of the semigroup $S_\Phi(\cdot)$ and we prove in particular that the spectral bound $s(L_\Phi)$ of L_Φ is a dominant eigenvalue and a first order pole of the resolvent of L_Φ. As a consequence we obtain that the rescaled semigroup $(e^{-s(L_\Phi)t} S_\Phi(t))_{t\geq 0}$ converges to the unique steady-state solution as t goes to infinity on a weighted L^1-space.

2. Construction of the semigroup solution of (1.6)

In this section we prove that the operator

$$\begin{gathered} L_\Phi \varphi = (A_\Phi + B)\varphi = (A_m + B)\varphi, \\ D(L_\Phi) = D(A_\Phi) := \{\varphi \in [W^1(0,1)]^{N+3}, \;\; \Gamma_0\varphi = \Phi\Gamma_1\varphi = \overline{\Phi}\varphi\} \end{gathered}$$

generates a C_0-semigroup $S_\Phi(\cdot)$ on X. Thus the Cauchy problem (1.6) is wellposed. Here $W^1(0,1) = \{\varphi \in L^1(0,1) : \frac{\partial\varphi}{\partial x} \in L^1(0,1)\}$ is the first Sobolev space equipped with the norm

$$\|\varphi\|_{W^1(0,1)} := \|\varphi\|_{L^1(0,1)} + \|\frac{\partial\varphi}{\partial x}\|_{L^1(0,1)}.$$

First, it is known that the operator A_0, defined by

$$A_0\varphi = A_m\varphi, \quad D(A_0) = \{\varphi \in [W^1(0,1)]^{N+3}, \; \Gamma_0\varphi = 0\},$$

generates the positive C_0-semigroup $(T_0(t))_{t\geq 0}$, given by

$$T_0(t)\varphi(x) = \chi_{(t,1)}(x)\varphi(x-t)$$

with $\chi_{(t,1)}(x) := \begin{cases} 1, & \text{if } x \geq t, \\ 0, & \text{otherwise.} \end{cases}$

We show now that the operator A_Φ generates a C_0-semigroup $(T_\Phi(t))_{t\geq 0}$ on X. To this purpose we give the expression of the resolvent of A_Φ.

Lemma 2.1. *For $\lambda > \log(1+\alpha)$, the resolvent $R(\lambda, A_\Phi)$ of A_Φ is given by*

$$R(\lambda, A_\Phi)g = (\lambda - A_\Phi)^{-1}g = e^{-\lambda \cdot}(Id - e^{-\lambda}\Phi)^{-1}\Phi\Gamma_1(\lambda - A_0)^{-1}g + (\lambda - A_0)^{-1}g, \tag{2.1}$$

for $g \in X$.

Proof. Let $\lambda > \log|\Phi| = \log(1+\alpha)$, $\psi \in \mathbb{C}^{N+3}$ and $g \in X$. The general solution of the equation

$$\begin{gathered} \lambda\varphi + \frac{\partial}{\partial x}\varphi = g, \\ \Gamma_0\varphi = \psi. \end{gathered} \tag{2.2}$$

is

$$\varphi(x) = e^{-\lambda x}\psi + (\lambda - A_0)^{-1}g(x). \tag{2.3}$$

We have to show that the solution of (2.2) satisfies the boundary condition $\psi = \Phi\Gamma_1\varphi$. So, by (2.3) we obtain

$$\psi = e^{-\lambda}\Phi\psi + \Phi\Gamma_1(\lambda - A_0)^{-1}g.$$

Hence, $[Id - e^{-\lambda}\Phi]\psi = \Phi\Gamma_1(\lambda - A_0)^{-1}g$. Since $e^{-\lambda}|\Phi| < 1$, it follows that the equation (2.2) with the boundary condition $\Gamma_0\varphi = \Phi\Gamma_1\varphi$ has a unique solution given by

$$\varphi(x) = e^{-\lambda x}(Id - e^{-\lambda}\Phi)^{-1}\Phi\Gamma_1(\lambda - A_0)^{-1}g + (\lambda - A_0)^{-1}g(x).$$

Moreover, φ is in $(W^1(0,1))^{N+3}$ which implies that $\varphi \in D(A_\Phi)$ and this proves (2.1). $\square$

Now, we show that operator A_Φ generates a C_0-semigroup on X.

Theorem 2.2. *On X the operator A_Φ generates a C_0-semigroup $(T_\Phi(t))_{t\geq 0}$ satisfying*

$$\|T_\Phi(t)\|_{\mathcal{L}(X)} \leq (1+\alpha)e^{t\log(1+\alpha)}. \tag{2.4}$$

Proof. On X we define a new norm

$$|||\varphi||| := \int_0^1 (1+\alpha)^x|\varphi(x)|dx, \quad \varphi \in X.$$

Since

$$\|\varphi\| \leq |||\varphi||| \leq (1+\alpha)\|\varphi\|, \quad \varphi \in X, \tag{2.5}$$

these two norms are equivalent. Take $\lambda > \log(1+\alpha)$, $g \in X$ and set $\varphi = R(\lambda, A_\Phi)g$. Multiplying (2.2) by $(1+\alpha)^x sign(\varphi)(x)$ and integrating by parts, we find

$$\begin{aligned} \lambda|||\varphi||| &= \lambda\int_0^1 (1+\alpha)^x|\varphi(x)|dx \\ &\leq -\int_0^1 (1+\alpha)^x\frac{\partial}{\partial x}|\varphi(x)|dx + \int_0^1 (1+\alpha)^x|g(x)|dx \\ &\leq |||g||| + \log(1+\alpha)|||\varphi||| + |\Gamma_0\varphi| - (1+\alpha)|\Gamma_1\varphi| \\ &= |||g||| + \log(1+\alpha)|||\varphi||| + |\Gamma_0\varphi| - |\Phi||\Gamma_1\varphi| \\ &\leq |||g||| + \log(1+\alpha)|||\varphi|||. \end{aligned}$$

Consequently,

$$|||R(\lambda, A_\Phi)g||| \le \frac{1}{\lambda - \log(1+\alpha)}|||g|||.$$

Since $D(A_\Phi)$ is dense in X, the Hille-Yosida theorem implies that A_Φ generates a C_0-semigroup $T_\Phi(\cdot)$ satisfying

$$|||T_\Phi(t)||| \le e^{t\log(1+\alpha)}, \quad t \ge 0.$$

Now the estimate (2.4) follows from (2.5) and this completes the proof. □

Since $B \in \mathcal{L}(X)$, by the bounded perturbation theorem (cf. [3, Theorem III.1.3]) we obtain the following generation result for the operator L_Φ.

Theorem 2.3. *The operator L_Φ generates a C_0-semigroup $(S_\Phi(t))_{t\ge 0}$ on X satisfying*

$$\|S_\Phi(t)\|_{\mathcal{L}(X)} \le (1+\alpha)e^{t(\log(1+\alpha)+(1+\alpha)\|B\|)}.$$

In the remainder part of this section, we give an explicit formula for the semigroup $T_\Phi(\cdot)$. For this purpose we define, on the space $[W^1(0,1)]^{N+3}$, the linear operator $\mathcal{T}_\Phi(t)$ by

$$\mathcal{T}_\Phi(t)\varphi(x) := \chi_{[0,t]}(x)\Phi\Gamma_1 T_0(t-x)\varphi, \quad x \in (0,1),\ 0 \le t \le 1 \tag{2.6}$$

for $\varphi \in [W^1(0,1)]^{N+3}$, where $\chi_{[0,t]}$ is the characteristic function of the interval $[0,t]$ defined by

$$\chi_{[0,t]}(x) = \begin{cases} 0, & \text{if } t < x,\\ 1, & \text{otherwise.}\end{cases}$$

For $\varphi \in [W^1(0,1)]^{N+3}$ we have

$$\begin{aligned}
\|\mathcal{T}_\Phi(t)\varphi\| &= \int_0^1 |\chi_{[0,t]}(x)\Phi\Gamma_1 T_0(t-x)\varphi|\,dx\\
&\le (1+\alpha)\int_0^t |\Gamma_1 T_0(t-x)\varphi|\,dx\\
&\le (1+\alpha)\int_0^t |\chi(1,t-x)\varphi(1-t+x)|\,dx\\
&\le (1+\alpha)\int_0^1 |\varphi(1-x)|\,dx\\
&= (1+\alpha)\|\varphi\|.
\end{aligned} \tag{2.7}$$

Since $[W^1(0,1)]^{N+3}$ is dense in X, the operator $\mathcal{T}_\Phi(t)$, $t \in [0,1]$, can be extended to a bounded linear operator on X which will be also denoted by $\mathcal{T}_\Phi(t)$.

Lemma 2.4. *The family $(\mathcal{T}_\Phi(t))_{0\le t\le 1}$ satisfies:*

(i) *$\mathcal{T}_\Phi(0) = 0$, and $\|\mathcal{T}_\Phi(t)\|_{\mathcal{L}(X)} \le (1+\alpha)$ for all $t \in [0,1]$,*
(ii) *for all $t, s \in [0,1]$ such that $s+t \in [0,1]$, $\mathcal{T}_\Phi(t)\mathcal{T}_\Phi(s) = 0$.*

Proof. (i) It is easy to see that $\mathcal{T}_\Phi(0) = 0$. The estimate has been proved above (see (2.7)).

(ii) Let $\varphi \in [W^1(0,1)]^{N+3}$, $t, s \in [0,1]$ such that $s+t \in [0,1]$, and set $\psi = \mathcal{T}_\Phi(s)\varphi$. Then

$$\begin{aligned}
\psi(x) &= \chi_{[0,s]}(x)\Phi(T_0(s-x)\varphi)(1)\\
&= \chi_{[0,s]}(x)\Phi\varphi(1-s+x)
\end{aligned}$$

$$=: \chi_{[0,s]}(x)\Phi y(x)$$

with $y(x) := \varphi(1-s+x) \in \mathbb{C}^{N+3}$. Hence,

$$\begin{aligned}
\mathcal{T}_\Phi(t)\psi(x) &= (\mathcal{T}_\Phi(t)\chi_{[0,s]}\Phi y(\cdot))(x) \\
&= \chi_{[0,t]}(x)\Phi\Gamma_1 T_0(t-x)\chi_{[0,s]}\Phi y(\cdot) \\
&= \chi_{[0,t]}(x)\Phi\chi_{[0,s]}(1-t+x)\Phi y(1-t+x) = 0,
\end{aligned}$$

since $\chi_{[0,s]}(1-t+x) = 0$ for all $x \in (0,1)$. The denseness of $[W^1(0,1)]^{N+3}$ in X completes the proof. $\square$

To show the main result of this section, we define some auxiliary operators. For any $t \geq 0$ there exists $n \in \mathbb{N}$ and $r \in [0, \frac{1}{2})$ such that $t = \frac{n}{2} + r$. We define the operators $\overline{B}_\Phi(t)$, $t \geq 0$, by

$$\overline{B}_\Phi(t) := (B_\Phi(1/2))^n B_\Phi(r),$$

where $B_\Phi(t) = T_0(t) + \mathcal{T}_\Phi(t)$ for $t \in [0,1]$.

Lemma 2.5. *The family $(\overline{B}_\Phi(t))_{t\geq 0}$ is a C_0-semigroup on X.*

Proof. The uniqueness of the decomposition $t = \frac{n}{2} + r$ with $n \in \mathbb{N}$ and $r \in [0, \frac{1}{2})$ implies that the operators $\overline{B}_\Phi(t)$, $t \geq 0$, are well defined. Moreover, from the boundedness of $B_\Phi(t)$ follows that $\overline{B}_\Phi(t)$, $t \geq 0$, are bounded linear operators on X, and the following holds

$$\overline{B}_\Phi(0) = B_\Phi(0) = T_0(0) + \mathcal{T}_\Phi(0) = Id.$$

We propose now to show the semigroup property. First, we start with the case $t, s \in [0,1]$ with $s + t \in [0,1]$ and prove that

$$B_\Phi(t)B_\Phi(s)\varphi = B_\Phi(t+s)\varphi \tag{2.8}$$

for $\varphi \in X$. In fact, for $\varphi \in [W^1(0,1)]^{N+3}$ (and hence by density for $\varphi \in X$), we have

$$\begin{aligned}
&B_\Phi(t)B_\Phi(s)\varphi(x) \\
&= (T_0(t) + \mathcal{T}_\Phi(t))(T_0(s) + \mathcal{T}_\Phi(s))\varphi(x) \\
&= T_0(t+s)\varphi(x) + \mathcal{T}_\Phi(t)T_0(s)\varphi(x) + T_0(t)\mathcal{T}_\Phi(s)\varphi(x) \\
&= T_0(t+s)\varphi(x) + \chi_{[0,t]}(x)\Phi\Gamma_1 T_0(t+s-x)\varphi + \chi_{[t,1]}(x)\mathcal{T}_\Phi(s)\varphi(x-t) \\
&= T_0(t+s)\varphi(x) + [\chi_{[0,t]}(x)\chi_{[t+s,1]}(x) + \chi_{[0,t]}(x)\chi_{[0,t+s]}(x)]\Phi\Gamma_1 T_0(t+s-x)\varphi \\
&\quad + \chi_{[t,1]}(x)\chi_{[0,t+s]}(x)\Phi\Gamma_1 T_0(t+s-x)\varphi \\
&= B_\Phi(t+s)\varphi(x).
\end{aligned}$$

Next, by an easy computation one sees that

$$\begin{aligned}
\Big(\mathcal{T}_\Phi(r)T_0(\frac{1}{2})\varphi + T_0(r)\mathcal{T}_\Phi(\frac{1}{2})\varphi\Big)(x) &= \Big(T_0(\frac{1}{2})\mathcal{T}_\Phi(r)\varphi + \mathcal{T}_\Phi(\frac{1}{2})T_0(r)\varphi\Big)(x) \\
&= \chi_{[0,r+\frac{1}{2}]}(x)\Phi\Gamma_1 T_0(r + \frac{1}{2} - x)\varphi
\end{aligned}$$

for all $\varphi \in X$. This shows that

$$B_\Phi(r)B_\Phi(1/2) = B_\Phi(1/2)B_\Phi(r) \quad \text{for all } r \in [0, \frac{1}{2}]. \tag{2.9}$$

Now, the semigroup property

$$\overline{B}_\Phi(t+s) = \overline{B}_\Phi(t)\overline{B}_\Phi(s), \quad t,s \geq 0$$

follows from (2.8) and (2.9). For the strong continuity, let us consider $t \in (0, \frac{1}{2})$ and $\varphi \in X$. Then $\overline{B}_\Phi(t)\varphi - \varphi = (T_0(t)\varphi - \varphi) + \mathcal{T}_\Phi(t)\varphi \to 0$ as $t \to 0^+$, since $T_0(\cdot)$ is strongly continuous and $\|\mathcal{T}_\Phi(t)\varphi\| \leq (1+\alpha)\int_{1-t}^1 |\varphi(x)|\,dx$. $\square$

Theorem 2.6. *The semigroups $T_\Phi(\cdot)$ and $\overline{B}_\Phi(\cdot)$ coincide.*

Proof. We denote by C the generator of the C_0-semigroup $\overline{B}_\Phi(\cdot)$. Let $\varphi \in D(A_\Phi)$, $t \in (0,1)$ and set $\psi = \varphi - \Gamma_0\varphi$. Then

$$\begin{aligned}
&\frac{1}{t}(\overline{B}_\Phi(t)\varphi - \varphi) + \varphi' \\
&= \frac{1}{t}(T_0(t)\psi - \psi) + \psi' + \frac{1}{t}(\chi_{(t,1)}(\cdot) - 1)\Gamma_0\varphi + \frac{1}{t}\mathcal{T}_\Phi(t)\varphi \\
&= \frac{1}{t}(T_0(t)\psi - \psi) + \psi' - \frac{1}{t}\chi_{(0,t)}(\cdot)\Gamma_0\varphi + \frac{1}{t}\chi_{(0,t)}(\cdot)\Phi\varphi(1-t+\cdot).
\end{aligned}$$

Since $\psi \in D(A_0)$ and $\Gamma_0\varphi = \Phi\Gamma_1\varphi$, it follows that

$$\lim_{t\to 0^+} \frac{1}{t}(\overline{B}_\Phi(t)\varphi - \varphi) + \varphi' = 0.$$

Hence, $D(A_\Phi) \subset D(C)$ and $C|_{D(A_\Phi)} = A_\Phi$. Since C and A_Φ are both generators, we deduce that $A_\Phi = C$ and therefore $T_\Phi(\cdot) = \overline{B}_\Phi(\cdot)$. $\square$

3. Irreducibility and some spectral properties

In this section we study the irreducibility of the semigroups $T_\Phi(\cdot)$ and $S_\Phi(\cdot)$, and we characterize the growth bound $\omega_0(T_\Phi)$. We begin by proving the irreducibility. To this purpose we need the following lemma.

Lemma 3.1. *Assume that A generates an irreducible C_0-semigroup $T(\cdot)$ on a Banach lattice X and $B \in \mathcal{L}(X)$ is such that $e^{tB} \geq 0$, $t \geq 0$. Then the perturbed semigroup $S(\cdot)$ is irreducible.*

Proof. Since the semigroup $(e^{tB})_{t\geq 0}$ is positive, it follows that $B + \|B\|Id \geq 0$ (cf. [9, Theorem 1.11.C-II]). Hence the semigroup generated by $A+B+\|B\|Id$ satisfies

$$e^{t\|B\|}S(t) \geq T(t), \quad t \geq 0.$$

Thus the irreducibility of $T(\cdot)$ implies that the semigroup $(e^{t\|B\|}S(t))_{t\geq 0}$ is irreducible. Hence, $S(\cdot)$ is irreducible too. $\square$

As a consequence we obtain the following result.

Proposition 3.2. *The semigroups $(T_\Phi(t))_{t\geq 0}$ and $(S_\Phi(t))_{t\geq 0}$ are irreducible.*

Proof. Let $\lambda \geq \ln(1+\alpha)$ and $\varphi > 0$. By Lemma 2.1 we have

$$\begin{aligned}
(\lambda - A_\Phi)^{-1}\varphi &= e^{-\lambda\cdot}(Id - e^{-\lambda}\Phi)^{-1}\Phi\Gamma_1(\lambda - A_0)^{-1}\varphi + (\lambda - A_0)^{-1}\varphi \\
&\geq e^{-\lambda\cdot}(Id - e^{-\lambda}\Phi)^{-1}\Phi\Gamma_1(\lambda - A_0)^{-1}\varphi \\
&\geq e^{-\lambda\cdot}\sum_{n=0}^{\infty}(e^{-\lambda}\Phi)^n\Phi\Gamma_1(\lambda - A_0)^{-1}\varphi \\
&\geq e^{-\lambda\cdot}\Phi\Gamma_1(\lambda - A_0)^{-1}\varphi
\end{aligned}$$

$$= e^{-\lambda\cdot}\Phi\Big(\int_0^1 e^{\lambda(s-1)}\varphi(s)\,ds\Big) > 0,$$

since $(\lambda - A_0)^{-1}\varphi(x) = \int_0^x e^{\lambda(s-x)}\varphi(s)\,ds$ and $\Phi > 0$. Hence $(\lambda - A_\Phi)^{-1}$ is irreducible and therefore $T_\Phi(\cdot)$ is irreducible.

Now, we decompose B as $B = B_0 + B_1$ with

$$B_0 = \begin{pmatrix} 0 & \eta\Psi & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & \alpha & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & D & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & 0 & D & 0 & . & . & . & 0 & 0 & 0 \\ 0 & 0 & 0 & D & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & D & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & D & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & -\mu(.) \end{pmatrix}.$$

Since B_1 is a real multiplication operator on X, it follows that $(e^{tB_1})_{t\geq 0}$ is a positive semigroup on X. Thus, by the positivity of B_0, we get the positivity of $(e^{tB})_{t\geq 0}$ on X. Hence, the irreducibility of $S_\Phi(\cdot)$ follows now from Lemma 3.1. □

Proposition 3.3. *The growth bound of the semigroups $T_\Phi(\cdot)$ satisfies*

$$\omega_0(T_\Phi) = 0.$$

Proof. Since $\sigma(A_0) = \emptyset$, it follows from the proof of Lemma 2.1 that

$$\lambda \in \sigma(A_\Phi) \Longleftrightarrow 1 \in \sigma(e^{-\lambda}\Phi).$$

An easy computation shows that

$$\det(Id - e^{-\lambda}\Phi) = (1 - e^{-\lambda})(1 - qe^{-\lambda})^{N+2}.$$

Hence, $1 \in \sigma(e^{-\lambda}\Phi) \Leftrightarrow e^{\lambda} = 1$ or $e^{\lambda} = q$. This implies that $\{\Re\lambda : \lambda \in \sigma(A_\Phi)\} = \{0, \log q\}$ and thus

$$s(A_\Phi) = \omega_0(T_\Phi) = 0,$$

since $q \in (0, 1)$. □

4. The spectral bound of the generator of $S_\Phi(\cdot)$

In this section we are interested in studying some spectral properties of the generator L_Φ of the semigroup $S_\Phi(\cdot)$ on X. In particular we show that $0 < s(L_\Phi) = \omega_0(S_\Phi) > 0$ is a dominant eigenvalue and a first order pole of the resolvent of L_Φ. Here, as in [10], we use an abstract framework developed by Greiner [5].

On the product space $\mathcal{X} := X \times \mathbb{C}^{N+3}$, we define the operators

$$\mathcal{A}_0 := \begin{pmatrix} L_m & 0 \\ -\Gamma_0 & 0 \end{pmatrix} \quad \text{with } D(\mathcal{A}_0) := D(L_m) \times \{0\},$$

$$\mathcal{B} := \begin{pmatrix} 0 & 0 \\ \overline{\Phi} & 0 \end{pmatrix} \quad \text{with } D(\mathcal{B}) := D(L_m) \times \mathbb{C}^{N+3},$$
$$\mathcal{A} := \mathcal{A}_0 + \mathcal{B} = \begin{pmatrix} L_m & 0 \\ \overline{\Phi} - \Gamma_0 & 0 \end{pmatrix} \quad \text{with } D(\mathcal{A}) := D(L_m) \times \{0\}.$$

Set $\mathcal{X}_0 := X \times \{0\} = \overline{D(\mathcal{A}_0)}$. Since $\Gamma_0 \in \mathcal{L}(D(A_m), \mathbb{C}^{N+3})$ is surjective one can define for $\gamma \in \rho(L_0)$ the operator $\mathcal{D}_\gamma := \big(\Gamma_0|_{\ker(\gamma - L_m)}\big)^{-1} \in \mathcal{L}(\mathbb{C}^{N+3}, \ker(\gamma - L_m))$ called the *Dirichlet* operator. Moreover,

$$R(\gamma, \mathcal{A}_0) = \begin{pmatrix} R(\gamma, L_0) & D_\gamma \\ 0 & 0 \end{pmatrix}.$$

The part $\mathcal{A}|_{\mathcal{X}_0}$ of $\mathcal{A}$ in $\mathcal{X}_0$ is given by

$$D(\mathcal{A}|_{\mathcal{X}_0)}) = D(L_\Phi) \times \{0\} \quad \text{and} \quad \mathcal{A}|_{\mathcal{X}_0} = \begin{pmatrix} L_\Phi & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, $\mathcal{A}|_{\mathcal{X}_0}$ can be identified with the operator $(L_\Phi, D(L_\Phi))$. Furthermore, for $\gamma \in \rho(L_0)$, the following characteristic equation holds (cf. [10, Page 11])

$$\gamma \in \sigma_p(L_\Phi) \Leftrightarrow 1 \in \sigma_p(\overline{\Phi}\mathcal{D}_\gamma) = \sigma(\overline{\Phi}\mathcal{D}_\gamma) \tag{4.1}$$

and if in addition there exists $\beta \in \mathbb{C}$ such that $1 \in \rho(\overline{\Phi}\mathcal{D}_\beta)$, then

$$\gamma \in \sigma(L_\Phi) \Leftrightarrow 1 \in \sigma(\overline{\Phi}\mathcal{D}_\gamma). \tag{4.2}$$

Let us consider the operators D_0, D_1 and D_2 defined on $W_0^{1,1}(0,1) := \{\varphi \in W^{1,1}(0,1) : \varphi(0) = 0\}$ by $D_0\varphi = -\varphi'$, $D_1\varphi = -\varphi' - (\alpha + \mu(\cdot))\varphi$ and $D_2\varphi = -\varphi' - \mu(\cdot)\varphi$, $\varphi \in W_0^{1,1}(0,1)$. Then, for any $\gamma \in \mathbb{C}$, we have

$$(R(\gamma, D_0)\varphi)(x) = e^{-\gamma x} \int_0^x e^{\gamma s} \varphi(s) ds,$$
$$(R(\gamma, D_1)\varphi)(x) = e^{-(\gamma+\alpha)x - \int_0^x \mu(\sigma) d\sigma} \int_0^x e^{(\gamma+\alpha)s + \int_0^s \mu(\sigma) d\sigma} \varphi(s) ds,$$
$$(R(\gamma, D_2)\varphi)(x) = e^{-\gamma x - \int_0^x \mu(\sigma) d\sigma} \int_0^x e^{\gamma s + \int_0^s \mu(\sigma) d\sigma} \varphi(s) ds$$

for $\varphi \in L^1(0,1)$ and $x \in [0,1]$. Set

$$r_{1,1} = R(\gamma, D_0),$$
$$r_{1,2} = \eta R(\gamma, D_0) \Psi R(\gamma, D_1),$$
$$r_{j,k} = \alpha^{j-k} R(\gamma, D_1)^{j-k+1}, \quad 2 \le k \le j \le N+2,$$
$$r_{N+3,k} = \alpha^{N+3-k} R(\gamma, D_2) R(\gamma, D_1)^{N+3-k}, \quad 2 \le k \le N+3.$$

Then the resolvent of L_0 can be computed explicitly as the following lemma shows.

Lemma 4.1. *For the operator $(L_0, D(L_0))$ we have $\rho(L_0) = \mathbb{C}$ and*

$$R(\gamma, L_0) = \begin{pmatrix} r_{1,1} & r_{1,2} & 0 & 0 & \dots & 0 & 0 \\ 0 & r_{2,2} & 0 & 0 & \dots & 0 & 0 \\ 0 & r_{3,2} & r_{3,3} & 0 & \dots & 0 & 0 \\ 0 & r_{4,2} & r_{4,3} & r_{4,4} & \dots & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & r_{N+2,2} & r_{N+2,3} & r_{N+2,4} & \dots & r_{N+2,N+2} & 0 \\ 0 & r_{N+3,2} & r_{N+3,2} & r_{N+3,4} & \dots & r_{N+3,N+2} & r_{N+3,N+3} \end{pmatrix}.$$

One can also characterize $\ker(\gamma - L_m)$ for any $\gamma \in \mathbb{C}$ and therefore one obtains an explicit formula for the Dirichlet operator $\mathcal{D}_\gamma$. To this purpose, for $\gamma \in \mathbb{C}$, set

$$\begin{gathered} \epsilon_k^\gamma(x) := \frac{\alpha^k}{k!} x^k e^{-(\gamma+\alpha)x - \int_0^x \mu(s)ds}, \quad 0 \le k \le N, \\ d_{1,1}^\gamma := \frac{\eta}{\gamma}(1 - e^{-\gamma x}) \int_0^1 \mu(x) \epsilon_0^\gamma(x) dx, \\ d_{N+3,k}^\gamma := \exp(-\gamma \cdot - \int_0^\cdot \mu(s)ds) - \sum_{n=0}^{N+1-k} \epsilon_n^\gamma, \quad 1 \le k \le N+1, \\ d_{N+3,N+2}^\gamma := \exp(-\gamma \cdot - \int_0^\cdot \mu(s)ds). \end{gathered}$$

Lemma 4.2. *For $\gamma \in \mathbb{C}$, the Dirichlet operator $\mathcal{D}_\gamma$ is given by*

$$\mathcal{D}_\gamma = \begin{pmatrix} e^{-\gamma x} & d_{1,1}^\gamma & 0 & 0 & \dots & 0 & 0 \\ 0 & \epsilon_0^\gamma & 0 & 0 & \dots & 0 & 0 \\ 0 & \epsilon_1^\gamma & \epsilon_0^\gamma & 0 & \dots & 0 & 0 \\ 0 & \epsilon_2^\gamma & \epsilon_1^\gamma & \epsilon_0^\gamma & \dots & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & \epsilon_N^\gamma & \epsilon_{N-1}^\gamma & \epsilon_{N-2}^\gamma & \dots & \epsilon_0^\gamma & 0 \\ 0 & d_{N+3,1}^\gamma & d_{N+3,2}^\gamma & d_{N+3,3}^\gamma & \dots & d_{N+3,N+1}^\gamma & d_{N+3,N+2}^\gamma \end{pmatrix}.$$

By setting

$$\begin{gathered} a_{k,j}^\gamma = 0 \text{ if } \ 0 \le k \le N \text{ and } j \ge k+2, \\ a_{0,0}^\gamma = e^{-\gamma}, \\ a_{1,0}^\gamma = \alpha e^{-\gamma}, \\ a_{0,1}^\gamma = d_{1,1}^\gamma(1), \\ a_{1,1}^\gamma = \alpha d_{1,1}^\gamma(1) + q\epsilon_0^\gamma(1) + \eta\epsilon_1^\gamma(1), \\ a_{1,2}^\gamma = \eta\epsilon_0^\gamma(1), \\ a_{2,2}^\gamma = q\epsilon_0^\gamma(1) + \eta\epsilon_1^\gamma(1), \\ a_{k,1}^\gamma = q\epsilon_{k-1}^\gamma(1) + \eta\epsilon_k^\gamma(1), \qquad 2 \le k \le N \text{ if } N \ge 2, \\ a_{N+2,k}^\gamma = q d_{N+3,k}^\gamma(1), \qquad 1 \le k \le N+2, \\ b_{N+1,k}^\gamma = q\epsilon_{N-k+1}^\gamma(1) + \eta d_{N+3,k}^\gamma(1), \quad 1 \le k \le N+1, \end{gathered}$$

$$b^{\gamma}_{N+1,N+2} = \eta d^{\gamma}_{N+3,N+2}(1),$$

one deduces the expression of $\overline{\Phi}\mathcal{D}_\gamma$.

Lemma 4.3. *For $\gamma \in \mathbb{C}$, the matrix $\overline{\Phi}\mathcal{D}_\gamma$ is equal to*

$$\begin{pmatrix}
a^{\gamma}_{0,0} & a^{\gamma}_{0,1} & 0 & 0 & \dots & . & 0 & 0 & 0 \\
a^{\gamma}_{1,0} & a^{\gamma}_{1,1} & a^{\gamma}_{1,2} & 0 & \dots & . & 0 & 0 & 0 \\
0 & a^{\gamma}_{2,1} & a^{\gamma}_{2,2} & a^{\gamma}_{1,2} & \dots & . & 0 & 0 & 0 \\
0 & a^{\gamma}_{3,1} & a^{\gamma}_{2,1} & a^{\gamma}_{2,2} & \dots & . & 0 & 0 & 0 \\
0 & a^{\gamma}_{4,1} & a^{\gamma}_{3,1} & a^{\gamma}_{2,1} & \dots & . & 0 & 0 & 0 \\
. & . & . & . & \dots & . & . & . & . \\
. & . & . & . & \dots & . & . & . & . \\
0 & a^{\gamma}_{N-1,1} & . & . & \dots & a^{\gamma}_{2,2} & a^{\gamma}_{1,2} & 0 & 0 \\
0 & a^{\gamma}_{N,1} & a^{\gamma}_{N-1,1} & . & \dots & a^{\gamma}_{2,1} & a^{\gamma}_{2,2} & a^{\gamma}_{1,2} & 0 \\
0 & b^{\gamma}_{N+1,1} & b^{\gamma}_{N+1,2} & b^{\gamma}_{N+1,3} & \dots & . & b^{\gamma}_{N+1,N} & b^{\gamma}_{N+1,N+1} & b^{\gamma}_{N+1,N+2} \\
0 & a^{\gamma}_{N+2,1} & a^{\gamma}_{N+2,2} & a^{\gamma}_{N+2,3} & \dots & . & a^{\gamma}_{N+2,N} & a^{\gamma}_{N+2,N+1} & a^{\gamma}_{N+2,N+2}
\end{pmatrix}.$$

Remark 4.4. By setting $\overline{\Phi}\mathcal{D}_\gamma = (\alpha^{(\gamma)}_{ij})_{1\le i,j\le N+3}$, $\gamma > 0$, we have $\lim_{\gamma\to+\infty} \alpha^{(\gamma)}_{ij} = 0$. Hence, there is $\beta > 0$ such that $r(\overline{\Phi}\mathcal{D}_\beta) < 1$. This implies that $1 \in \rho(\overline{\Phi}\mathcal{D}_\beta)$. So, by (4.1), (4.2) and Lemma 4.1, we get, for any $\gamma \in \mathbb{C}$,

$$\gamma \in \sigma(L_\Phi) \Leftrightarrow 1 \in \sigma(\overline{\Phi}\mathcal{D}_\gamma) = \sigma_p(\overline{\Phi}\mathcal{D}_\gamma) \Leftrightarrow \gamma \in \sigma_p(L_\Phi). \tag{4.3}$$

In particular we obtain

$$\sigma(L_\Phi) = \sigma_p(L_\Phi)$$

and if $1 \in \rho(\overline{\Phi}\mathcal{D}_\gamma)$, then

$$R(\gamma, L_\Phi) = R(\gamma, L_0) + \mathcal{D}_\gamma(Id_{\mathbb{C}^{N+3}} - \overline{\Phi}\mathcal{D}_\gamma)^{-1}\overline{\Phi}R(\gamma, L_0) \tag{4.4}$$

(cf. [10, Proposition 1.8]).

The following result shows that $s(L_\Phi) > 0$.

Proposition 4.5. *There exists $\gamma_0 > 0$ such that $1 = r(\overline{\Phi}\mathcal{D}_{\gamma_0})$ and therefore*

$$s(L_\Phi) = \gamma_0 > 0.$$

Proof. Since $\overline{\Phi}\mathcal{D}_0 = (\alpha^{(0)}_{ij})_{1\le i,j\le N+3}$ is an irreducible matrix, it follows from [13, Proposition 6.3., Chap.I] that $r(\overline{\Phi}\mathcal{D}_0) > \max_{1\le i\le N+3} \alpha^{(0)}_{ii}$. In particular,

$$r(\overline{\Phi}\mathcal{D}_0) > a^0_{0,0} = 1. \tag{4.5}$$

On the other hand, by the explicit expression of $\overline{\Phi}\mathcal{D}_\beta$ one can see that the function $0 < \beta \mapsto r(\overline{\Phi}\mathcal{D}_\beta)$ is decreasing and $\lim_{\beta\to+\infty} r(\overline{\Phi}\mathcal{D}_\beta) = 0$. Thus, by continuity and (4.5), there exists a unique $\gamma_0 > 0$ such that $r(\overline{\Phi}\mathcal{D}_{\gamma_0}) = 1 \in \sigma(\overline{\Phi}\mathcal{D}_{\gamma_0})$. Hence, from (4.3) we get $\gamma_0 \in \sigma(L_\Phi)$.

Now, take $\lambda > \gamma_0$ and set $\overline{\Phi}\mathcal{D}_\lambda = (\alpha^{(\lambda)}_{ij})_{1\le i,j\le N+3}$. Since $0 \le \alpha^{(\lambda)}_{ij} \le \alpha^{(\gamma_0)}_{ij}$ and $\alpha^{(\lambda)}_{11} < \alpha^{(\gamma_0)}_{11}$, it follows from [13, Page 22] that

$$r(\overline{\Phi}\mathcal{D}_\lambda) < r(\overline{\Phi}\mathcal{D}_{\gamma_0}) = 1.$$

Then, by the positivity of $\overline{\Phi}\mathcal{D}_\lambda$ and (4.4), we obtain $\lambda \in \rho(L_\Phi)$ and $R(\lambda, L_\Phi) \ge 0$. Since $s(L_\Phi) = \inf\{\mu \in \rho(L_\Phi) : R(\mu, L_\Phi) \ge 0\}$ (cf. [12, Remark 2.3.5]), we get $s(L_\Phi) < \lambda$ and hence $s(L_\Phi) \le \gamma_0$. Thus, since $\gamma_0 \in \sigma(L_\Phi)$, it follows that $s(L_\Phi) = \gamma_0$. □

The first main result of this paper shows that the spectral bound of L_Φ is a dominant spectral value.

Theorem 4.6. *The spectral bound $s(L_\Phi)$ of L_Φ is a first order pole of the resolvent and the boundary spectrum of L_Φ is given by*

$$\sigma_b(L_\Phi) = \sigma(L_\Phi) \cap \{\Re\lambda = s(L_\Phi)\} = \{s(L_\Phi)\}.$$

Proof. It follows from (4.4) and the compactness of $\overline{\Phi}R(\gamma, L_0)$, $\Re\gamma > s(L_\Phi)$, that

$$r_{\mathrm{ess}}(R(\gamma, L_\Phi)) = r_{\mathrm{ess}}(R(\gamma, L_0)), \quad \Re\gamma > s(L_\Phi).$$

Since $\sigma(L_0) = \emptyset$, we deduce from the spectral theorem for the resolvent (cf. [3]) that $r_{\mathrm{ess}}(R(\gamma, L_0)) = 0$ and hence

$$r_{\mathrm{ess}}(R(\gamma, L_\Phi)) = 0, \quad \Re\gamma > s(L_\Phi).$$

This implies that $\frac{1}{\lambda - s(L_\Phi)}$ is a pole of finite algebraic multiplicity for any $\lambda > s(L_\Phi)$. By [9, Proposition 2.5.A-III] we deduce that $s(L_\Phi)$ is a pole of finite algebraic multiplicity and the first assertion is proved by applying [9, Proposition 3.5.C-III], since $S_\Phi(\cdot)$ is irreducible (see Proposition 3.2). For the second assertion we note first that, by

Proposition 4.5, $s(L_\Phi) = \gamma_0 > 0$. Let us consider $a \in \mathbb{R}$ such that

$$|a| > \sqrt{\frac{4\gamma_0^2}{(1 - e^{-\gamma_0})^2} - \gamma_0^2} =: \xi_0.$$

Then, it is easy to see that

$$|d_{1,1}^{\gamma_0 + ia}(1)| < d_{1,1}^{\gamma_0}(1).$$

Hence,

$$|\alpha_{ij}^{(\gamma_0 + ia)}| \le \alpha_{ij}^{(\gamma_0)} \quad \text{and} \quad |\alpha_{12}^{(\gamma_0 + ia)}| < \alpha_{12}^{(\gamma_0)}$$

for all $i, j = 1, \dots, N + 3$, where $(\alpha_{ij}^{(\gamma)})_{1 \le i,j \le N+3} = \overline{\Phi}\mathcal{D}_\gamma$, $\gamma \in \mathbb{C}$. So, by [13, Page 22] and Proposition 4.5 we obtain

$$r(\overline{\Phi}\mathcal{D}_{\gamma_0 + ia}) < r(\overline{\Phi}\mathcal{D}_{\gamma_0}) = 1.$$

Thus, by (4.3), we get $\gamma_0 + ia \in \rho(L_\Phi)$ for any $a \in \mathbb{R}$ with $|a| > \xi_0$. This means that $\sigma_b(L_\Phi)$ is bounded. On the other hand, using [9, Proposition 2.9.C-III] and [9, Proposition 2.10.C-III], we obtain that $\sigma_b(L_\Phi)$ is cyclic, i.e., if $a + ib \in \sigma_b(L_\Phi)$, $a, b \in \mathbb{R}$, then $a + ikb \in \sigma_b(L_\Phi)$ for all $k \in \mathbb{Z}$. Now, the boundedness of $\sigma_b(L_\Phi)$ gives the second assertion. $\square$

Now, we deduce the asymptotic behavior of the semigroup $(S_\Phi(t))_{t \ge 0}$.

Theorem 4.7. *There exists $0 \ll w \in [L^\infty(0,1)]^{N+3}$ such that the rescaled semigroup $(e^{-s(L_\Phi)t}S_\Phi(t))_{t \ge 0}$ converges to the unique steady-state solution as t goes to infinity in the weighted space $L^1_w := [L^1(0,1; w dx)]^{N+3}$; i.e., there is $0 \ll \psi \in L^1_w$ and $0 \ll \widehat{w} \in (L^1_w)^*$ such that*

$$\lim_{t \to \infty} e^{-s(L_\Phi)t} S_\Phi(t)\varphi = \langle \widehat{w}, \varphi \rangle_{L^1_w} \psi$$

for all $\varphi \in L^1_w$, where the limit is in L^1_w equipped with the weighted norm

$$\|\varphi\|_w := \sum_{i=0}^{N+2} \int_0^1 \varphi_i(x) w_i(x)\, dx.$$

Proof. Since, by Theorem 4.6, $s(L_\Phi)$ is a first order pole of the resolvent, it follows from [9, Proposition 3.5.C-III] that there is a strictly positive eigenvector w of L_Φ^* corresponding to $s(L_\Phi)$. Hence, $e^{-s(L_\Phi)t}S_\Phi(t)^*w = w$ and therefore

$$\|e^{-s(L_\Phi)t}S_\Phi(t)\|_w \leq 1 \quad \text{for all } t \geq 0.$$

On the other hand, we know from Theorem 4.6, Remark 4.4 and Proposition 4.1 that $s(L_\Phi) \in \sigma_p(L_\Phi)$ and $S_\Phi(\cdot)$ is irreducible. So, we deduce that the set $\{e^{-s(L_\Phi)t}S_\Phi(t) : t \geq 0\}$ is relatively weakly compact in L^1_w (cf. [8, Lemma 3.10]). Now, the assertion follows as in [8, Theorem 3.11]. □

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