Electronic Journal of Differential Equations, Vol. 2009(2009), No. 136, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF LOCAL AND GLOBAL SOLUTIONS TO SOME IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. First, by using Schauder's fixed-point theorem we establish the existence uniqueness of locals for some fractional differential equation with a finite number of impulses. On the other hand, by using Brouwer's fixed-point theorem, we establish existence of the global solutions under suitable assumptions.

1. Introduction

The concept of fractional calculus can be considered as a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. However, great efforts must be done before the ordinary derivatives could be truly interpreted as a special case of the fractional derivatives. For more details, we refer to the books by Oldham and Spanier [8] and by Miller and Ross [6].

Actually, fractional derivatives have been extensively applied in many fields, for example in Probability, Viscoelasticity, Electronics, Economics, Mechanics as well as Biology.

Some results on quantitative and qualitative theory of some fractional differential equations are obtained, we may cite the references [3, 5, 6, 8, 9]. On the other hand, the theory of impulsive differential equations is also an important area of research which has been investigated in the last few years by great number of mathematicians. We recall that the impulsive differential equations may better model phenomena and dynamical processes subject to a great changes in short times issued, for instance, in Physics, Biotechnology, Automatics and Robotics. To learn more about the most recent used techniques for this kind of problems we refer to the book of Benchohra et al [2].

So, we propose to study fractional differential equation subject to a finite number of impulses. As we know there just few authors have investigated this subject [7]. We have obtained some results regarding local existence and uniqueness for some fractional integrodifferential problem with a finite number of impulses. For the existence and uniqueness of local solutions we use the Schauder's fixed-point theorem, while we use Brouwer's fixed-point theorem for the global solutions.

2000 *Mathematics Subject Classification.* 26A33, 34A12, 34A37.
Key words and phrases. Fractional derivative; impulsive conditions; fixed point; local solution; global solution.

Submitted July 3, 2009. Published October 21, 2009.
Supported by the LMA, University of Badji Mokhtar Annaba, Algeria.

2. Preliminaries

Among the definitions of fractional derivatives we recall the Riemann-Liouville definiton as follows.

$$D^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t_0}^{t}(t-s)^{-\alpha+n-1}u(s)\,ds$$

where $\Gamma(\cdot)$ is the well known gamma function and $\alpha\in(n-1,n)$, with n being an integer. One may observe that the derivative of a constant is not at all equal to zero which can cause serious problems in both views, theoretical and practical. For this reason we prefer to use Caputo's definition which gives better results than those of Riemann-Liouville. So we define Caputo's derivative of order $\alpha\in(n-1,n)$ of a function $u(t)$ by

$$D^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-s)^{-\alpha+n-1}\frac{d^n}{ds^n}u(s)\,ds.$$

Also, we use the fractional integral operator of order $\alpha>0$ given by

$$D^{-\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}u(s)\,ds.$$

We shall consider the fractional differential equation

$$D^{\alpha}u(t)=f(t,u(t));\quad t\in[t_0,t_0+\tau],\ t\neq t_k,\ k=1,\dots,m; \tag{2.1}$$

with the initial condition

$$D^{\alpha-1}u(t_0)=u_0;\quad (t-t_0)^{1-\alpha}u(t)\big|_{t=t_0}=\frac{u_0}{\Gamma(\alpha)}; \tag{2.2}$$

subject to the impulsive conditions

$$\begin{gathered}D^{\alpha-1}(u(t_k^+)-u(t_k^-))=I_k(t);\quad t=t_k,\ k=1,\dots,m;\\ (t-t_k)^{1-\alpha}u(t)\big|_{t=t_k}=\frac{I_k(t_k)}{\Gamma(\alpha)},\quad k=1,\dots,m.\end{gathered} \tag{2.3}$$

We set the following assumptions

- (A1) $t>t_0\geq 0$, α is a real number such that $0<\alpha\leq 1$, u_0 is a real constant vector of $\mathbb{R}^n$ (the usual real n-dimensional Euclidean space equipped with its Euclidean norm $\|.\|$);
- (A2) $f(t,u):I\times\mathbb{R}^n\to\mathbb{R}^n$; $I_k(t):I\to\mathbb{R}^n$, $k=1,\dots,m$, with $I=[t_0,t_0+\tau]$;
- (A3) $t_k\in I$, $k=1,\dots,m$ and $t_0<t_1<\dots<t_k<\dots<t_m$.

We introduce the following spaces:
$\mathcal{PC}(I,\mathbb{R}^n)=\{u:I\to\mathbb{R}^n: u(t)$ is continuous at $t\neq t_0$, $t\neq t_k$, and left continuous at $t=t_k$, and (t_0^+) and $u(t_k^+)$ exist for $k=1,\dots,m\}$;
$\mathcal{PC}_{\alpha}(I,\mathbb{R}^n)=\{u\in\mathcal{PC}(I,\mathbb{R}^n):\lim_{t\to t_0^+}(t-t_0)^{\alpha}u(t)$ and $\lim_{t\to t_k^+}(t-t_k)^{\alpha}u(t)$ exist and are finite for $k=1,\dots,m$, $\alpha>0\}$. This is a Banach space with respect to the norm

$$\|u\|_{\alpha}=\sup_{t\in I'}(t-t_0)^{\alpha+1}\prod_{i=1}^{m}(t-t_i)^{\alpha+1}\|u(t)\|,$$

where $I'=(t_0,t_0+\tau]\backslash\{t_k\}_{k=1,2,\dots}$.

We begin with the following Lemma.

Lemma 2.1. *If f and I_k, $k = 1, \dots m$ are continuous functions, then $u(t)$ is a solution to problem (2.1)-(2.3) in $\mathcal{PC}_{1-\alpha}([t_0, t_0+\tau], \mathbb{R}^n)$ if and only if $u(t)$ satisfies the integrodifferential equation*

$$\begin{aligned} u(t) = \frac{u_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1} f(s,u(s))\,ds \\ + \frac{1}{\Gamma(\alpha)}\sum_{t_0<t_k\le t}(t-t_k)^{\alpha-1} I_k(t_k). \end{aligned} \tag{2.4}$$

Proof. Let $u(t)$ be a solution of problem (2.1)-(2.3). Using the fractional integral of order $\alpha > 0$ and the properties of derivative of order $\alpha > 0$, and then applying D^{-1} to (2.1) we obtain

$$\begin{aligned} D^{-1}(D^{\alpha}u(t)) = \int_{t_0}^{t} f(s,u(s))\,ds = \int_{t_0}^{t}\frac{d}{dt}D^{-(1-\alpha)}u(s)\,ds \\ = D^{\alpha-1}u(t) - u_0 - \sum_{t_0<t_k\le t} I_k(t_k). \end{aligned}$$

So

$$D^{\alpha-1}u(t) = u_0 + \int_{t_0}^{t} f(s,u(s))\,ds + \sum_{t_0<t_k\le t} I_k(t_k). \tag{2.5}$$

Next, applying the operator $D^{1-\alpha}$ to $D^{\alpha-1}u(t)$ we obtain

$$\begin{aligned} u(t) = D\Big(\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}u_0\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,u(s))\,ds \\ + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\sum_{t_0<t_k\le s} I_k(t_k)\,ds\Big), \end{aligned}$$

which gives the integral equation (2.4). On the other hand, from (2.2) and (2.3), it follows that

$$\begin{gathered} \lim_{t\to t_0^+}(t-t_0)^{1-\alpha}u(t) = \frac{u_0}{\Gamma(\alpha)}, \\ \lim_{t\to t_k^+}(t-t_k)^{\alpha}u(t) = \frac{I_k(t_k)}{\Gamma(\alpha)}, \quad k = 1,2,\dots, \end{gathered} \tag{2.6}$$

which proves that $u(t) \in \mathcal{PC}_{1-\alpha}([t_0, t_0+\tau])$.

Let $u(t)$ be a solution to the integral equation (2.4) in $\mathcal{PC}_{\alpha}([t_0, t_0+\tau])$. Performing D^{α} to the integral equation (2.4) we get for $t \ne t_0$ and $t \ne t_k$, $k = 1,\dots,m$;

$$D^{\alpha}u(t) = DD^{\alpha}D^{-\alpha}u_0 + DD^{\alpha}D^{-\alpha}\sum_{t_0<t_k\le t} I_k(t_k) + D^{\alpha}D^{-\alpha}f(t,u(t)) = f(t,u(t)),$$

and for $t = t_0$, and $t = t_k$, $k = 1,\dots,m$, we apply $D^{\alpha-1}$ to the integral equation (2.4) to obtain (2.5) which in turn gives

$$\begin{gathered} D^{\alpha-1}u(t_0) = u_0 \\ D^{\alpha-1}(u(t_k^+) - u(t_k^-)) = I_k(t_k), \quad k = 1,\dots,m. \end{gathered}$$

Now, since $u(t) \in \mathcal{PC}_{1-\alpha}([t_0, t_0+\tau])$, it satisfies the limits (2.6) from which we get conditions (2.2) and (2.3). $\square$

3. Existence and uniqueness of a local solution

We need the following Schauder's fixed-point theorem.

Theorem 3.1. *If U is a closed , bounded, convex subset of a Banach space X and the mapping $A : U \to U$ is completely continuous, then A has a fixed point in U.*

Let us denote the right hand side of (2.4) by $Au(t)$ which we write as $Au(t) = u_1(t) + Bu(t)$, where

$$u_1(t) = \frac{u_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k \le t} (t-t_k)^{\alpha-1} I_k(t_k),$$

$$Bu(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, u(s))\, ds.$$

Theorem 3.2. *If $f \in C([t_0, t_0+\tau] \times \mathbb{R}^n, \mathbb{R}^n)$ and there exist positive constants N, b_k, $k = 1, \dots, m$; such that $\|f(t,u)\| \le N$, $\|I_k(t_k)\| \le b_k$, $k = 1, \dots, m$, then there exists at least one solution $u(t)$ of the problem (2.1)-(2.3) in $\mathcal{PC}_{1-\alpha}([t_0, t_0+\mu])$ for some positive constant*

$$\mu = \min\Big(\tau,\ \big(\frac{\alpha\beta}{N}\Gamma(\alpha)\big)^{1/(m(2-\alpha)+2)}\Big).$$

Proof. It is easy to see that the set

$$U_{1-\alpha}^{\beta} = \{u \in \mathcal{PC}_{1-\alpha}([t_0, t_0+\mu], \mathbb{R}^n) : \|u(t) - u_1(t)\|_{1-\alpha} \le \beta\}$$

is not empty because $u_1(t) \in U_{1-\alpha}^{\beta}$; on the other hand, it is a closed, bounded, convex subset of the Banach space $\mathcal{PC}_{1-\alpha}([t_0, t_0+\mu], \mathbb{R}^n)$.

Next, we define the operator A on $U_{1-\alpha}^{\beta}$ by

$$\begin{aligned} Au(t) &= \frac{u_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int t_0{}^t (t-s)^{\alpha-1} f(s, u(s))\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{t_0 < t_k \le t} (t-t_k)^{\alpha-1} I_k(t_k). \end{aligned} \tag{3.1}$$

To prove that A maps $U_{1-\alpha}^{\beta}$ into itself we see that, for every $u \in U_{1-\alpha}^{\beta}$, we have

$$\begin{aligned} (t-t_0)^{2-\alpha} \prod_{i=1}^{m} (t-t_i)^{2-\alpha} \|Au(t) - u_1(t)\| &\le \frac{\mu^{(m+1)(2-\alpha)}}{\Gamma(\alpha)} N \int_{t_0}^{t} (t-s)^{\alpha-1}\, ds \\ &\le \frac{\mu^{m(2-\alpha)+2}}{\alpha\Gamma(\alpha)} N \le \beta. \end{aligned}$$

Hence

$$\|Au(t) - u_1(t)\|_{1-\alpha} \le \beta, \tag{3.2}$$

which implies $AU_{1-\alpha}^{\beta} \subset U_{1-\alpha}^{\beta}$.

To see that $AU^{\beta}_{1-\alpha}$ is uniformly bounded in $\mathcal{PC}_{1-\alpha}([t_0, t_0+\mu])$ we note that

$$\begin{aligned}
&(t-t_0)^{2-\alpha}\prod_{i=1}^{m}(t-t_i)^{2-\alpha}\|u_1(t)\|\\
&\le \frac{\|u_0\|}{\Gamma(\alpha)}(t-t_0)\prod_{i=1}^{m}(t-t_i)^{2-\alpha}\\
&\quad+\frac{(t-t_0)^{2-\alpha}}{\Gamma(\alpha)}\sum_{t_0<t_k\le t}\prod_{i=1}^{m}(t-t_i)^{2-\alpha}(t-t_k)^{\alpha-1}\|I_k(t_k)\|;\\
&\le \frac{\|u_0\|}{\Gamma(\alpha)}\mu^{m(2-\alpha)+1}+\frac{\mu^{2-\alpha}\mu^{(m-1)(2-\alpha)+1}}{\Gamma(\alpha)}\sum_{k=1}^{m}b_k.
\end{aligned}$$

Hence,

$$\|u_1(t)\|_{1-\alpha}\le \frac{\mu^{m(2-\alpha)+1}}{\Gamma(\alpha)}(b+\|u_0\|), \tag{3.3}$$

where $b=\sum_{t_0<t_k\le t} b_k$. From (3.2) and (3.3), we deduce that

$$\|u(t)\|_{1-\alpha}\le \frac{\mu^{m(2-\alpha)+1}}{\Gamma(\alpha)}(\|u_0\|+b)+\frac{\mu^{m(2-\alpha)+2}}{\alpha\Gamma(\alpha)}N,$$

for every $u\in U^{\beta}_{1-\alpha}$.

We shall prove in the next step that $AU^{\beta}_{1-\alpha}$ is equicontinuous in $\mathcal{PC}_{1-\alpha}([t_0, t_0+\mu])$. We observe on the one hand that the derivative of $u_1(t)$ is uniformly bounded in $\mathcal{PC}_{1-\alpha}([t_0, t_0+\mu])$ because

$$\begin{aligned}
&(t-t_0)^{2-\alpha}\prod_{i=1}^{m}(t-t_i)^{2-\alpha}\|u_1'(t)\|\\
&\le \frac{1-\alpha}{\Gamma(\alpha)}\|u_0\|\prod_{i=1}^{m}(t-t_i)^{2-\alpha}\\
&\quad+\frac{1-\alpha}{\Gamma(\alpha)}(t-t_0)^{2-\alpha}\sum_{t_0<t_k\le t}\prod_{i=1}^{m}(t-t_i)^{2-\alpha}(t-t_k)^{\alpha-2}\|I_k(t_k)\|
\end{aligned}$$

giving

$$\|u_1'(t)\|_{1-\alpha}\le \frac{1-\alpha}{\Gamma(\alpha)}\mu^{m(2-\alpha)}(\|u_0\|+b\mu^{-(2-\alpha)}).$$

On the other hand, we have for $t_0<s_1<s_2<t_0+\mu$,

$$\begin{aligned}
&(t-t_0)^{2-\alpha}\prod_{i=1}^{m}(t-t_i)^{2-\alpha}\|Bu(s_2)-Bu(s_1)\|\\
&\le \frac{\mu^{(m+1)(2-\alpha)}}{\Gamma(\alpha)}N\Big|\int_{t_0}^{s_2}(s_2-s)^{\alpha-1}-\int_{t_0}^{s_1}(s_1-s)^{\alpha-1}\,ds\Big|\\
&\le \frac{\mu^{(m+1)(2-\alpha)}}{\Gamma(\alpha)}N\Big|\int_{t_0}^{s_1}((s_2-s)^{\alpha-1}-(s_1-s)^{\alpha-1})\,ds+\int_{s_1}^{s_2}(s_2-s)^{\alpha-1}\,ds\Big|,
\end{aligned}$$

so that

$$\|Bu(s_2)-Bu(s_1)\|_{1-\alpha} \le \frac{\mu^{(m+1)(2-\alpha)}}{\Gamma(\alpha)} N[2(s_2-s_1)^\alpha+|(s_2-t_0)^\alpha-(s_1-t_0)^\alpha|]; \quad (3.4)$$

that is, $Bu(t)$ is equicontinuous, and so $Au(t)$ is equicontinuous in $\mathcal{PC}_{1-\alpha}([t_0, t_0+\mu], \mathbb{R}^n)$. Hence, $\overline{AU^\beta_{1-\alpha}}$ is compact in $\mathcal{PC}_{1-\alpha}([t_0, t_0+\mu])$ showing that A is completely continuous. Therefore, we conclude by Schauder's theorem that A has at least one fixed-point in $U^\beta_{1-\alpha}$ which is exactly a solution to (2.1)-(2.3) in view of lemma 2.1. The proof is now complete. □

Theorem 3.3. *Besides the hypotheses of theorem 3.1, we suppose that there exists a constant L such that*

$$0 < L < \frac{\alpha\Gamma(\alpha)}{\mu^\alpha}, \quad (3.5)$$

where μ is defined as in theorem 3.2, and

$$\|f(t,u)-f(t,w)\| \le L\|u-w\|, \quad \textit{for every } u,w \in \mathbb{R}^n.$$

Then, the solution $u(t)$ of (2.1)-(2.3) is unique in $\mathcal{PC}_{1-\alpha}([t_0, t_0+\mu], \mathbb{R}^n)$.

Proof. In virtue of theorem 3.1 there exists at least one solution $u(t)$ of (2.1)-(2.3) in $\mathcal{PC}_{1-\alpha}([t_0, t_0+\mu], \mathbb{R}^n)$.

First, suppose to the contrary that there exist two different solutions u and w in $\mathcal{PC}_{1-\alpha}([t_0, t_0+\mu], \mathbb{R}^n)$ which satisfy the integral equation (2.4). It is easy to see that

$$\begin{aligned} \|u(t)-w(t)\| &\le \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1}\|f(s,u(s))-f(s,w(s))\|\,ds \\ &\le \frac{\mu^\alpha}{\alpha\Gamma(\alpha)} L\|u(t)-w(t)\|. \end{aligned}$$

It follows that

$$\|u(t)-w(t)\|_{1-\alpha} \le \frac{\mu^\alpha}{\alpha\Gamma(\alpha)} L\|u(t)-w(t)\|_{1-\alpha},$$

and taking into account condition (3.5) we obtain

$$\|u(t)-w(t)\|_{1-\alpha} = 0.$$

So, the two solutions are identical in $\mathcal{PC}_{1-\alpha}([t_0, t_0+\mu], \mathbb{R}^n)$ which completes the proof. □

To illustrate the foregoing results we propose the following example:

Example 3.4. On the interval $[0,1]$, Consider the impulsive fractional differential initial-value problem

$$\begin{gathered} D^{1/2}u(t) = \frac{e^{-t}}{t+2}\sin u(t); \quad t \ne \frac{k}{k+1},\ k=1,2; \\ D^{-1/2}u(0)=0; \quad t^{1/2}u(t)\big|_{t=0}=0; \\ D^{-1/2}(u(t_k^+)-u(t_k^-)) = t_k\cos t_k; \quad t_k = \frac{k}{k+1},\ k=1,2 \\ (t-\frac{k}{k+1})^{1/2}u(t)\big|_{t=\frac{1}{k+1}} = \frac{k}{\sqrt{\pi}(k+1)}\cos(\frac{k}{k+1}), \quad k=1,2. \end{gathered} \quad (3.6)$$

We see that $f(t,u)=\frac{e^{-t}}{t+2}\sin u\in C([0,1]\times\mathbb{R};\mathbb{R})$; $I_k(t)\in C([0,1];\mathbb{R})$, and since $\Gamma(1/2)=\sqrt{\pi}$, then the solution of (3.6) satisfies the integral equation

$$\begin{aligned} u(t)=&\frac{1}{\sqrt{\pi}}\int_{t_0}^{t}(t-s)^{-1/2}\frac{e^{-s}}{s+2}\sin u(s)\,ds\\ &+\frac{1}{\sqrt{\pi}}\sum_{0<t_k\le t}\big(t-\frac{k}{k+1}\big)^{-1/2}\frac{k}{k+1}\cos\big(\frac{k}{k+1}\big). \end{aligned}\tag{3.7}$$

Since $|f(t,u)|=|\frac{e^{-t}}{(t+2)}\sin u|\le\frac{1}{2}$, for every $t\in[0,1]$ and $u\in\mathbb{R}$, and

$$|f(t,u)-f(t,w)|\le\frac{1}{2}|u-w|,\text{ for every } u,w\in\mathbb{R},$$

condition (3.5) is easily satisfied and so in view of theorems 3.2 and 3.3, (3.6) admits a unique solution $u(t)$ in $\mathcal{PC}_{1/2}([0,1],\mathbb{R})$.

4. Existence of a global solution

In this part we shall prove the existence of a global solution to (2.1)-(2.3) under suitable assumptions, by using the following Brouwer's fixed-point theorem.

Theorem 4.1. *Set Ω be a closed, bounded, convex non empty subset of X a Banach space and let $A:\Omega\to X$ be a continuous mapping. If $A(\Omega)\subset\Omega$, then A has a fixed-point in Ω.*

Consider the scalar fractional differential equation

$$D^{\alpha}v(t)=g(t,v(t));\quad t\in[t_0,+\infty[;\ t\neq t_k,\ k=1,\dots,m;\ 0<\alpha\le 1;\tag{4.1}$$

subject to the initial conditions

$$D^{\alpha-1}v(t)=v_0;\quad (t-t_0)^{1-\alpha}v(t)\big|_{t=t_0}=\frac{1}{\Gamma(\alpha)}v_0;\tag{4.2}$$

and the impulsive conditions

$$\begin{gathered} D^{\alpha-1}v(t_k^+)-D^{\alpha-1}v(t_k^-)=J_k(t_k);\quad k=1,\dots,m\\ (t-t_k)^{1-\alpha}v(t)\big|_{t=t_k}=\frac{1}{\Gamma(\alpha)}J_k(v(t_k)),\quad k=1,\dots,m. \end{gathered}\tag{4.3}$$

We assume that v_0 is a positive constant; $f(t,u)\in C([t_0,+\infty[\times\mathbb{R}^n,\mathbb{R}^n)$; $g(t,v)\in C([t_0,+\infty[\times\mathbb{R}_+,\mathbb{R}_+)$, $I_k(t)\in C([t_0,+\infty[,\mathbb{R}^n)$ and $J_k(t)\in C([t_0,+\infty[,\mathbb{R}_+)$; $k=1,\dots,m$.

In view of theorem 3.1 the solution of (4.1)-(4.3) satisfies the integral equation

$$\begin{aligned} v(t)=&\frac{v_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}g(s,v(s))\,ds\\ &+\frac{1}{\Gamma(\alpha)}\sum_{t_0<t_k\le t}(t-t_k)^{\alpha-1}J_k(t_k). \end{aligned}\tag{4.4}$$

in the Banach space $\mathcal{PC}_{1-\alpha}([t_0,+\infty[,\mathbb{R})$ endowed with the norm

$$|v|_{1-\alpha}=\sup_{t\in[t_0,+\infty[\setminus\{t_k\}_{k=0,1,\dots m}}(t-t_0)^{2-\alpha}\prod_{i=1}^{m}(t-t_i)^{2-\alpha}|v(t)|.$$

Theorem 4.2. *Assume that $\|f(t,u)\| \le g(t,\|u\|)$ for every $t \ge t_0$ and $u \in \mathbb{R}^n$, where $g(t,v)$ is nonnegative and nondecreasing in v, for each $t \ge t_0$, and*

$$\|I_k(t_k)\| \le J_k(t_k), \quad \text{for } t = t_k, \quad k = 1,\dots m.$$

If (4.1)-(4.3) has a positive solution $v(t)$ in $\mathcal{PC}_{1-\alpha}([t_0,+\infty[,\mathbb{R})$, then (2.1)-(2.3) has at least a solution in $\mathcal{PC}_{1-\alpha}([t_0,+\infty[,\mathbb{R}^n)$ such that $\|u\|_{1-\alpha} \le |v|_{1-\alpha}$, for each $u_0 \in \mathbb{R}^n$ satisfying $\|u_0\| \le v_0$.

Proof. To apply Brouwer's theorem we use the Banach space $\mathcal{PC}_{1-\alpha}([t_0,+\infty[,\mathbb{R}^n)$, $0 < \alpha \le 1$, which we equip with the norm

$$\|u\|_{1-\alpha} = \sup_{t\in[t_0,+\infty[\setminus\{t_k\}_{k=0,1,\dots m}} (t-t_0)^{2-\alpha}\prod_{i=1}^{m}(t-t_i)^{2-\alpha}\|u(t)\|.$$

Next, define a subset of $\mathcal{PC}_{1-\alpha}([t_0,+\infty[,\mathbb{R}^n)$ by

$$\begin{aligned}\mathcal{V}_{1-\alpha} = \big\{&u \in \mathcal{PC}_{1-\alpha}([t_0,+\infty[,\mathbb{R}^n) : \|u\|_{1-\alpha} \le |v|_{1-\alpha};\\ &\text{where } v(t) \text{ is a positive solution of (4.1)-(4.3)}\big\}.\end{aligned}$$

It is not difficult to verify that $\mathcal{V}_{1-\alpha}$ is a closed, convex, and bounded subsect of $\mathcal{PC}_{1-\alpha}([t_0,+\infty[,\mathbb{R}^n)$.

The operator A defined by (3.1) is continuous, and so, it remains to prove that $A(\mathcal{V}_{1-\alpha}) \subset \mathcal{V}_{1-\alpha}$. For each $u \in \mathcal{V}_{1-\alpha}$, by (3.1), we have

$$\begin{aligned}\|Au(t)\| \le{}& \prod_{i=1}^{m}(t-t_i)^{\alpha+1}\frac{\|u_0\|}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\|f(s,u(s))\|\,ds\\ &+ (t-t_0)^{\alpha+1}\prod_{i=1,i\ne k}^{m}(t-t_i)^{\alpha+1}\frac{1}{\Gamma(\alpha)}\sum_{t_0<t_k\le t}\|I_k(t_k)\|\,.\end{aligned}$$

From the assumptions, we obtain

$$\begin{aligned}\|Au(t)\| \le{}& \frac{v_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}g(s,v(s))\,ds\\ &+ \frac{1}{\Gamma(\alpha)}\sum_{t_0<t_k\le t}(t-t_k)^{\alpha-1}J_k(t_k)\end{aligned} \tag{4.5}$$

It follows from (4.4) that

$$\|Au(t)\| \le v(t), \quad t \ge t_0;$$

and since

$$\begin{gathered}\lim_{t\to t_0^+}(t-t_0)^{1-\alpha}\|Au(t)\| \le \lim_{t\to t_0^+}(t-t_0)^{1-\alpha}v(t),\\ \lim_{t\to t_K^+}(t-t_k)^{1-\alpha}\|Au(t)\| \le \lim_{t\to t_K^+}(t-t_k)^{1-\alpha}v(t); \quad k = 1,\dots,m,\end{gathered}$$

it follows that

$$\|Au\|_{1-\alpha} \le |v|_{1-\alpha}.$$

Hence, $A(\mathcal{V}_{1-\alpha}) \subset \mathcal{V}_{1-\alpha}$. Hence, as all the requirements of Brouwer's fixed-point theorem are satisfied, then A has a fixed point in $\mathcal{V}_{1-\alpha}$ which is the solution of (2.1)-(2.3) such that $\|u\|_{1-\alpha} \le |v|_{1-\alpha}$. □

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Corrigendum posted on August 31, 2010.

First, we apologize for the misprints in the original article. Now we correct those misprints and present a new proof of the global existence result, using Schauder's fixed-point theorem instead of Brouwer's theorem.

- Page 2, line 6: Replace "For this reason we prefer to use Caputo's definition which gives better results than those of Riemann-Liouville" by "For this reason, and despite our use of the Riemann-Liouville derivative, many authors prefer to use Caputo's definition"
- Page 2, line 10: Replace $D^\alpha u(t)$ by ${}^cD^\alpha u(t)$
- Page 2, in the second condition of (2.3): Replace $u(t)$ by $u(t^+)$
- Page 2, line 7 from the bottom: Replace (t_0^+) by $u(t_0^+)$
- Page 3, Eq. (2.6): Replace $\lim_{t\to t_k^+}(t-t_k)^\alpha u(t)$ by $\lim_{t\to t_k^+}(t-t_k)^{1-\alpha}u(t)$
- Page 3 line 1: Insert the sentence:

The notation $\mathcal{PC}([t_0,t_0+\tau])$ stands for $\mathcal{PC}([t_0,t_0+\tau],\mathbb{R}^n)$ throughout this article.

- Page 6, last line: Replace $u(t)$ by $u(t^+)$
- Page 7: Delete the entire Theorem 4.1.
- Page 7, in the second condition of (4.3): Replace $v(t)$ by $v(t^+)$
- Page 7, line 5 from the bottom: Replace "theorem 3.1" by "Lemma 2.1"
- Page 7 after Eq. (4.4), insert the paragraph:

Since, $\mathcal{PC}_{1-\alpha}([t_0,+\infty[,\mathbb{R}^n)$ is not a Banach space, we introduce the Banach space

$$\begin{aligned}&\mathcal{PC}^b_{1-\alpha}([t_0,+\infty[,\mathbb{R}^n)\\&=\{u\in\mathcal{PC}_{1-\alpha}([t_0,+\infty[,\mathbb{R}^n):\sup_{t\in J^*}\prod_{i=0}^m(t-t_i)^{2-\alpha}\|u(t)\|<+\infty\},\end{aligned}$$

where $J^*=[t_0,+\infty[\setminus\{t_k\}_{k=0,\dots,m}$. This space is endowed with the norm $\|\cdot\|_{\alpha-1}$ defined on page 8, which is still valid for $n=1$. Therefore, it is clear that the solution of (4.1)-(4.3) satisfies (4.4) in $\mathcal{PC}^b_{1-\alpha}([t_0,+\infty[,\mathbb{R}_+)$.

- Page 7, line 2 from the bottom: Replace $\mathcal{PC}_{1-\alpha}([t_0,+\infty[,\mathbb{R})$ by $\mathcal{PC}^b_{1-\alpha}([t_0,+\infty[,\mathbb{R}_+)$

- Page 8, line 4: Replace $\mathcal{PC}_{1-\alpha}([t_0,+\infty[,\mathbb{R})$ by $\mathcal{PC}^b_{1-\alpha}([t_0,+\infty[,\mathbb{R}_+)$,
- Page 8, line 5: Replace $\mathcal{PC}_{1-\alpha}([t_0,+\infty[,\mathbb{R}^n)$ by $\mathcal{PC}^b_{1-\alpha}([t_0,+\infty[,\mathbb{R}^n)$
- Page 8: Replace the proof of Theorem 4.2 by the following proof.

Proof of Theorem 4.2. To apply Schauder's theorem we have to establish that the operator A, defined by (3.1), is completely continuous. To prove that claim we define the set

$$\begin{aligned}\mathcal{V}_{1-\alpha} = \Big\{ & u \in \mathcal{PC}^b_{1-\alpha}([t_0,+\infty[,\mathbb{R}^n) : D^{\alpha-1}u(t_0) = u_0,\\ & \|u(t)\| \le v(t),\ t \neq t_k,\ k = 0,\dots,m,\\ & \sup_{t\in J^*}\prod_{i=0}^{m}(t-t_i)^{2-\alpha}\|u(t)\| \le \sup_{t\in J^*}\prod_{i=0}^{m}(t-t_i)^{2-\alpha}v(t),\\ & v(t) \text{ being a positive solution of (4.1)-(4.3) in } \mathcal{PC}^b_{1-\alpha}([t_0,+\infty[,\mathbb{R}_+)\Big\}.\end{aligned}$$

It is not difficult to verify that $\mathcal{V}_{1-\alpha}$ is not empty, closed, convex and bounded in $\mathcal{PC}^b_{1-\alpha}([t_0,+\infty[,\mathbb{R}^n)$. The operator A is continuous, and so for each $u \in \mathcal{V}_{1-\alpha}$, we have for each $t \neq t_k$, $k = 0,\dots,m$,

$$\begin{aligned}\|Au(t)\| \le (t-t_0)^{\alpha-1}\frac{\|u_0\|}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\|f(s,u(s))\|ds\\ + \frac{1}{\Gamma(\alpha)}\sum_{t_0<t_k\le t}(t-t_k)^{\alpha-1}\|I_k(t_k)\|.\end{aligned}$$

From the assumptions of theorem 4.2, we obtain

$$\|Au(t)\| \le v(t), \quad t \neq t_k,\ k = 0,\dots,m,$$

and since

$$\lim_{t\to t_k^+}(t-t_k)^{1-\alpha}\|Au(t)\| \le \lim_{t\to t_k^+}(t-t_k)^{1-\alpha}v(t), \quad k = 0,\dots,m,$$

we have

$$\sup_{t\in J^*}\prod_{i=0}^{m}(t-t_i)^{2-\alpha}\|Au(t)\| \le \sup_{t\in J^*}\prod_{i=0}^{m}(t-t_i)^{2-\alpha}v(t).$$

Hence, $A(\mathcal{V}_{1-\alpha}) \subset \mathcal{V}_{1-\alpha}$. The elements of $A\mathcal{V}_{1-\alpha}$ are uniformly bounded in $\mathcal{PC}^b_{1-\alpha}([t_0,+\infty[,R^n)$ because

$$\prod_{i=0}^{m}(t-t_i)^{2-\alpha}\|Au(t)\| \le \prod_{i=0}^{m}(t-t_i)^{2-\alpha}v(t) < \infty.$$

Next, we prove that the elements in the set $A\mathcal{V}_{1-\alpha}$ are equicontinuous in the space $\mathcal{PC}^b_{1-\alpha}([t_0,+\infty[,R^n)$. To do this, we show that the derivative of $u_1(t)$ (defined in section 3) is uniformly bounded in $\mathcal{PC}^b_{1-\alpha}([t_0,+\infty[,R^n)$. Note that

$$\begin{aligned}\|u_1'(t)\| &\le \frac{(1-\alpha)}{\Gamma(\alpha)}\Big(v_0(t-t_0)^{\alpha-2} + \sum_{t_0<t_k\le t}(t-t_k)^{\alpha-2}J_k(t_k)\Big)\\ &\le \frac{(1-\alpha)}{\Gamma(\alpha)}v(t); \quad t \neq t_k,\ k = 0,\dots,m,\end{aligned}$$

and since $v(t)$ is a solution of (4.1)-(4.3) in $\mathcal{PC}^b_{1-\alpha}([t_0,+\infty[,\mathbb{R}_+)$, we have

$$\|u_1'(t)\|_{1-\alpha} \leq |v(t)|_{1-\alpha}\frac{(1-\alpha)}{\Gamma(\alpha)} < +\infty.$$

On the other hand, regarding the set $Bu(t)$ defined in section 3, the uniform convergence in $\mathcal{PC}^b_{1-\alpha}([t_0,+\infty[,\mathbb{R}_+)$ is equivalent to the uniform convergence in $\mathcal{PC}^b_{1-\alpha}([t_0,T_p],R^n)$, for each T_p, with $[t_0,t_m]\subset[t_0,T_p]$ and $\lim_{p\to\infty} T_p=\infty$. Thus, for each s_1 and s_2 different from t_k, $k=0,\dots,m$, satisfying $t_0<s_1<s_2<T_p$, we have

$$\begin{aligned}
&\|Bu(s_2)-Bu(s_1)\|\\
&\leq \frac{1}{\Gamma(\alpha)}\Big[\int_{t_0}^{s_2}(s_2-s)^{\alpha-1}-\int_{t_0}^{s_1}(s_1-s)^{\alpha-1}\Big]\|f(s,u(s))\|ds\\
&\leq \frac{1}{\Gamma(\alpha)}\Big[\int_{t_0}^{s_1}(s_2-s)^{\alpha-1}-(s_1-s)^{\alpha-1}+\int_{s_1}^{s_2}(s_2-s)^{\alpha-1}\Big]g(s,v(s))ds,
\end{aligned}$$

and since $g(s,v(s))$ is continuous and positive, we obtain

$$\|Bu(s_2)-Bu(s_1)\|\leq\frac{G}{\Gamma(\alpha)}[2(s_2-s_1)^\alpha+|(s_2-t_0)^\alpha-(s_1-t_0)^\alpha|],$$

where

$$G=\sup_{t\in[t_0,+\infty[} g(t,v(t)).$$

Clearly, the right hand side tends to zero as $s_1\to s_2$. We infer that

$$\|Bu(s_2)-Bu(s_1)\|_{1-\alpha}\to 0,\quad \text{as } s_1\to s_2.$$

Therefore, $\overline{A\mathcal{V}_{1-\alpha}}$ is compact and A is completely continuous. We conclude by Schauder's theorem that A has at least one fixed point in $\mathcal{V}_{1-\alpha}$ which is the solution to the given problem (2.1)-(2.3). Furthermore, it satisfies the estimate $\|u\|_{1-\alpha}\leq |v|_{1-\alpha}$. The proof is complete. □

End of the corrigendum.

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