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REGULARITY OF WEAK SOLUTIONS TO THE MAGNETO-HYDRODYNAMICS EQUATIONS IN TERMS OF THE DIRECTION OF VELOCITY

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Abstract. In this article, we study the regularity of weak solutions to the 3D incompressible magneto-hydrodynamics equations. We obtain a new class of regularity criteria in terms of the direction of the velocity. Our result extend some results known for incompressible Navier-Stokes equations.

1. Introduction

We consider the 3D incompressible magneto-hydrodynamics (MHD) equation

$$\begin{gathered}
\frac{\partial u}{\partial t} - \nu\Delta u + u\cdot\nabla uc = -\nabla p - \frac{1}{2}\nabla b^2 + b\cdot\nabla b,\\
\frac{\partial b}{\partial t} - \eta\Delta b + u\cdot\nabla b = b\cdot\nabla u,\\
\nabla\cdot u = \nabla\cdot b = 0,\\
u(0,x) = u_0(x), \quad b(0,x) = b_0(x)
\end{gathered} \tag{1.1}$$

Here u, b describe the flow velocity vector and the magnetic field vector respectively, and p is pressure. While u_0, b_0 are the given initial velocity and initial magnetic field respectively, with $\nabla\cdot u_0 = 0$, $\nabla\cdot b_0 = 0$. Without loss of generality, we set $\nu = \eta = 1$ in the rest of the paper (it can be achieved by rescaling). If $\nu = \eta = 0$, (1.1) is called the ideal MHD equations.

It is well known that there exist a global Leray-Hopf weak solution $(u,b) \in L^\infty(0,\infty; L^2(\mathbb{R}^3)) \cap L^2(0,\infty; \dot{H}(\mathbb{R}^3))$ if the initial data $(u_0, b_0) \in L^2(\mathbb{R}^3)$. Using the standard energy method, it can be easily proved that the solution satisfies the energy inequality

$$\|u(t)\|^2 + \|b(t)\|^2 + \int_0^T (\|\nabla u(s)\|_{L^2}^2 + \|\nabla b(s)\|_{L^2}^2) ds \le \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2, \quad T \ge 0.$$

Up to now, it is unknown whether solutions of (1.1) on $(0,T)$ will develop finite time singularities even if the initial data is sufficiently smooth. This problem, global regularity issue, has been thoroughly studied for the 3D Navier-Stokes equations and many of these results can be extended to the 3D MHD equations. Serrin [6]

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showed that a weak solution of 3D incompressible Navier-Stokes equations with initial data $u_0 \in L^2(\mathbb{R}^3)$ lying $L^p(0,\infty;L^q(\mathbb{R}^3))$ is smooth in the spatial direction if $p,q \geq 1$ and $2/p+3/q < 1$. He and Xin [5] extended this criteria to MHD equations, precisely they showed that under the condition

$$u \in L^p(0,T;L^q(\mathbb{R}^3)) \quad \text{for } 1/p + 3/2q \leq 1/2 \text{ and } q > 3,$$

then the solution remains smooth on $[0,T]$.

Another class of regularity criteria which involves the gradient of u for the 3D Navier-Stoke equations was introduced by Beirão de Veiga [2] . He showed that any Leray-Hopf solution is smooth given $\nabla u \in L^p(0,T;L^q(\mathbb{R}^3))$ with $2/p+3/q = 2$, $3/2 < q < \infty$. Beale, Kato and Majda[1] dealt with the vorticity $\omega = \nabla \times u$ and proved the regularity under the condition $\omega \in L^1(0,T;L^\infty(\mathbb{R}^3))$. He, Xin [5] and Zhou [9] respectively extended the result of Beirão de Veiga [2] to the MHD equations, they obtained some condition of ∇u alone to determine the regularity of the MHD equations. Precisely, the showed that under the condition

$$\nabla u \in L^{p}(0,T;L^q(\mathbb{R}^3)) \quad \text{with } 2/p+3/q \leq 1,\ 3/2 < q \leq \infty,$$

then the solution can be extended to $t = T$. Caflisch, Kapper and Steele [3] extended the well known result of [1] to the 3D ideal MHD equations, they showed under the condition

$$\int_0^T (\|\nabla \times u(t)\|_{L^\infty} + \|\nabla \times b(t)\|_{L\infty})dt < \infty$$

then the solutions remains smooth on $[0,T]$.

Constantin and Fefferman [4] used the direction of the vorticity $\omega/|\omega|$ to describe the regularity criterion to the Navier-Stokes equations. They showed that under a Lipschitz-like regularity assumption on $\omega/|\omega|$, the solution is smooth. Under the framework of Constantin and Fefferman, Zhou [10, 11] get some more relaxed regularity criterion in terms of the direction of vorticity. Inspired by the initial work of [4], He and Xin[5] extended the result to the MHD equations. They showed that if there exist three positive constant K, ρ, Ω such that

$$|\omega(x+y,t) - \omega(x,t)| \leq K|\omega(x+y,t)||y|^{1/2}$$

holds if both $|y| \leq \rho$ and $|\omega(x,t)| \geq \Omega$ for any $t \in [0,T]$, then the solution is remains smooth on $[0,T]$, where $\omega = \nabla \times u$ is the vorticity of the velocity. Zhou [12] also studied the regularity criterion for generalized magneto-hydrodynamics equations in term of the vorticity field and get similar result.

Of the same spirit in [4], Vasseur[7] used the direction of the velocity $u/|u|$ to describe the regularity criterion to the Navier-Stokes equations. He showed that if the initial value $u_0 \in L^2(\mathbb{R}^3)$, and $\operatorname{div}(u/|u|) \in L^p(0,\infty;L^q(\mathbb{R}^3))$ with

$$\frac{2}{p} + \frac{3}{q} \leq \frac{1}{2}, \quad q \geq 6,\ p \geq 4$$

then u is smooth on $(0,\infty) \times \mathbb{R}^3$.

We restricted ourselves to $u/|u|$ to study the regularities of the weak solutions of (1.1). We followed the same method of [7] to get our result:

Theorem 1.1. *Let u, b be a Leray-Hopf solution to MHD equation* (1.1) *with the initial value $u_0, b_0 \in H^1(\mathbb{R}^3)$. If*

$$b \in L^\alpha(0,T;L^\beta(\mathbb{R}^3)) \quad \text{with } \frac{2}{\alpha} + \frac{3}{\beta} \leq 1, \beta > 3,$$

and

$$\operatorname{div}(u/|u|) \in L^p(0,T;L^q(\mathbb{R}^3)) \quad with \quad \frac{2}{p}+\frac{3}{q} \le \frac{1}{2},\ q \ge 6, p \ge 4,$$

then u, b *is smooth on* $(0,\infty)\times\mathbb{R}^3$.

This result shows that it is sufficient to control the norm of b and the rate of change the direction of the velocity to get full regularity of the solution. The proof is standard, based on energy methods.

First, we need to introduced some definitions and symbols.

Definition 1.2 ([8])**.** A measurable vector pair (u,b) is called a weak solution to the the generalized magneto-hydrodynamics equations (1.1), if it satisfies the following properties

(1) $u \in L^\infty([0,T);L^2(\mathbb{R}^3)) \cap L^2([0,T);H(\mathbb{R}^3))$,
$b \in L^\infty([0,T);L^2(\mathbb{R}^3)) \cap L^2([0,T);H(\mathbb{R}^3))$;

(2) (u,b) satisfies (1.1) in the sense of distribution; that is,

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^3}\Big(\frac{\partial\phi}{\partial t}+u\cdot\nabla\phi\Big)u\,dx\,dt+\int_{\mathbb{R}^3}u_0\phi(0,x)\,dx\\
&=\int_0^T\int_{\mathbb{R}^3}(\nabla u:\nabla\phi+b\cdot\nabla\phi\cdot b)\,dx\,dt\,,\\
&\int_0^T\int_{\mathbb{R}^3}\Big(\frac{\partial\phi}{\partial t}+u\cdot\nabla\phi\Big)b\,dx\,dt+\int_{\mathbb{R}^3}b_0\phi(0,x)\,dx\\
&=\int_0^T\int_{\mathbb{R}^3}(\nabla b:\nabla\phi+b\cdot\nabla\phi\cdot u)\,dx\,dt
\end{aligned}$$

for all $\phi \in C_0^\infty(\mathbb{R}^3\times[0,T))$ with $\nabla\cdot\phi = 0$, and

$$\int_0^T\int_{\mathbb{R}^3}u\cdot\nabla\phi\,dx\,dt=0,\quad \int_0^T\int_{\mathbb{R}^3}b\cdot\nabla\phi\,dx\,dt=0$$

for every $\phi \in C_0^\infty(\mathbb{R}^3\times[0,T))$.

(3) The energy inequality holds; that is,

$$\begin{aligned}
\|u(t)\|_{L^2}^2+2\int_0^t\|\nabla u\|_{L^2}^2ds\ &\le \|u_0\|_{L^2}^2,\\
\|b(t)\|_{L^2}^2+2\int_0^t\|\nabla b\|_{L^2}^2ds\ &\le \|b_0\|_{L^2}^2,
\end{aligned}$$

for all $t \in [0,T)$.

The space $L^{p,q}$ consists of functions f for which $\|f\|_{L^{p,q}} < +\infty$, where

$$\|f\|_{L^{p,q}}=\begin{cases}\Big(\int_0^T\|u(\tau,\cdot)\|_{L^q}^p\,d\tau\Big)^{1/p}, & \text{if } 1\le p<+\infty,\\ \operatorname{ess\,sup}_{0<\tau<t}\|u(\tau,\cdot)\|_{L^q}, & \text{if } p=+\infty\end{cases}$$

where

$$\|u(\tau,\cdot)\|_{L^q}=\begin{cases}\Big(\int_0^T|u(\tau,x)|^q\,dx\Big)^{1/q}, & \text{if } 1\le q<+\infty,\\ \operatorname{ess\,sup}_{x\in\mathbb{R}^3}\|u(\tau,\cdot)\|_{L^q} & \text{if } q=+\infty\end{cases}$$

2. Proof of Theorem 1.1

Proof. Multiplying the first equation by $|u|^2u$, and the second equation by $|b|^2b$. Integrate the first equation over $\mathbb{R}^3$, and after suitable integration by parts, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3}\frac{|u|^4}{4}\,dx+\int_{\mathbb{R}^3}(|\nabla u|^2|u|^2+2|u|^2|\nabla|u\|^2)\,dx\\&=\int_{\mathbb{R}^3}2pu|u|\cdot\nabla|u|dx-\int_{\mathbb{R}^3}b\cdot\nabla(|u|^2u)\cdot b\,dx\,.\end{aligned}\tag{2.1}$$

Integrate the second equation over $\mathbb{R}^3$, and after suitable integration by parts, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3}\frac{|b|^4}{4}\,dx+\int_{\mathbb{R}^3}(|b|^2|\nabla b|^2+2|b|^2|\nabla|b\|^2)\,dx=\int_{\mathbb{R}^3}(b\cdot\nabla u)\cdot|b|^2bdx\tag{2.2}$$

Adding (2.1) and (2.2) yields

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3}\frac{|u|^4+|b|^4}{4}\,dx+\int_{\mathbb{R}^3}(|\nabla u|^2|u|^2+2|u|^2|\nabla|u\|^2)dx\\&+\int_{\mathbb{R}^3}(|b|^2|\nabla b|^2+2|b|^2|\nabla|b\|^2)\,dx\\&=\int_{\mathbb{R}^3}(2pu|u|\cdot\nabla|u|-b\cdot\nabla(|u|^2u)\cdot b-(b\cdot\nabla|b|^2b)\cdot u)\,dx\end{aligned}\tag{2.3}$$

Next we estimate the right hand terms one by one. Because

$$-\Delta p=\sum_{i,j=1}^{3}\partial_i\partial_j(u_iu_j-b_ib_j).$$

The Calderon-Zygmund inequality tells us that there exists a absolute constant C such that

$$\|p\|_{L^q}\le C(\|u\|_{L^{2q}}^2+\|b\|_{L^{2q}}^2),\quad\text{for }1<q<\infty.$$

By generalized Hölder's inequality and Young's inequality, we get

$$\begin{aligned}2\int_{\mathbb{R}^3}pu|u|\cdot\nabla|u|&\le 2\int_{\mathbb{R}^3}|p\|u|^2|\frac{u}{|u|}\cdot\nabla|u\|\,dx\\&\le 2\|p\|_{L^r}\|u\|_{L^{2r}}^2\|\frac{u}{|u|}\cdot\nabla|u\|\|_{L^{\bar q}}\\&\le 2(\|u\|_{L^{2r}}^4+\|u\|_{L^{2r}}^2\|b\|_{L^{2r}}^2)\|\frac{u}{|u|}\cdot\nabla|u\|\|_{L^{\bar q}}\\&\le C||\frac{u}{|u|}\cdot\nabla|u|||_{L^{\bar p}}(\|u\|_{L^{2r}}^4+\|b\|_{L^{2r}}^4).\end{aligned}\tag{2.4}$$

Where $2/r+1/\bar q=1, 2\le r<6$, and here we use the fact

$$|u|\mathrm{div}(u/|u|)=-\frac{u}{|u|}\cdot\nabla|u|.$$

Write $\|u\|_{L^{2r}}^4=\||u|^2\|_{L^r}$, then interpolation inequality of L^p spaces and Sobolev imbedding theorem gives

$$\|u|^2\|_{L^r}\le\||u|^2\|_{L^2}^{2\theta}\||u|^2\|_{L^6}^{2(1-\theta)}\le\||u|^2\|_{L^2}^{2\theta}\|\nabla|u|^2\|_{L^2}^{2(1-\theta)}.$$

Where $\theta/2 + (1-\theta)/6 = 1/r$. Using Young's inequality again, we obtain

$$\begin{aligned}
&\|\frac{u}{|u|}\cdot\nabla|u|\|_{L^{\bar p}}(\|u\|^4_{L^{2r}} + \|b\|^4_{L^{2r}}) \\
&\le C\|\frac{u}{|u|}\cdot\nabla|u|\|^{1/\theta}_{L^{\bar p}}(\|u\|^4_{L^4} + \|b\|^4_{L^4}) + \frac{1}{8}(\|\nabla|u|^2\|^2_{L^2} + \|\nabla|b|^2\|^2_{L^2}) \qquad (2.5)\\
&= C\|\frac{u}{|u|}\cdot\nabla|u|\|^{1/\theta}_{L^{\bar p}}(\|u\|^4_{L^4} + \|b\|^4_{L^4}) + \frac{1}{2}(\||u|\nabla|u|\|^2_{L^2} + \||b|\nabla|b|\|^2_{L^2}).
\end{aligned}$$

Assume that $\mathrm{div}(u/|u|) \in L^p(L^q)$, $u \in L^a(L^b)$. Then we have $|u|\mathrm{div}(u/|u|) \in L^{\bar p}(L^{\bar q})$ where

$$\frac{1}{\bar p} = \frac{1}{a} + \frac{1}{p}, \quad \frac{1}{\bar q} = \frac{1}{b} + \frac{1}{q}$$

If $1/\theta \le \bar p$, then

$$\int_0^T \|\frac{u}{|u|}\cdot\nabla|u|\|^{1/\theta}_{L^{\bar p}}\,dx < \infty.$$

Now we seek conditions for $1/\theta \le \bar p$. By the relation

$$\begin{gathered}
\frac{2}{r} + \frac{1}{\bar q} = 1,\\
\frac{\theta}{2} + \frac{1-\theta}{6} = \frac{1}{r},\\
\frac{1}{\theta} \le \bar p,
\end{gathered}$$

we get $2/\bar p + 3/\bar q \le 2$. From the definition of weak solution, we know $2/a + 3/b = 3/2$. So

$$\frac{2}{\bar p} + \frac{3}{\bar q} = \frac{2}{a} + \frac{3}{b} + \frac{2}{p} + \frac{3}{q} = \frac{3}{2} + \frac{2}{p} + \frac{3}{q} \le 2;$$

that is,

$$\frac{2}{p} + \frac{3}{q} \le \frac{1}{2}.$$

Next we estimate the second term of (2.3). As in [8], we have

$$-\int_{\mathbb{R}^3} b\cdot\nabla(|u|^2u)\cdot b dx \le \int_{\mathbb{R}^3} |b|^2|u||\nabla|u|^2|\,dx$$

Using Cauchy's inequality with ε, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^3} |b|^2|u||\nabla|u|^2|\,dx &\le C\int_{\mathbb{R}^3}|b|^4|u|^2\,dx + \frac{1}{8}\|\nabla|u|^2\|^2_{L^2}\\
&= C\int_{\mathbb{R}^3}|b|^4|u|^2\,dx + \frac{1}{2}\||u|\nabla|u|\|^2_{L^2}\,.
\end{aligned}$$

By generalized Hölder inequality, interpolation inequality of L^p spaces, Sobolev imbedding theorem and Young's inequality, we obtain

$$\begin{aligned}
C\int_{\mathbb{R}^3}|b|^4|u|^2dx &\le C\|b\|^2_{L^\beta}\|b\|^2_{L^{2r}}\|u\|^2_{L^{2r}}\\
&\le C\|b\|^2_{L^\beta}(\|b\|^4_{L^{2r}} + \|u\|^4_{L^{2r}})\\
&\le C\|b\|^{2/\xi}_{L^\beta}(\|b\|^4_{L^4} + \|u\|^4_{L^4}) + \frac{1}{2}(\||u|\nabla|u|\|^2_{L^2} + \||b|\nabla|b|\|^2_{L^2})\,,
\end{aligned}$$

where

$$
\begin{gathered}
\frac{1}{\beta}+\frac{1}{r}=\frac{1}{2} \\
\frac{\xi}{2}+\frac{1-\xi}{6}=\frac{1}{r} \\
2 \leq r<6
\end{gathered}
\tag{2.6}
$$

We need $2/\xi \leq \alpha$ so that

$$
\int_0^T \|b\|_{L^\beta}^{2/\xi} dx<\infty .
$$

By the relation (2.6), this is satisfied if $2/\alpha+3/\beta \leq 1$. And $\beta>3$ implies $2 \leq r<6$. Due to the above argument,

$$
-\int_{\mathbb{R}^3} b \cdot \nabla(|u|^2 u) \cdot b dx \leq C\|b\|_{L^\beta}^{2/\xi}(\|b\|_{L^4}^4+\|u\|_{L^4}^4)+(\||u|\nabla|u|\|_{L^2}^2+\frac{1}{2}\||b|\nabla|b|\|_{L^2}^2)
\tag{2.7}
$$

The last term of (2.3) can be treated in the same way,

$$
-\int_{\mathbb{R}^3} b \cdot \nabla(|b|^2 b) \cdot u dx \leq C\|b\|_{L^\beta}^{2/\xi}(\|b\|_{L^4}^4+\|u\|_{L^4}^4)+(\frac{1}{2}\||u|\nabla|u|\|_{L^2}^2+\||b|\nabla|b|\|_{L^2}^2)
\tag{2.8}
$$

Combining (2.3), (2.4), (2.5), (2.7), (2.8), we obtain

$$
\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^3} \frac{|u|^4+|b|^4}{4} dx+\int_{\mathbb{R}^3}(|\nabla u|^2|u|^2+|b|^2|\nabla b|^2) dx \\
&\leq C(\|b\|_{L^\beta}^{2/\xi}+\||u| \operatorname{div}(u/|u|)\|_{L^q}^{1/\theta}+1)(\|b\|_{L^b}^4+\|u\|_{L^b}^4) \\
&=C(\|b\|_{L^\beta}^{2/\xi}+\||u| \operatorname{div}(u/|u|)\|_{L^q}^{1/\theta}+1) \int_{\mathbb{R}^3} \frac{|u|^4+|b|^4}{4} dx
\end{aligned}
$$

Let $A(t)=C(\|b\|_{L^\beta}^{2/\xi}+\||u| \operatorname{div}(u/|u|)\|_{L^q}^{1/\theta}+1)$, then the Gronwell inequality implies that, whenever T is finite,

$$
\|u(T)\|_{L^4}^4+\|b(T)\|_{L^4}^4 \leq C(\|u_0\|_{L^4}^4+\|b_0\|_{L^4}^4) \exp(t \sup_{t \in[0, T)} A(t)).
$$

This shows that the solution (u, b) can be extended to $t=T$. This completes the proof. $\square$

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