Electronic Journal of Differential Equations, Vol. 2009(2009), No. 128, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

HOMOCLINIC SOLUTIONS FOR A CLASS OF SECOND ORDER NON-AUTONOMOUS SYSTEMS

RONG YUAN, ZIHENG ZHANG

Abstract. This article concerns the existence of homoclinic solutions for the second order non-autonomous system

$$\ddot{q} + A\dot{q} - L(t)q + W_q(t, q) = 0,$$

where A is a skew-symmetric constant matrix, $L(t)$ is a symmetric positive definite matrix depending continuously on $t \in \mathbb{R}$, $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. We assume that $W(t, q)$ satisfies the global Ambrosetti-Rabinowitz condition, that the norm of A is sufficiently small and that L and W satisfy additional hypotheses. We prove the existence of at least one nontrivial homoclinic solution, and the existence of infinitely many homoclinic solutions if $W(t, q)$ is even in q. Recent results in the literature are generalized and improved.

1. Introduction

The purpose of this work is to study the existence of *homoclinic* solutions for the second order non-autonomous system

$$\ddot{q} + A\dot{q} - L(t)q + W_q(t, q) = 0, \tag{1.1}$$

where A is a skew-symmetric constant matrix, $L(t)$ is a symmetric and positive definite matrix depending continuously on $t \in \mathbb{R}$, $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. A solution $q(t)$ of (1.1) is called a homoclinic solution (to 0) if $q \in C^2(\mathbb{R}, \mathbb{R}^n)$, $q(t) \to 0$ and $\dot{q}(t) \to 0$ as $t \to \pm\infty$. If $q(t) \not\equiv 0$, $q(t)$ is called a nontrivial homoclinic solution.

When $A = 0$, (1.1) is the second order Hamiltonian system. Assuming that $L(t)$ and $W(t, q)$ are independent of t or T-periodic in t, the existence of homoclinic solutions for the Hamiltonian system (1.1) has been studied via critical point theory and variational methods, see for instance [2, 4, 6, 8, 9, 15, 17] and the references therein; a more general case is considered in [10]. In this case, the existence of homoclinic solutions can be obtained by taking the limit of periodic solutions of approximating problems. If $L(t)$ and $W(t, q)$ are neither independent of t not periodic in t, compactness arguments derived from Sobolev imbedding theorem are not available for the study of (1.1), see [1, 5, 7, 11, 12, 13, 14, 18] and the references therein.

2000 *Mathematics Subject Classification.* 34C37, 35A15, 37J45.
Key words and phrases. Homoclinic solutions; critical point; variational methods; mountain pass theorem.

Submitted October 14, 2008. Published October 7, 2009.
Supported by National Natural Science Foundation of China and RFDP.

When $A \neq 0$, as far as we know, the existence of homoclinic solutions of (1.1) has not been studied. Our basic hypotheses on L and W are:

(H1) $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$, $L(t)$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$, and there is a continuous function $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(t) > 0$ for all $t \in \mathbb{R}$, $\big(L(t)q, q\big) \geq \alpha(t)|q|^2$, and $\alpha(t) \to +\infty$ as $|t| \to +\infty$.

(H2) There exists a constant $\mu > 2$ such that for every $t \in \mathbb{R}$ and $q \in \mathbb{R}^n \backslash \{0\}$
$$0 < \mu W(t, q) \leq \big(W_q(t, q), q\big).$$

(H3) $W_q(t, q) = o(|q|)$ as $|q| \to 0$ uniformly with respect to $t \in \mathbb{R}$.

(H4) There exists $\overline{W} \in C(\mathbb{R}^n, \mathbb{R})$ such that $|W_q(t, q)| \leq |\overline{W}(q)|$ for every $t \in \mathbb{R}$ and $q \in \mathbb{R}^n$.

Remark 1.1. From (H1), we see that there is a constant $\beta > 0$ such that
$$\big(L(t)q, q\big) \geq \beta|q|^2 \quad \text{for all } t \in \mathbb{R} \text{ and } q \in \mathbb{R}^n. \tag{1.2}$$
(H2) is called the global Ambrosetti-Rabinowitz condition due to Ambrosetti and Rabinowitz (e.g., [3]). Combining (H2) with (H3), we see that $W(t, q) \geq 0$ for all $(t, q) \in \mathbb{R} \times \mathbb{R}^n$, $W(t, 0) = 0$, $W_q(t, 0) = 0$. Moreover, $W(t, q) = o(|q|^2)$ as $|q| \to 0$ uniformly with respect to t, which implies that for any $\varepsilon > 0$ there is $\delta > 0$ such that
$$W(t, q) \leq \varepsilon|q|^2 \quad \text{for } (t, q) \in \mathbb{R} \times \mathbb{R}^n,\ |q| \leq \delta. \tag{1.3}$$

In addition, we need the following hypothesis on A.

(H5) $\|A\| < \sqrt{\beta}$, where β is defined in (1.2).

Now we state our main result.

Theorem 1.2. *Assume* (H1)–(H5). *Then* (1.1) *possesses at least one nontrivial homoclinic solution. Moreover, if we assume that $W(t, q)$ is even in q; i.e.,*

(H6) *$W(t, -q) = W(t, q)$ for all $t \in \mathbb{R}$ and $q \in \mathbb{R}^n$,*

then (1.1) *has infinitely many distinct homoclinic solutions.*

Remark 1.3. From Remark 1.1, we know that there exists $\beta > 0$ such that (1.2) holds. However, since we do not have an explicit estimate on β, we simply assume that $\|A\|$ is sufficiently small. Furthermore, when $A = 0$, our main result is just [13, Theorems 1 and 2].

To overcome the lack of compactness in standard Sobolev imbedding theorems, we employ a compact imbedding theorem obtained in [13]. In Section 2 we state and prove preliminary results. Section 3 is devoted to the proof of Theorem 1.2.

2. Preliminaries

Let
$$E = \Big\{q \in H^1(\mathbb{R}, \mathbb{R}^n) : \int_{\mathbb{R}} \big[|\dot{q}(t)|^2 + \big(L(t)q(t), q(t)\big)\big] dt < +\infty\Big\}.$$
This vector space is a Hilbert space when endowed with the inner product
$$(x, y) = \int_{\mathbb{R}} \big[\big(\dot{x}(t), \dot{y}(t)\big) + \big(L(t)x(t), y(t)\big)\big] dt$$
and the corresponding norm $\|x\|^2 = (x, x)$. Note that
$$E \subset H^1(\mathbb{R}, \mathbb{R}^n) \subset L^p(\mathbb{R}, \mathbb{R}^n)$$

for all $p \in [2, +\infty]$ with the imbedding being continuous. In particular, for $p = +\infty$, there exists a constant $C > 0$ such that

$$\|q\|_\infty \le C\|q\|, \quad \forall q \in E. \tag{2.1}$$

Here $L^p(\mathbb{R}, \mathbb{R}^n)$ $(2 \le p < +\infty)$ and $H^1(\mathbb{R}, \mathbb{R}^n)$ denote the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norms

$$\|q\|_p := \Big(\int_{\mathbb{R}} |q(t)|^p dt \Big)^{1/p} \quad \text{and} \quad \|q\|_{H^1} := \big(\|q\|_2^2 + \|\dot{q}\|_2^2 \big)^{1/2}$$

respectively. $L^\infty(\mathbb{R}, \mathbb{R}^n)$ is the Banach space of essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^n$ equipped with the norm

$$\|q\|_\infty := \operatorname{ess\,sup}\{|q(t)| : t \in \mathbb{R}\}.$$

Lemma 2.1 ([13, Lemma 1]). *Assume L satisfies* (H1). *Then the embedding of E in $L^2(\mathbb{R}, \mathbb{R}^n)$ is compact.*

Lemma 2.2 ([13, Lemma 2]). *Assume* (H1), (H3), (H4). *If $q_k \rightharpoonup q_0$ (weakly) in E, then $W_q(t, q_k) \to W_q(t, q_0)$ in $L^2(\mathbb{R}, \mathbb{R}^n)$.*

Lemma 2.3. *Under Assumption* (H2)*, for every $t \in \mathbb{R}$, we have*

$$W(t, q) \le W\Big(t, \frac{q}{|q|}\Big)|q|^\mu, \quad \text{if } 0 < |q| \le 1, \tag{2.2}$$

$$W(t, q) \ge W\Big(t, \frac{q}{|q|}\Big)|q|^\mu, \quad \text{if } |q| \ge 1. \tag{2.3}$$

Proof. It suffices to show that for every $q \neq 0$ and $t \in \mathbb{R}$ the function $(0, \infty) \ni \zeta \to W(t, \zeta^{-1}q)\zeta^\mu$ is non-increasing, which is an immediate consequence of (H2). $\square$

Remark 2.4. From Lemma 2.3, we see that there exists $\alpha_0(t) > 0$ such that

$$W(t, q) \ge \alpha_0(t)|q|^\mu \quad \text{for all } (t, q) \in \mathbb{R} \times \mathbb{R}^n, \ |q| \ge 1.$$

Now we introduce more notation and some definitions. Let $\mathcal{B}$ be a real Banach space, $I \in C^1(\mathcal{B}, \mathbb{R})$, which means that I is a continuously Fréchet-differentiable functional defined on $\mathcal{B}$.

Definition 2.5 ([16]). $I \in C^1(\mathcal{B}, \mathbb{R})$ is said to satisfy the (PS) condition if any sequence $\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{B}$, for which $\{I(u_j)\}_{j\in\mathbb{N}}$ is bounded and $I'(u_j) \to 0$ as $j \to +\infty$, possesses a convergent subsequence in $\mathcal{B}$.

Moreover, let B_r be the open ball in $\mathcal{B}$ with the radius r and centered at 0 and ∂B_r denote its boundary. We obtain the existence and multiplicity of homoclinic solutions of (1.1) by use of the following well-known Mountain Pass Theorems, see [16].

Lemma 2.6 ([16, Theorem 2.2]). *Let $\mathcal{B}$ be a real Banach space and $I \in C^1(\mathcal{B}, \mathbb{R})$ satisfying the (PS) condition. Suppose that $I(0) = 0$ and*

- (A1) *there exist constants ρ, $\alpha > 0$ such that $I|_{\partial B_\rho} \ge \alpha$,*
- (A2) *there exists $e \in \mathcal{B} \setminus \overline{B}_\rho$ such that $I(e) \le 0$.*

Then I possesses a critical value $c \ge \alpha$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0, 1], \mathcal{B}) : g(0) = 0, g(1) = e\}.$$

Lemma 2.7 ([16, Theorem 9.12]). *Let $\mathcal{B}$ be an infinite dimensional real Banach space and $I \in C^1(\mathcal{B}, \mathbb{R})$ be even satisfying the (PS) condition and $I(0) = 0$. If $\mathcal{B} = V \oplus X$, where V is finite dimensional, and I satisfies*

- (A3) *there exist constants ρ, $\alpha > 0$ such that $I|_{\partial B_\rho \cap X} \geq \alpha$ and*
- (A4) *for each finite dimensional subspace $\tilde{\mathcal{B}} \subset \mathcal{B}$, there is an $R = R(\tilde{\mathcal{B}})$ such that $I \leq 0$ on $\tilde{\mathcal{B}} \backslash B_{R(\tilde{\mathcal{B}})}$,*

then I has an unbounded sequence of critical values.

3. Proof of Theorem 1.2

Now we establish the corresponding variational framework to obtain homoclinic solutions of (1.1). Take $\mathcal{B} = E$ and define the functional $I : E \to \mathbb{R}$ by

$$\begin{aligned} I(q) &= \int_{\mathbb{R}} \Big[\frac{1}{2}|\dot{q}(t)|^2 + \frac{1}{2}\big(Aq(t), \dot{q}(t)\big) + \frac{1}{2}\big(L(t)q(t), q(t)\big) - W(t, q(t))\Big] dt \\ &= \frac{1}{2}\|q\|^2 + \frac{1}{2}\int_{\mathbb{R}} \big(Aq(t), \dot{q}(t)\big) dt - \int_{\mathbb{R}} W(t, q(t)) dt. \end{aligned} \tag{3.1}$$

Lemma 3.1. *Under the conditions of Theorem 1.2, we have*

$$I'(q)v = \int_{\mathbb{R}} \Big[\big(\dot{q}(t), \dot{v}(t)\big) - \big(A\dot{q}(t), v(t)\big) + \big(L(t)q(t), v(t)\big) - \Big(W_q(t, q(t)), v(t)\Big)\Big] dt, \tag{3.2}$$

for all q, $v \in E$, which yields, using the skew-symmetry of A,

$$\begin{aligned} I'(q)q &= \|q\|^2 - \int_{\mathbb{R}} \big(A\dot{q}(t), q(t)\big) dt - \int_{\mathbb{R}} \Big(W_q(t, q(t)), q(t)\Big) dt \\ &= \|q\|^2 + \int_{\mathbb{R}} \big(Aq(t), \dot{q}(t)\big) dt - \int_{\mathbb{R}} \Big(W_q(t, q(t)), q(t)\Big) dt. \end{aligned} \tag{3.3}$$

Moreover, I is a continuously Fréchet-differentiable functional defined on E; i.e., $I \in C^1(E, \mathbb{R})$ and any critical point of I on E is a classical solution of (1.1) *with $q(\pm\infty) = 0 = \dot{q}(\pm\infty)$.*

Proof. We begin by showing that $I : E \to \mathbb{R}$. By (1.3), there exist constants $M > 0$ and $R_1 > 0$ such that

$$W(t, q) \leq M|q|^2 \quad \text{for all } (t, q) \in \mathbb{R} \times \mathbb{R}^n,\ |q| \leq R_1. \tag{3.4}$$

Letting $q \in E$, then $q \in C^0(\mathbb{R}, \mathbb{R}^n)$ (see, e.g., [17]), the space of continuous functions q on $\mathbb{R}$ such that $q(t) \to 0$ as $|t| \to +\infty$; i.e., $E \subset C^0(\mathbb{R}, \mathbb{R}^n)$. Therefore there is a constant $R_2 > 0$ such that $|t| \geq R_2$ implies that $|q(t)| \leq R_1$. Hence, by (3.4), we have

$$0 \leq \int_{\mathbb{R}} W(t, q(t)) dt \leq \int_{-R_2}^{R_2} W(t, q(t)) dt + M \int_{|t| \geq R_2} |q(t)|^2 dt < +\infty. \tag{3.5}$$

Combining (3.1) and (3.5), we show that $I : E \to \mathbb{R}$.

Next we prove that $I \in C^1(E, \mathbb{R})$. Rewrite I as $I = I_1 - I_2$, where

$$I_1 := \frac{1}{2}\int_{\mathbb{R}} \Big[|\dot{q}(t)|^2 + \big(Aq(t), \dot{q}(t)\big) + \big(L(t)q(t), q(t)\big)\Big] dt, \quad I_2 := \int_{\mathbb{R}} W(t, q(t)) dt.$$

It is easy to check that $I_1 \in C^1(E, \mathbb{R})$, and by using the skew-symmetry of A, we have

$$I_1'(q)v = \int_{\mathbb{R}} \Big[\big(\dot{q}(t), \dot{v}(t)\big) - \big(A\dot{q}(t), v(t)\big) + \big(L(t)q(t), v(t)\big)\Big] dt. \tag{3.6}$$

Therefore it is sufficient to consider I_2. In the process we will see that

$$I_2'(q)v = \int_{\mathbb{R}} \big(W_q(t,q(t)), v(t)\big)dt, \tag{3.7}$$

which is defined for all q, $v \in E$. For any given $q \in E$, let us define $J(q) : E \to \mathbb{R}$ as following

$$J(q)v = \int_{\mathbb{R}} \big(W_q(t,q(t)), v(t)\big)dt, \quad v \in E.$$

It is obvious that $J(q)$ is linear. Now we show that $J(q)$ is bounded. Indeed, for any given $q \in E$, there exists a constant $M_1 > 0$ such that $\|q\| \le M_1$ and, by (2.1), $\|q\|_\infty \le CM_1$. According to (H3) and (H4), there is a constant $b_1 > 0$ (dependent on q) such that

$$|W_q(t,q(t))| \le b_1|q(t)| \quad \text{for all } t \in \mathbb{R},$$

which by (1.2) and the Hölder inequality yields

$$|J(q)v| = \Big|\int_{\mathbb{R}} \big(W_q(t,q(t)), v(t)\big)dt\Big| \le b_1\|q\|_2\,\|v\|_2 \le \frac{b_1}{\beta}\|q\|\,\|v\|. \tag{3.8}$$

Moreover, for q and $v \in E$, by the Mean Value Theorem, we have

$$\int_{\mathbb{R}} W(t,q(t)+v(t))dt - \int_{\mathbb{R}} W(t,q(t))dt = \int_{\mathbb{R}} \big(W_q(t,q(t)+h(t)v(t)), v(t)\big)dt,$$

where $h(t) \in (0,1)$. Therefore, by Lemma 2.2 and the Hölder inequality, we have

$$\begin{aligned} &\int_{\mathbb{R}} \big(W_q(t,q(t)+h(t)v(t)), v(t)\big)dt - \int_{\mathbb{R}} \big(W_q(t,q(t)), v(t)\big)dt \\ &= \int_{\mathbb{R}} \big(W_q(t,q(t)+h(t)v(t)) - W_q(t,q(t)), v(t)\big)dt \to 0 \end{aligned} \tag{3.9}$$

as $v \to 0$. Combining (3.8) and (3.9), we see that (3.7) holds. It remains to prove that I_2' is continuous. Suppose that $q \to q_0$ in E and note that

$$I_2'(q)v - I_2'(q_0)v = \int_{\mathbb{R}} \big(W_q(t,q(t)) - W_q(t,q_0(t)), v(t)\big)dt.$$

By Lemma 2.2 and the Hölder inequality, we obtain

$$I_2'(q)v - I_2'(q_0)v \to 0 \quad \text{as } q \to q_0,$$

which implies the continuity of I_2' and we show that $I \in C^1(E,\mathbb{R})$.

Lastly, we check that critical points of I are classical solutions of (1.1) satisfying $q(t) \to 0$ and $\dot q(t) \to 0$ as $|t| \to +\infty$. It is well known that $E \subset C^0(\mathbb{R},\mathbb{R}^n)$ (the space of continuous functions q on $\mathbb{R}$ such that $q(t) \to 0$ as $|t| \to +\infty$). On the other hand, if q is a critical point of I, for any $v \in E \subset C^0(\mathbb{R},\mathbb{R}^n)$, by (3.2) we have

$$\begin{aligned} \int_{\mathbb{R}} \Big[\big(\dot q(t), \dot v(t)\big) - \big(A\dot q(t), v(t)\big)\Big]dt &= \int_{\mathbb{R}} \big(\dot q(t) + Aq(t), \dot v(t)\big)dt \\ &= -\int_{\mathbb{R}} \big(L(t)q(t) - W_q(t,q(t)), v(t)\big)dt, \end{aligned}$$

which implies that $L(t)q - W_q(t,q)$ is the weak derivative of $\dot q + Aq$. Since $L \in C(\mathbb{R},\mathbb{R}^{n^2})$, $W \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and $E \subset C^0(\mathbb{R},\mathbb{R}^n)$, we see that $\dot q + Aq$ is continuous, which yields that $\dot q$ is continuous and $q \in C^2(\mathbb{R},\mathbb{R}^n)$; i.e., q is a classical solution of (1.1). Moreover, it is easy to check that q satisfies $\dot q(t) \to 0$ as $|t| \to +\infty$ since $\dot q$ is continuous. $\square$

Lemma 3.2. *Under Assumption* (H1)-(H5)*, I satisfies the (PS) condition.*

Proof. Assume that $\{u_j\}_{j\in\mathbb{N}} \subset E$ is a sequence such that $\{I(u_j)\}_{j\in\mathbb{N}}$ is bounded and $I'(u_j) \to 0$ as $j \to +\infty$. Then there exists a constant $C_1 > 0$ such that

$$|I(u_j)| \le C_1, \quad \|I'(u_j)\|_{E^*} \le C_1 \tag{3.10}$$

for every $j \in \mathbb{N}$.

We firstly prove that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in E. By (3.1), (3.3), (H2) and the Hölder inequality, we have

$$\begin{aligned}
\big(\frac{\mu}{2}-1\big)\|u_j\|^2 &= \mu I(u_j) - I'(u_j)u_j \\
&\quad + \int_{\mathbb{R}} \big(\mu W(t,u_j(t)) - \big(W_q(t,u_j(t)),u_j(t)\big)\big)dt \\
&\quad - \big(\frac{\mu}{2}-1\big)\int_{\mathbb{R}} \big(Au_j(t),\dot u_j(t)\big)dt \\
&\le \mu I(u_j) - I'(u_j)u_j + \big(\frac{\mu}{2}-1\big)\frac{\|A\|}{\sqrt{\beta}}\|u_j\|^2.
\end{aligned} \tag{3.11}$$

Combining this inequality with (3.10), we obtain

$$\big(\frac{\mu}{2}-1\big)\big(1-\frac{\|A\|}{\sqrt{\beta}}\big)\|u_j\|^2 \le \mu I(u_j) - I'(u_j)u_j \le \mu C_1 + C_1\|u_j\|. \tag{3.12}$$

Since $\mu > 2$ and $\|A\| < \sqrt{\beta}$, the inequality (3.12) shows that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in E. By Lemma 2.1, the sequence $\{u_j\}_{j\in\mathbb{N}}$ has a subsequence, again denoted by $\{u_j\}_{j\in\mathbb{N}}$, and there exists $u \in E$ such that

$$\begin{gathered}
u_j \rightharpoonup u, \quad \text{weakly in } E, \\
u_j \to u, \quad \text{strongly in } L^2(\mathbb{R},\mathbb{R}^n).
\end{gathered}$$

Hence

$$\big(I'(u_j) - I'(u)\big)(u_j - u) \to 0,$$

and by Lemma 2.2 and the Hölder inequality, we have

$$\int_{\mathbb{R}} \Big(W_q(t,u_j(t)) - W_q(t,u(t)), u_j(t) - u(t)\Big)dt \to 0,$$

and

$$\big|\int_{\mathbb{R}} \big(A\dot u_j(t) - A\dot u(t), u_j(t) - u(t)\big)dt\big| \le \|A\|\|\dot u_j - \dot u\|\|u_j - u\|_2 \to 0$$

as $j \to +\infty$. On the other hand, an easy computation shows that

$$\begin{aligned}
&\big(I'(u_j) - I'(u), u_j - u\big) \\
&= \|u_j - u\|^2 - \int_{\mathbb{R}} \big(A\dot u_j(t) - A\dot u(t), u_j(t) - u(t)\big)dt \\
&\quad - \int_{\mathbb{R}} \Big(W_q(t,u_j(t)) - W_q(t,u(t)), u_j(t) - u(t)\Big)dt.
\end{aligned}$$

Consequently, $\|u_j - u\| \to 0$ as $j \to +\infty$. □

Now we can give the proof of Theorem 1.2, we divide the proof into several steps.

Proof of Theorem 1.2.
Step 1 It is clear that $I(0)=0$ by Remark 1.1 and $I\in C^1(E,\mathbb{R})$ satisfies the (PS) condition by Lemmas 3.1 and 3.2.

Step 2 We now show that there exist constants $\rho>0$ and $\alpha>0$ such that I satisfies the condition (A1) of Lemma 2.6. By (1.3), for all $\varepsilon>0$, there exists $\delta>0$ such that $W(t,q)\le\varepsilon|q|^2$ whenever $|q|\le\delta$. Letting $\rho=\frac{\delta}{C}$ and $\|q\|=\rho$, we have $\|q\|_\infty\le\delta$, where $C>0$ is defined in (2.1). Hence $W(t,q(t))\le\varepsilon|q(t)|^2$ for all $t\in\mathbb{R}$. Integrating on $\mathbb{R}$, we get

$$\int_{\mathbb{R}} W(t,q(t))dt\le\varepsilon\|q\|_2^2\le\frac{\varepsilon}{\beta}\|q\|^2.$$

In consequence, combining this with (3.1), we obtain that, for $\|q\|=\rho$,

$$\begin{aligned}I(q)&=\frac{1}{2}\|q\|^2+\frac{1}{2}\int_{\mathbb{R}}\big(Aq(t),\dot q(t)\big)dt-\int_{\mathbb{R}}W(t,q(t))dt\\&\ge\frac{1}{2}\|q\|^2-\frac{1}{2}\frac{\|A\|}{\sqrt{\beta}}\|q\|^2-\frac{\varepsilon}{\beta}\|q\|^2\\&=\big(\frac{1}{2}-\frac{1}{2}\frac{\|A\|}{\sqrt{\beta}}-\frac{\varepsilon}{\beta}\big)\|q\|^2.\end{aligned}\tag{3.13}$$

Setting $\varepsilon=\frac{1}{4\beta}(1-\frac{\|A\|}{\sqrt{\beta}})$, the inequality (3.13) implies

$$I|_{\partial B_\rho}\ge\frac{1}{4}\big(1-\frac{\|A\|}{\sqrt{\beta}}\big)\frac{\delta^2}{C^2}=\alpha>0.$$

Step 3 It remains to prove that there exists $e\in E$ such that $\|e\|>\rho$ and $I(e)\le 0$, where ρ is defined Step 2. By (3.1), we have, for every $m\in\mathbb{R}\setminus\{0\}$ and $q\in E\setminus\{0\}$,

$$\begin{aligned}I(m\,q)&=\frac{m^2}{2}\|q\|^2+\frac{m^2}{2}\int_{\mathbb{R}}\big(Aq(t),\dot q(t)\big)dt-\int_{\mathbb{R}}W(t,m\,q(t))dt\\&\le\frac{m^2}{2}\big(1+\frac{\|A\|}{\sqrt{\beta}}\big)-\int_{\mathbb{R}}W(t,m\,q(t))dt.\end{aligned}$$

Take some $Q\in E$ such that $\|Q\|=1$. Then there exists a subset Ω of positive measure of $\mathbb{R}$ such that $Q(t)\neq 0$ for $t\in\Omega$. Take $m>0$ such that $m|Q(t)|\ge 1$ for $t\in\Omega$. Then, by (H5) and Remark 2.4, we obtain that

$$I(m\,Q)\le\frac{m^2}{2}\big(1+\frac{\|A\|}{\sqrt{\beta}}\big)-m^\mu\int_\Omega\alpha_0(t)|Q(t)|^\mu dt.\tag{3.14}$$

Since $\alpha_0(t)>0$ and $\mu>2$, (3.14) implies that $I(mQ)<0$ for some $m>0$ such that $m|Q(t)|\ge 1$ for $t\in\Omega$ and $\|mQ\|>\rho$, where ρ is defined in Step 2. By Lemma 2.6, I possesses a critical value $c\ge\alpha>0$ given by

$$c=\inf_{g\in\Gamma}\max_{s\in[0,1]}I(g(s)),$$

where

$$\Gamma=\{g\in C([0,1],E):g(0)=0,\,g(1)=e\}.$$

Hence there is $q\in E$ such that $I(q)=c$, $I'(q)=0$.

Step 4 Now suppose that $W(t,q)$ is even in q; i.e., (H6) holds, which implies that I is even. Furthermore, we already know that $I(0)=0$ and $I\in C^1(E,\mathbb{R})$ satisfies the (PS) condition in Step 1.

To apply Lemma 2.7, it suffices to prove that I satisfies the conditions (A3) and (A4) of Lemma 2.7. Here we take $V = \{0\}$ and $X = E$. (A3) is identically the same as in Step 2, so it is already proved. Now we prove that (A4) holds. Let $\tilde{E} \subset E$ be a finite dimensional subspace. From Step 3 we know that, for any $Q \in \tilde{E} \subset E$ such that $\|Q\| = 1$, there is $m_Q > 0$ such that

$$I(mQ) < 0 \quad \text{for every } |m| \geq m_Q > 0.$$

Since $\tilde{E} \subset E$ is a finite dimensional subspace, we can choose an $R = R(\tilde{E}) > 0$ such that

$$I(q) < 0, \quad \forall q \in \tilde{E} \backslash B_R.$$

Hence, by Lemma 2.7, I possesses an unbounded sequence of critical values $\{c_j\}_{j\in\mathbb{N}}$ with $c_j \to +\infty$. Let q_j be the critical point of I corresponding to c_j, then (1.1) has infinitely many distinct homoclinic solutions.

References

[1] C. O. Alves, P. C . Carrião and O. H. Miyagaki; *Existence of homoclinic orbits for asymptotically periodic systems involving Duffing-like equation*, Appl. Math. Lett., **16** (2003), no. 5, 639-642.

[2] A. Ambrosetti and V. Coti Zelati; *Multiple homoclinic orbits for a class of conservative systems*, Rend. Sem. Mat. Univ. Padova., **89** (1993), 177-194.

[3] A. Ambrosetti and P. H. Rabinowitz; *Dual variational methods in critical point theory and applications*, J. Funct. Anal., **14** (1973), no. 4, 349-381.

[4] P. Caldiroli and P. Montecchiari; *Homoclinic orbits for second order Hamiltonian systems with potential changing sign*, Comm. Appl. Nonlinear Anal., **1** (1994), no. 2, 97-129.

[5] P. C. Carrião and O. H. Miyagaki; *Existence of homoclinic solutions for a class of time-dependent Hamiltonian systems*, J. Math. Anal. Appl., **230** (1999), no. 1, 157-172.

[6] V. Coti Zelati and P. H. Rabinowitz; *Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials*, J. Amer. Math. Soc., **4** (1991), no. 4, 693-727.

[7] Y. Ding; *Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems*, Nonlinear Anal., **25** (1995), no. 11, 1095-1113.

[8] Y. Ding and M. Girardi; *Periodic and homoclinic solutions to a class of Hamiltonian systems with the potentials changing sign*, Dynam. Systems Appl., **2** (1993), no. 1, 131-145.

[9] A. Flavia; *Periodic and homoclinic solutions to a class of Hamiltonian systems with indefinite potential in sign*, Boll. Un. Mat. Ital., B (7) **10** (1996), no. 2, 303-324.

[10] M. Izydorek and J. Janczewska; *Homoclinic solutions for a class of the second order Hamiltonian systems*, J. Differential Equations, **219** (2005), no. 2, 375-389.

[11] P. Korman and A. C. Lazer; *Homoclinic orbits for a class of symmetric Hamiltonian systems*, Electron. J. Differential Equations, (1994), no. 01, 1-10.

[12] Y. Lv and C. Tang; *Existence of even homoclinic orbits for a class of Hamiltonian systems*, Nonlinear Anal., **67** (2007), no. 7, 2189-2198.

[13] W. Omana and M. Willem; *Homoclinic orbits for a class of Hamiltonian systems*, Differential Integral Equations, **5** (1992), no. 5, 1115-1120.

[14] Z. Ou and C. Tang; *Existence of homoclinic solution for the second order Hamiltonian systems*; J. Math. Anal. Appl., **291** (2004), no. 1, 203-213.

[15] E. Paturel; *Multiple homoclinic orbits for a class of Hamiltonian systems*, Calc. Val. Partial Differential Equations, **12** (2001), no. 2, 117-143.

[16] P. H. Rabinowitz; *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, in: CBMS Reg. Conf. Ser. in. Math., vol. 65, American Mathematical Society, Providence, RI, 1986.

[17] P. H. Rabinowitz; *Homoclinic orbits for a class of Hamiltonian systems*, Proc. Roy. Soc. Edinburgh Sect. A., **114** (1990), no. 1-2, 33-38.

[18] P. H. Rabinowitz and K. Tanaka; *Some results on connecting orbits for a class of Hamiltonian systems*, Math. Z., **206** (1991), no. 3, 473-499.

Rong Yuan
School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China
E-mail address: ryuan@bnu.edu.cn

Ziheng Zhang
School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China
E-mail address: zhzh@mail.bnu.edu.cn