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UNBOUNDED UPPER AND LOWER SOLUTION METHOD FOR THIRD-ORDER BOUNDARY-VALUE PROBLEMS ON THE HALF-LINE

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Abstract. In this article, we prove the existence of unbounded upper and lower solutions of third-order boundary-value problems on the half-line. Here the Nagumo conditions play an important role in the nonlinear term involved in the second-order derivatives.

1. Introduction

Boundary-value problems on the half-line arise naturally in the study of radially symmetric solutions of nonlinear elliptic equations and various physical phenomena, see [2, 3, 4, 5, 13, 14], such as the theory of drain flows, plasma physics, unsteady flow of gas through a semi-infinite porous media, in determining the electrical potential in an isolated neutral atom.

Recently an increasing interest in studying the existence of solutions and positive solutions to boundary-value problems for second-order differential equations on the half-line is observed; see for example [6, 9, 10, 13, 16, 17, 19, 20, 21]. However, to the best knowledge of the authors, no work has been done for the third-order boundary-value problems on the half-line. It is well known that the study of third-order boundary-value problems is very important. For finite interval, there are many results, see [8, 11, 12]. So it is necessary to discuss the existence of the three-order boundary-value problems on the half-line.

In this paper, we are concerned with the existence of solutions for the following boundary-value problem on the half-line for the third-order differential equation

$$\begin{gathered} u'''(t) + a(t)f(t, u(t), u'(t), u''(t)) = 0, \quad t \in (0, +\infty), \\ u(0) = u'(0) = 0, \quad \lim_{t\to+\infty} u''(t) =: u''(+\infty) = 0, \end{gathered} \tag{1.1}$$

where $a : (0, +\infty) \to (0, +\infty)$, $f : [0 + \infty) \times \mathbb{R}^3 \to \mathbb{R}$ are continuous. By using the upper and lower solutions method, the authors present sufficient conditions for the existence of unbounded solutions to (1.1).

This paper is organized as follows. In section 2, some definitions and lemmas are given. We establish an upper and lower solution theory for (1.1) in section 3.

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Sufficient conditions are given for the existence of solutions. In section 4, we give an example to demonstrate our main results.

2. Preliminaries

In this section, we introduce some necessary definitions and preliminary results that will be used to prove our main results. Let

$$E = \Big\{x \in C^2[0,+\infty) : \sup_{0\le t<+\infty} \frac{|x(t)|}{(1+t)^2} < +\infty, \sup_{0\le t<+\infty} |x'(t)| < +\infty, \lim_{t\to+\infty} x''(t) \text{ exists } \Big\},$$

with the norm $\|x\| = \max\{\|x\|_1, \|x'\|_\infty, \|x''\|_\infty\}$, where $\|x\|_1 = \sup_{t\in[0,+\infty)} |\frac{x(t)}{(1+t)^2}|$, $\|x'\|_\infty = \sup_{t\in[0,+\infty)} |x'(t)|$, $\|x''\|_\infty = \sup_{t\in[0,+\infty)} |x''(t)|$. By standard arguments, we can prove that $(E, \|\cdot\|)$ is a Banach space.

Definition 2.1. A function $\alpha \in E \cap C^3(0,+\infty)$ is called a lower solution of (1.1) if

$$\begin{gathered} \alpha'''(t) + a(t)f(t,\alpha(t),\alpha'(t),\alpha''(t)) \ge 0, \quad t \in (0,+\infty), \\ \alpha(0) \le 0, \quad \alpha'(0) \le 0, \quad \alpha''(+\infty) \le 0. \end{gathered}$$

Similarly we define an upper solution $\beta \in E \cap C^3(0,+\infty)$ of (1.1) by reversing the above inequalities.

Remark 2.2. If

$$\alpha'(t) \le \beta'(t), \quad \text{for every } t \in [0,+\infty), \tag{2.1}$$

then by integrating (2.1) and using the boundary conditions of Definition 2.1, we can easily obtain that $\alpha(t) \le \beta(t)$ for all $t \in [0,+\infty)$.

Definition 2.3. Given a pair of upper and lower solutions $\beta, \alpha \in E \cap C^3(0,+\infty)$ of (1.1) satisfying $\alpha'(t) \le \beta'(t)$, $t \in [0,+\infty)$. A continuous function $f : [0,+\infty)\times\mathbb{R}^3 \to \mathbb{R}$ is said to satisfy the Nagumo condition with respect to the pair of functions α, β, if there exists a nonnegative function $\phi \in C[0,+\infty)$ and a positive one $h \in C[0,+\infty)$ such that

$$|f(t,x,y,z)| \le \phi(t)h(|z|), \tag{2.2}$$

for all $0 \le t < +\infty$, $\alpha(t) \le x \le \beta(t)$, $\alpha'(t) \le y \le \beta'(t)$, $z \in \mathbb{R}$ and

$$\int_0^{+\infty} \frac{s}{h(s)} ds = +\infty. \tag{2.3}$$

The above Nagumo conditions provide a priori estimate for the second-order derivative u'' of a class of the solutions of problem (1.1).

Now we consider the following boundary-value problem for third-order differential equation on the half-line:

$$\begin{gathered} u'''(t) + \sigma(t) = 0, \quad t \in (0,+\infty), \\ u(0) = u'(0) = 0, \quad u''(+\infty) = 0, \end{gathered} \tag{2.4}$$

where $\sigma \in C[0,+\infty)$.

Lemma 2.4. *Let $\sigma \in C[0,+\infty)$ and $\int_0^\infty \sigma(t)dt < \infty$. Then $u \in C^2[0,+\infty) \cap C^3(0,+\infty)$ is a solution of (2.4) if and only if u is a solution of the following integral equation:*

$$u(t) = \int_0^\infty G(t,s)\sigma(s)ds, \quad t \in [0,+\infty), \tag{2.5}$$

where

$$G(t,s) = \begin{cases} \frac{1}{2}s(2t-s), & 0 \le s \le t, \\ \frac{1}{2}t^2, & t \le s < \infty. \end{cases}$$

Proof. It is easy to show that the general solution for the equation in boundary-value problem (2.4) is

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2\sigma(s)ds + At^2 + Bt + C, \quad t \in [0,+\infty), \tag{2.6}$$

where A, B, C are constants. By the boundary condition of (2.4), we get $A = \frac{1}{2}\int_0^\infty \sigma(s)ds$, $B = C = 0$. Substituting the expressions of A, B and C into (2.6), we know that (2.5) holds. $\square$

Let $C_l := \{y \in C[0,+\infty) : \lim_{t\to+\infty} y(t) \text{ exists }\}$. For $y \in C_l$, define $\|y\| := \sup_{t\in[0,+\infty)} |y(t)|$. Then C_l is a Banach space (see [1]).

Lemma 2.5 ([7, 18])**.** *Let $M \subset C_l$. Then M is relatively compact if the following conditions hold:*

(a) *M is bounded in C_l;*
(b) *the functions belonging to M are locally equicontinuous on $[0,+\infty)$;*
(c) *the functions from M are equiconvergent; that is, given $\epsilon > 0$, there corresponds $T(\epsilon) > 0$ such that $|x(t) - x(+\infty)| < \epsilon$ for all $t > T(\epsilon)$ and $x \in M$.*

By Lemma 2.5, similar to the proof of [17, Theorem 2.2], we easily obtain the following result.

Lemma 2.6. *Let $M \subset E$. Then M is relatively compact if the following conditions hold:*

(i) *M is bounded in E;*
(ii) *the functions belonging to $\{y : y = \frac{x}{(1+t)^2}, x \in M\}$, $\{z : z = x'(t), x \in M\}$, and $\{w : w = x''(t), x \in M\}$ are locally equicontinuous on $[0,+\infty)$;*
(iii) *the functions from $\{y : y = \frac{x}{(1+t)^2}, x \in M\}$, $\{z : z = x'(t), x \in M\}$, and $\{w : w = x''(t), x \in M\}$ are equiconvergent at $+\infty$.*

3. Main result

In this section, we study the existence of solution to (1.1).

Theorem 3.1. *Assume that there are $\alpha, \beta \in E \cap C^3(0,+\infty)$ lower and upper solutions of (1.1), respectively, such that (2.1) holds. Let $f : [0,+\infty) \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function satisfying the Nagumo condition with respect to the pair of functions α, β, and verifying*

$$f(t,\alpha(t),y,z) \le f(t,x,y,z) \le f(t,\beta(t),y,z), \tag{3.1}$$

for $(t,x,y,z)\in[0,+\infty)\times[\alpha(t),\beta(t)]\times\mathbb{R}^2$. *If*

$$\int_0^{+\infty}\max\{s,1\}a(s)ds<+\infty,\quad \int_0^{+\infty}\max\{s,1\}a(s)\phi(s)ds<+\infty, \tag{3.2}$$

then (1.1) *has at least one solution* $u\in E\cap C^3(0,+\infty)$ *satisfying*

$$\alpha(t)\le u(t)\le\beta(t),\quad \alpha'(t)\le u'(t)\le\beta'(t),\quad |u''(t)|\le N\quad \textit{for all } t\in[0,+\infty),$$

where N *is a constant dependent only on* α,β,a *and* ϕ.

Proof. Define the auxiliary functions

$$\omega_0(t,x)=\begin{cases}\alpha(t), & x<\alpha(t),\\ x(t), & \alpha(t)\le x\le\beta(t),\\ \beta(t), & x>\beta(t);\end{cases}\quad \omega_1(t,y)=\begin{cases}\alpha'(t), & y<\alpha'(t),\\ y(t), & \alpha'(t)\le y\le\beta'(t),\\ \beta'(t), & y>\beta'(t).\end{cases}$$

Consider the boundary-value problem

$$\begin{gathered}u'''(t)+a(t)f^*(t,u(t),u'(t),u''(t))=0,\quad t\in(0,+\infty),\\ u(0)=u'(0)=0,\quad \lim_{t\to+\infty}u''(t)=0,\end{gathered} \tag{3.3}$$

where

$$f^*(t,x,y,z)=f(t,\omega_0(t,x),\omega_1(t,y),z)+\frac{\omega_1(t,y)-y}{1+|\omega_1(t,y)-y|}. \tag{3.4}$$

For each $u\in E$, we have by (2.2), (3.2) and (3.4) that

$$\begin{aligned}&\Big|\int_0^{\infty}a(s)f^*(s,u(s),u'(s),u''(s))ds\Big|\\ &\le\int_0^{\infty}a(s)[\phi(s)h(|u''(s)|)+1]ds\\ &\le\int_0^{\infty}a(s)(H_0\phi(s)+1)ds\\ &\le\int_0^{\infty}\max\{s,1\}a(s)(H_0\phi(s)+1)ds<+\infty,\end{aligned} \tag{3.5}$$

where $H_0=\max_{0\le t\le\|u''\|_\infty}h(t)$. From (3.5) and Lemma 2.4, we know that u is a solution of (3.3) if and only if u solves the operator equation $u=Tu$. Here, the operator T is defined by

$$(Tu)(t)=\int_0^{\infty}G(t,s)a(s)f^*(s,u(s),u'(s),u''(s))ds,\quad u\in E,\ t\in[0,+\infty). \tag{3.6}$$

We claim that $T:E\to E$ is completely continuous.

Step 1: $T:E\to E$ is well defined. For $u\in E$, we get by (3.5) that

$$\int_1^{\infty}sa(s)(H_0\phi(s)+1)ds\le\int_0^{\infty}\max\{s,1\}a(s)(H_0\phi(s)+1)ds<+\infty, \tag{3.7}$$

which implies

$$\lim_{t\to+\infty}ta(t)(H_0\phi(t)+1)=0. \tag{3.8}$$

Since

$$\int_t^{\infty}a(s)(H_0\phi(s)+1)ds\le\int_t^{\infty}sa(s)(H_0\phi(s)+1)ds,\quad t\ge1, \tag{3.9}$$

by (3.7) and (3.9), we have

$$\lim_{t\to\infty}\int_t^{\infty} a(s)(H_0\phi(s)+1)ds = 0. \tag{3.10}$$

By the Lebesgue dominated convergence theorem, (3.8) and (3.10), we obtain

$$\begin{aligned}
&\lim_{t\to+\infty}\frac{|(Tu)(t)|}{(1+t)^2}\\
&\le \lim_{t\to+\infty}\int_0^{\infty}\frac{G(t,s)}{(1+t)^2}a(s)(H_0\phi(s)+1)ds\\
&= \lim_{t\to+\infty}\Big[\int_0^{t}\frac{\frac12 s(2t-s)}{(1+t)^2}a(s)(H_0\phi(s)+1)ds+\int_t^{\infty}\frac{\frac12 t^2}{(1+t)^2}a(s)(H_0\phi(s)+1)ds\Big]\\
&= \lim_{t\to+\infty}\frac{\int_0^t sa(s)(H_0\phi(s)+1)ds+\frac{t^2}{2}a(t)(H_0\phi(t)+1)}{2(1+t)}\\
&\quad+\lim_{t\to+\infty}\frac{t\int_t^{\infty}a(s)(H_0\phi(s)+1)ds-\frac{t^2}{2}a(t)(H_0\phi(t)+1)}{2(1+t)} \quad \text{(L'Hopital's rule)}\\
&= \lim_{t\to+\infty}\frac12 ta(t)(H_0\phi(t)+1)+\frac14\lim_{t\to+\infty}\frac{t}{1+t}ta(t)(H_0\phi(t)+1)\\
&\quad+\lim_{t\to+\infty}\frac12\Big[\int_t^{\infty}a(s)(H_0\phi(s)+1)ds+ta(t)(H_0\phi(t)+1)\Big] \quad \text{(L'Hopital's rule)}\\
&\quad-\frac14\lim_{t\to+\infty}\frac{t}{1+t}ta(t)(H_0\phi(t)+1)\\
&= \frac12\lim_{t\to+\infty}\int_t^{\infty}a(s)(H_0\phi(s)+1)ds = 0;
\end{aligned}$$

that is,

$$\lim_{t\to+\infty}\frac{(Tu)(t)}{(1+t)^2} = 0, \tag{3.11}$$

which implies

$$\sup_{0\le t<+\infty}\frac{|(Tu)(t)|}{(1+t)^2} < +\infty.$$

By (3.5), we get

$$\begin{aligned}
\sup_{0\le t<+\infty}|(Tu)'(t)| &= \sup_{0\le t<+\infty}\Big|\int_0^{\infty}\frac{\partial G(t,s)}{\partial t}a(s)f^*(s,u(s),u'(s),u''(s))ds\Big|\\
&= \sup_{0\le t<+\infty}\Big|\int_0^{t}sa(s)f^*(s,u(s),u'(s),u''(s))ds\\
&\quad+\int_t^{\infty}ta(s)f^*(s,u(s),u'(s),u''(s))ds\Big|\\
&\le \sup_{0\le t<+\infty}\Big[\int_0^{t}sa(s)(H_0\phi(s)+1)ds+\int_t^{\infty}ta(s)(H_0\phi(s)+1)ds\Big]\\
&\le \int_0^{\infty}sa(s)(H_0\phi(s)+1)ds\\
&\le \int_0^{\infty}\max\{s,1\}a(s)(H_0\phi(s)+1)ds < +\infty.
\end{aligned}$$

From (3.10), we have

$$\Big|\int_t^\infty a(s)f^*(s,u(s),u'(s),u''(s))ds\Big| \le \int_t^\infty a(s)(H_0\phi(s)+1)ds \to 0, \quad t\to+\infty.$$

Therefore,

$$\lim_{t\to+\infty}(Tu)''(t) = \lim_{t\to+\infty}\int_t^\infty a(s)f^*(s,u(s),u'(s),u''(s))ds = 0. \tag{3.12}$$

So $Tu\in E$.

Step 2: $T: E\to E$ is continuous. For any convergent sequence $u_n\to u$ in E, we have

$$u_n(t)\to u(t),\quad u_n'(t)\to u'(t),\quad u_n''(t)\to u''(t),\quad n\to+\infty,\ t\in[0,+\infty).$$

Now the continuity of f^* implies

$$|f^*(s,u_n(s),u_n'(s),u_n''(s)) - f^*(s,u(s),u'(s),u''(s))| \to 0, \quad n\to+\infty,\ \forall t\in[0,+\infty).$$

Since $u_n\to u$, we have $\sup_{n\in N}\|u_n''\|_\infty < +\infty$. Let

$$H_p = \max_{0\le t\le \max\{\|u''\|_\infty,\ \sup_{n\in N}\|u_n''\|_\infty\}} h(t).$$

Then

$$\begin{aligned}
&\int_0^\infty sa(s)|f^*(s,u_n(s),u_n'(s),u_n''(s)) - f^*(s,u(s),u'(s),u''(s))|ds \\
&\le 2\int_0^\infty sa(s)(H_p\phi(s)+1)ds < +\infty.
\end{aligned} \tag{3.13}$$

Hence, from the Lebesgue dominated convergence theorem and (3.13), we have

$$\begin{aligned}
&\|Tu_n - Tu\|_1 \\
&= \sup_{t\in\mathbb{R}^+}\frac{|(Tu_n)(t)-(Tu)(t)|}{(1+t)^2} \\
&= \sup_{t\in\mathbb{R}^+}\Big|\int_0^\infty \frac{G(t,s)}{(1+t)^2}a(s)(f^*(s,u_n(s),u_n'(s),u_n''(s)) - f^*(s,u(s),u'(s),u''(s))ds\Big| \\
&\le \sup_{t\in\mathbb{R}^+}\Big[\int_0^t \frac{\frac12 s(2t-s)}{(1+t)^2}a(s)|f^*(s,u_n(s),u_n'(s),u_n''(s)) - f^*(s,u(s),u'(s),u''(s))|ds \\
&\quad + \int_t^\infty \frac{\frac12 t^2}{(1+t)^2}a(s)|f^*(s,u_n(s),u_n'(s),u_n''(s)) - f^*(s,u(s),u'(s),u''(s))|ds\Big] \\
&\le \sup_{t\in\mathbb{R}^+}\Big[\int_0^t sa(s)|f^*(s,u_n(s),u_n'(s),u_n''(s)) - f^*(s,u(s),u'(s),u''(s))|ds \\
&\quad + \int_t^\infty \frac12 sa(s)|f^*(s,u_n(s),u_n'(s),u_n''(s)) - f^*(s,u(s),u'(s),u''(s))|ds\Big] \\
&\le \int_0^\infty sa(s)|f^*(s,u_n(s),u_n'(s),u_n''(s)) - f^*(s,u(s),u'(s),u''(s))|ds
\end{aligned} \tag{3.14}$$

which approaches zero as as $n \to \infty$. Also

$$\begin{aligned}
&\|(Tu_n)' - (Tu)'\|_\infty \\
&= \sup_{t\in\mathbb{R}^+} |(Tu_n)'(t) - (Tu)'(t)| \\
&= \sup_{t\in\mathbb{R}^+} \Big| \int_0^t sa(s)(f^*(s,u_n(s),u_n'(s),u_n''(s)) - f^*(s,u(s),u'(s),u''(s)))ds \\
&\quad + \int_t^\infty ta(s)(f^*(s,u_n(s),u_n'(s),u_n''(s)) - f^*(s,u(s),u'(s),u''(s)))ds\Big| \\
&\le \sup_{t\in\mathbb{R}^+} \Big[\int_0^t sa(s)|f^*(s,u_n(s),u_n'(s),u_n''(s)) - f^*(s,u(s),u'(s),u''(s))|ds \\
&\quad + \int_t^\infty sa(s)|f^*(s,u_n(s),u_n'(s),u_n''(s)) - f^*(s,u(s),u'(s),u''(s))|ds\Big] \\
&= \int_0^\infty sa(s)|f^*(s,u_n(s),u_n'(s),u_n''(s)) - f^*(s,u(s),u'(s),u''(s))|ds
\end{aligned} \tag{3.15}$$

which approaches zero as $n \to \infty$. From (3.13), we easily show that

$$\int_0^\infty a(s)|f^*(s,u_n(s),u_n'(s),u_n''(s)) - f^*(s,u(s),u'(s),u''(s))|ds < +\infty. \tag{3.16}$$

From the above inequality, we obtain

$$\begin{aligned}
&\|(Tu_n)'' - (Tu)''\|_\infty \\
&= \sup_{t\in\mathbb{R}^+} |(Tu_n)''(t) - (Tu)''(t)| \\
&= \sup_{t\in\mathbb{R}^+} \Big| \int_t^\infty a(s)(f^*(s,u_n(s),u_n'(s),u_n''(s)) - f^*(s,u(s),u'(s),u''(s)))ds\Big| \\
&\le \int_0^\infty a(s)|f^*(s,u_n(s),u_n'(s),u_n''(s)) - f^*(s,u(s),u'(s),u''(s))|ds
\end{aligned} \tag{3.17}$$

which approaches zero as $n \to \infty$. Therefore, by (3.14), (3.15) and (3.17), it follows that $\|Tu_n - Tu\| \to 0$, as $n \to +\infty$; so $T : E \to E$ is continuous.

Step 3: $T : E \to E$ is compact. Let A be any bounded subset of E, then for $u \in A$, let $H_q = \sup_{0\le t\le \|u''\|_\infty, u\in A} h(t) < +\infty$, similar to the proof of (3.14), (3.15) and (3.17), by (2.2) and (3.2) one has

$$\begin{aligned}
\|Tu\|_1 &= \sup_{t\in\mathbb{R}^+} \frac{|(Tu)(t)|}{(1+t)^2} \\
&\le \sup_{t\in\mathbb{R}^+} \int_0^\infty \frac{G(t,s)}{(1+t)^2} a(s)|f^*(s,u(s),u'(s),u''(s))|ds \\
&\le \int_0^\infty sa(s)|f^*(s,u(s),u'(s),u''(s))|ds \\
&\le \int_0^\infty sa(s)(H_q\phi(s)+1)ds < +\infty,
\end{aligned}$$

$$\begin{aligned}
\|(Tu)'\|_\infty &\le \int_0^\infty sa(s)|f^*(s,u(s),u'(s),u''(s))|ds \\
&\le \int_0^\infty sa(s)(H_q\phi(s)+1)ds < +\infty,
\end{aligned}$$

and

$$\begin{aligned}\|(Tu)''\|_\infty &\le \int_0^\infty a(s)|f^*(s,u(s),u'(s),u''(s))|ds\\ &\le \int_0^\infty a(s)(H_q\phi(s)+1)ds < +\infty,\end{aligned}$$

which implies that $\|Tu\| < +\infty$. Thus TA is uniformly bounded. Meanwhile, for any $B > 0$, if $t_1, t_2 \in [0, B]$, we have

$$\begin{aligned}&\Big|\frac{(Tu)(t_1)}{(1+t_1)^2} - \frac{(Tu)(t_2)}{(1+t_2)^2}\Big|\\ &= \Big|\int_0^\infty \Big(\frac{G(t_1,s)}{(1+t_1)^2} - \frac{G(t_2,s)}{(1+t_2)^2}\Big)a(s)f^*(s,u(s),u'(s),u''(s))ds\Big|\\ &\le \int_0^\infty \Big|\frac{G(t_1,s)}{(1+t_1)^2} - \frac{G(t_2,s)}{(1+t_2)^2}\Big|a(s)(H_q\phi(s)+1)ds\end{aligned}$$

which approaches zero as $t_1 \to t_2$. Also

$$\begin{aligned}&|(Tu)'(t_1) - (Tu)'(t_2)|\\ &= \Big|\int_0^{t_1} sa(s)f^*(s,u(s),u'(s),u''(s))ds + \int_{t_1}^\infty t_1a(s)f^*(s,u(s),u'(s),u''(s))ds\\ &- \int_0^{t_2} sa(s)f^*(s,u(s),u'(s),u''(s))ds - \int_{t_2}^\infty t_2a(s)f^*(s,u(s),u'(s),u''(s))ds\Big|\\ &\le \Big|\int_{t_1}^{t_2} sa(s)f^*(s,u(s),u'(s),u''(s))ds\Big| + \Big|t_1\int_{t_1}^{t_2} a(s)f^*(s,u(s),u'(s),u''(s))ds\Big|\\ &\quad + \Big|(t_2-t_1)\int_{t_2}^\infty a(s)f^*(s,u(s),u'(s),u''(s))ds\Big|\\ &\le \int_{t_1}^{t_2} sa(s)(H_q\phi(s)+1)ds + t_1\int_{t_1}^{t_2} a(s)(H_q\phi(s)+1)ds\\ &\quad + |t_2-t_1|\int_{t_2}^\infty a(s)(H_q\phi(s)+1)ds\end{aligned}$$

which approaches zero as $t_1 \to t_2$. Also

$$\begin{aligned}|(Tu)''(t_1) - (Tu)''(t_2)| &= \Big|\int_{t_1}^{t_2} a(s)f^*(s,u(s),u'(s),u''(s))ds\Big|\\ &\le \int_{t_1}^{t_2} a(s)(H_q\phi(s)+1)ds\end{aligned}$$

which approaches zero as $t_1 \to t_2$. As a result, TA is equicontinuous. From (3.11), we get

$$\Big|\frac{(Tu)(t)}{(1+t)^2} - \lim_{t\to+\infty}\frac{(Tu)(t)}{(1+t)^2}\Big| = \Big|\frac{(Tu)(t)}{(1+t)^2}\Big| \to 0, \quad \text{as } t \to +\infty.$$

Since

$$\begin{aligned}\Big|\int_t^\infty ta(s)f^*(s,u(s),u'(s),u''(s))ds\Big| &\le \int_t^\infty ta(s)(H_0\phi(s)+1)ds\\ &\le \int_t^\infty sa(s)(H_0\phi(s)+1)ds\end{aligned}$$

which approaches zero as $t \to +\infty$, we have

$$\begin{aligned}\lim_{t\to+\infty}(Tu)'(t) &= \lim_{t\to+\infty}\Big[\int_0^t sa(s)f^*(s,u(s),u'(s),u''(s)))ds \\ &\quad + \int_t^\infty ta(s)f^*(s,u(s),u'(s),u''(s)))ds\Big] \\ &= \int_0^\infty sa(s)f^*(s,u(s),u'(s),u''(s)))ds.\end{aligned}$$

Thus,

$$\begin{aligned}&|(Tu)'(t) - \lim_{t\to+\infty}(Tu)'(t)| \\ &= \Big|\int_t^\infty ta(s)f^*(s,u(s),u'(s),u''(s)))ds - \int_t^\infty sa(s)f^*(s,u(s),u'(s),u''(s)))ds\Big|\end{aligned}$$

which approaches zero as $t \to +\infty$. Moreover, by (3.12), we have

$$\begin{aligned}\Big|(Tu)''(t) - \lim_{t\to+\infty}(Tu)''(t)\Big| &= |(Tu)''(t)| \\ &= \Big|\int_t^\infty a(s)f^*(s,u(s),u'(s),u''(s)))ds\Big|\end{aligned}$$

which approaches zero as $t \to +\infty$. That is, TA is equiconvergent at infinity. Then TA is relatively compact. Hence, $T : E \to E$ is completely continuous. □

By the Schauder fixed point theorem, we can easily obtain that T has at least one fixed point $u \in E$. Thus u is a solution of (3.3).

Next, we show that u satisfies the inequalities

$$\alpha(t) \le u(t) \le \beta(t), \quad \alpha'(t) \le u'(t) \le \beta'(t), \quad \forall t \in \mathbb{R}^+,$$

which implies that u is a solution of (1.1). First, we show that $u'(t) \le \beta'(t)$ for all $t \in [0, +\infty)$. Suppose not, then

$$\sup_{0\le t<+\infty}(u'(t) - \beta'(t)) > 0.$$

Since $\lim_{t\to+\infty}(u''(t) - \beta''(t)) < 0$, there are two cases.

Case 1. There exists a $t_0 \in (0,\infty)$ such that

$$u'(t_0) - \beta'(t_0) = \sup_{t\in\mathbb{R}^+}(u'(t) - \beta'(t)) > 0.$$

So we have $u''(t_0) = \beta''(t_0)$ and

$$u'''(t_0) \le \beta'''(t_0). \tag{3.18}$$

By (3.1), (3.3) and (3.4), we get

$$\begin{aligned}u'''(t_0) &= -a(t_0)\Big[f(t_0,\omega_0(t_0,u),\omega_1(t_0,u'),u''(t_0)) + \frac{\omega_1(t_0,u') - u'(t_0)}{1+|\omega_1(t_0,u') - u'(t_0)|}\Big] \\ &= -a(t_0)\Big[f(t_0,\omega_0(t_0,u(t_0)),\beta'(t_0)),\beta''(t_0)) + \frac{\beta'(t_0) - u'(t_0)}{1+|\beta'(t_0) - u'(t_0)|}\Big] \\ &\ge -a(t_0)f(t_0,\beta(t_0),\beta'(t_0)),\beta''(t_0)) + a(t_0)\frac{u'(t_0) - \beta'(t_0)}{1+|u'(t_0) - \beta'(t_0)|} \\ &> -a(t_0)f(t_0,\beta(t_0),\beta'(t_0)),\beta''(t_0)) \ge \beta'''(t_0),\end{aligned}$$

which is a contradiction.

Case 2. $u'(0)-\beta'(0)=\lim_{t\to 0^+}(u'(t)-\beta'(t))=\sup_{t\in\mathbb{R}^+}(u'(t)-\beta'(t))>0$. By the boundary condition, we have the contradiction $u'(0)-\beta'(0)\le 0$. Consequently, $u'(t)\le\beta'(t)$ holds for all $t\in[0,+\infty)$. Using an analogous technique, we prove that $\alpha'(t)\le u'(t)$, for all $t\in[0,+\infty)$. By integration,

$$\alpha(t)\le u(t)\le\beta(t),\quad \forall t\in[0,+\infty).$$

Let $\sigma>0$ and choose

$$r\ge\max\Big\{\sup_{t\in[\sigma,+\infty)}\frac{\beta'(t)-\alpha'(0)}{t},\ \sup_{t\in[\sigma,+\infty)}\frac{\beta'(0)-\alpha'(t)}{t}\Big\} \tag{3.19}$$

and $N>r$, such that

$$\int_r^N\frac{s}{h(s)}ds\ge m\Big(\sup_{0\le t<+\infty}\beta'(t)-\inf_{0\le t\le+\infty}\alpha'(t)\Big), \tag{3.20}$$

where $m=\sup_{t\in[0,+\infty)}a(t)\phi(t)<+\infty$.

Remark 3.2. By condition (3.2), it is easy to know that $\int_0^{+\infty}a(s)\phi(s)ds<+\infty$. Thus we have $m<+\infty$.

Finally, we show that $|u''(t)|<N$ for $t\in[0,+\infty)$. If $|u''(t)|\le r$, for every $t\in[0,+\infty)$ then we have $|u''(t)|<N$. If $u''(t)>r$, for all $t\in[0,+\infty)$, then for any $R\ge\sigma$, by (3.19) we have

$$\frac{\beta'(R)-\alpha'(0)}{R}\ge\frac{u'(R)-u'(0)}{R}=\frac{\int_0^R u''(s)ds}{R}>r\ge\frac{\beta'(R)-\alpha'(0)}{R},$$

which is a contradiction. If $u''(t)<-r$, for every $t\in[0,+\infty)$, a similar contradiction can be obtained. So, there exists $t_0\in[0,+\infty)$ such that $|u''(t_0)|\le r$. Hence, there exists $[t_1,t_2]\subset[0,+\infty)$ such that $|u''(t_1)|=r$, $|u''(t)|>r$, $t\in(t_1,t_2]$ or $|u''(t_2)|=r$, $|u''(t)|>r$, $t\in[t_1,t_2)$. Without loss of generality, we suppose that $u''(t_1)=r$, $u''(t)>r$, $t\in(t_1,t_2]$. Then, by a convenient change of variable and applying assumptions (2.2) and (3.20), we have

$$\begin{aligned}
\int_{u''(t_1)}^{u''(t_2)}\frac{s}{h(s)}ds&=\int_{t_1}^{t_2}\frac{u''(t)}{h(u''(t))}u'''(t)dt\\
&=\int_{t_1}^{t_2}\frac{-a(t)f(t,u(t),u'(t),u''(t))u''(t)}{h(u''(t))}dt\\
&\le\int_{t_1}^{t_2}a(t)\phi(t)u''(t)dt\\
&\le m\int_{t_1}^{t_2}u''(t)dt=m(u'(t_2)-u'(t_1))\\
&\le m\Big(\sup_{t\in[0,+\infty)}\beta'(t)-\inf_{t\in[0,+\infty)}\alpha'(t)\Big)\\
&\le\int_r^N\frac{s}{h(s)}ds,
\end{aligned}$$

which concludes that $u''(t_2)\le N$. Since t_2 can be arbitrarily as long as $u''(t)>r$ we can conclude that, for every $t\in[0,+\infty)$ such that $u''(t)>r$, we have $u''(t)\le N$.

By a similar way, we can also obtain that if $u'(t_1) = -r$, $u'(t) < -r$, $t \in (t_1, t_2]$, then $u'(t) > -N$, $t \in [0, +\infty)$. Hence,

$$u'''(t) = -a(t)f^*(t, u(t), u'(t), u''(t)) = -a(t)f(t, u(t), u'(t), u''(t));$$

that is, u is a solution of (1.1).

4. An example

In this section, we give an example to illustrate our main result. Consider the boundary-value problem

$$\begin{gathered} u'''(t) + e^{-\gamma t}(t^3 + u^3(t))(1 - u'(t))(1 + \arctan((u''(t))^2) = 0, \quad t \in (0, +\infty), \\ u(0) = u'(0) = 0, \quad u''(+\infty) = 0, \end{gathered} \tag{4.1}$$

where γ is a positive constant. Set

$$a(t) = e^{-\gamma t}, \quad f(t, x, y, z) = (t^3 + x^3)(1 - y)(1 + \arctan(z^2)).$$

According to Definition 2.1, it is easy to check that $\alpha(t) = -t$, $\beta(t) = t$ are a pair of lower and upper solutions of (4.1). Moreover, we have $\alpha, \beta \in E$, $\alpha(t) \le \beta(t)$, $t \in [0, +\infty)$.

Obviously, f is continuous on $[0, +\infty) \times \mathbb{R}^3$ and increasing in x when $\alpha(t) \le x(t) \le \beta(t)$, $t \in [0, +\infty)$. Meanwhile, when $0 \le t < +\infty$, $-t \le x \le t$, $-1 \le y \le 1$, it holds

$$|f(t, x, y, z)| \le \phi(t)h(|z|),$$

where $\phi(t) = 4(1 + t^3)$ and $h(z) = 1 + z^2$. Since

$$\int_0^{+\infty} \frac{s}{h(s)} ds = \int_0^{+\infty} \frac{s}{1 + s^2} ds = +\infty,$$

f satisfies the Nagumo condition with respect to $-t, t$. Furthermore, we have

$$\int_0^{+\infty} \max\{s, 1\}a(s)ds = \int_0^1 e^{-\gamma s} ds + \int_1^{+\infty} s e^{-\gamma s} ds < +\infty$$

and

$$\int_0^{+\infty} \max\{s, 1\}a(s)\phi(s) = \int_0^1 4(1 + s^3)e^{-\gamma s} ds + \int_1^{+\infty} 4s(1 + s^3)e^{-\gamma s} ds < +\infty;$$

that is, (3.2) holds. Therefore, by Theorem 3.1, there exists at least one solution $u(t)$ for (4.1) such that

$$-t \le u(t) \le t, \quad -1 \le u'(t) \le 1, \quad t \in [0, +\infty).$$

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