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ANALYSIS OF ELECTRO-VISCOELASTIC ANTIPLANE CONTACT PROBLEM WITH TOTAL SLIP RATE DEPENDENT FRICTION

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To the memory of Professor Y. B. Boukhadra

Abstract. We consider a mathematical model which describes the antiplane shear deformation of a cylinder in frictional contact with a rigid foundation. The contact is bilateral and is modelled with a total slip rate dependent friction law. The material is assumed to be electro-viscoelastic and the foundation is assumed to be electrically conductive. First, we describe the classical formulation for the antiplane problem and we give the corresponding variational formulation which is given by a system coupling an evolutionary variational equality for the displacement field and a time-dependent variational equation for the electric potential field. Then we prove the existence of a unique weak solution to the model. The proof is based on arguments of variational inequalities and by using the Banach fixed-point Theorem.

1. Introduction

The contact between deformable bodies is a phenomenon frequently found in industry and in everyday life. The contact of the breaking pads with the wheel, of the tire with the road and the piston with the skirt are just simples examples. Considerable progress has been achieved recently in modelling, mathematical analysis and numerical simulations of various contact processes and, as a result, a general Mathematical Theory of Contact Mechanics (MTCM) is currently maturing. It is concerned with the mathematical structures which underlie general contact problems with different constitutive laws (i.e., different materials), varied geometries and settings, and different contact conditions, see for instance [6, 14, 15] and the references therein. The reason is that, owing to the inherent complicated nature, contact phenomena are modelled by difficult nonlinear problems, which explains the slow progress in their mathematical analysis.

Piezoelectric materials are characterized by the coupling between the mechanical and electrical properties. This coupling leads to the appearance of electric potential when mechanical stress is present and, conversely, mechanical stress is generated when electric potential is presented. The first effect is used in mechanical sensors

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and the reverse effect is used in actuators, in engineering control equipment. Piezoelectric materials for which the mechanical properties are elastic are called electro-elastic materials and those for which the mechanical properties are viscoelastic are called electro-viscoelastic material's. General models for electro-viscoelastic materials can be found in [3] In all these references the foundation was assumed to be electrically insulated. Antiplane problems [1, 4, 5, 13, 6] for piezoelectric materials were considered in [8, 2, 9, 12, 10, 7, 16]. We rarely actually load piezoelectric bodies so as to cause them to deform in antiplane shear; however, the governing equations and boundary conditions for antiplane shear problems involving piezoelectric materials are beautifully simple and the solution has many of the features of the more general case and may help us to solve the more complex problem too.

In this paper, as in [12, 10, 7, 16, 17]; there a model for the antiplane contact of an electro-elastic cylinder was considered under the assumption that the foundation is electrically conductive; the variational formulation of the model was derived and the existence of a unique solution to the model was proved by using arguments of evolutionary variational inequalities. Unlike [2, 11, 12, 6], in the present paper we consider a quasistatic contact problem between a rigid foundation and a cylinder. This problem is considered to be antiplane, i.e. the displacements parallel to the generators of the cylinder and is independent to the axial coordinate. Our interest is to describe a simple physical process in which both frictional contact, viscosity and piezoelectric effects are involved, and to show that the resulting model leads to a well-posed mathematical problem. Taking into account the frictional contact between a viscous piezoelectric body and an electrically conductive foundation in the study of an antiplane problem leads to a new and interesting mathematical model which has the virtue of relative mathematical simplicity without loss of essential physical relevance.

Our paper is structured as follows. In section 2 we present the model of the antiplane frictional contact of an electro-viscoelastic cylinder. In section 3 we introduce the notation, list the assumption on problem's data, derive the variational formulation of the problem and state our main existence and uniqueness result; i.e., Theorem 4.1. The proof of this result is carried out in several steps in Section 4 and is based on the argument of evolutionary variational inequalities and Banach's fixed point.

2. The antiplane contact problem

We consider a piezoelectric body $\mathcal{B}$ identified with a region in $\mathbb{R}^3$ it occupies in a fixed and undistorted reference configuration. We assume that $\mathcal{B}$ is a cylinder with generators parallel to the x_3-axes with a cross-section which is a regular region Ω in the x_1, x_2-plane, $Ox_1x_2x_3$ being a Cartesian coordinate system. The cylinder is assumed to be sufficiently long so that the end effects in the axial direction are negligible. Thus, $\mathcal{B} = \Omega \times (-\infty, +\infty)$. The cylinder is acted upon by body forces of density $\mathbf{f}_0$ and has volume free electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. To describe the boundary conditions, we denote by $\partial\Omega = \Gamma$ the boundary of Ω and we assume a partition of Γ into three open disjoint parts Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that the one-dimensional measure of Γ_1 and Γ_a, denoted $\operatorname{meas}\Gamma_1$ and $\operatorname{meas}\Gamma_a$, are positive. The cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty)$ and therefore the displacement field vanishes there. Surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2 \times (-\infty, +\infty)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (-\infty, +\infty)$ and a surface electrical charge of density q_b is prescribed on $\Gamma_b \times (-\infty, +\infty)$. The cylinder is in contact over $\Gamma_3 \times (-\infty, +\infty)$ with a conductive obstacle, the so called foundation. The contact is frictional and is modelled with Tresca's law. We are interested in the deformation of the cylinder on the time interval $[0, T]$.

Below in this paper the indices i and j denote components of vectors and tensors and run from 1 to 3, summation over two repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding spatial variable; also, a dot above represents the time derivative. We use $\mathcal{S}^3$ for the linear space of second order symmetric tensors on $\mathbb{R}^3$ or, equivalently, the space of symmetric matrices of order 3, and " $\cdot$ ", $\|\cdot\|$ will represent the inner products and the Euclidean norms on $\mathbb{R}^3$ and $\mathcal{S}^3$; we have:

$$\mathbf{u}\cdot\mathbf{v} = u_i v_i, \quad \|\mathbf{v}\| = (\mathbf{v}\cdot\mathbf{v})^{1/2} \quad \text{for all } \mathbf{u} = (u_i),\ \mathbf{v} = (v_i) \in \mathbb{R}^3,$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathcal{S}^3.$$

We assume that

$$\mathbf{f}_0 = (0, 0, f_0), \tag{2.1}$$

with $f_0 = f_0(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}$, and

$$\mathbf{f}_2 = (0, 0, f_2), \tag{2.2}$$

with $f_2 = f_2(x_1, x_2, t) : \Gamma_2 \times [0, T] \to \mathbb{R}$. The body forces (2.1)and the surface tractions (2.2) would be expected to give rise to a deformation of the elastic cylinder whose displacement, denoted by $\mathbf{u}$, is of the form

$$\mathbf{u} = (0, 0, u), \tag{2.3}$$

with $u = u(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}$. Such kind of deformation, associated to a displacement field of the form (2.3), is called an *antiplane shear*. We assume too that

$$q_0 = q_0(x_1, x_2, t), \tag{2.4}$$
$$q_2 = q_2(x_1, x_2, t), \tag{2.5}$$

with $q_0 : \Omega \times [0, T] \to \mathbb{R}$ and $q_2 : \Gamma_b \times [0, T] \to \mathbb{R}$. The electric charges (2.4), (2.5) would be expected to give rise to deformations and to electric charges of the piezoelectric cylinder corresponding to an electric potential field φ which is independent on x_3 and have the form

$$\varphi = \varphi(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}. \tag{2.6}$$

The infinitesimal strain tensor, denoted by $\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))$, is defined by

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\,(u_{i,j} + u_{j,i}), \quad i, j = 1, 2, 3, \tag{2.7}$$

where the index that follows a comma indicates a partial derivative with respect to the corresponding component of the spatial variable. Moreover, in the sequel, the convention of summation upon a repeated index is used. From (2.3) and (2.7) it follows that, in the case of the antiplane problem, the infinitesimal strain tensor becomes

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \begin{pmatrix} 0 & 0 & \frac{1}{2}u_{,1} \\ 0 & 0 & \frac{1}{2}u_{,2} \\ \frac{1}{2}u_{,1} & \frac{1}{2}u_{,2} & 0 \end{pmatrix}. \tag{2.8}$$

We also denote by $\mathbf{E}(\varphi) = (E_i(\varphi))$ the electric field and by $\mathbf{D} = (D_i)$ the electric displacement field where

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\,(u_{i,j} + u_{j,i}), \tag{2.9}$$

$$E_i(\varphi) = -\varphi_{,i}\,. \tag{2.10}$$

Let $\boldsymbol{\sigma} = (\sigma_{ij})$ denote the stress field. We suppose that the material's behavior is modelled by an electro-viscoelastic constitutive law of the form

$$\boldsymbol{\sigma} = 2\theta\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \zeta\,\mathrm{tr}\,\boldsymbol{\varepsilon}(\dot{\mathbf{u}})\,\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda\,\mathrm{tr}\,\boldsymbol{\varepsilon}(\mathbf{u})\,\mathbf{I} - \mathcal{E}^*\mathbf{E}(\varphi), \tag{2.11}$$

$$\mathbf{D} = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}) + \alpha\mathbf{E}(\varphi), \tag{2.12}$$

where ζ and θ are viscosity coefficients, λ and μ are the Lamé coefficients, tr $\boldsymbol{\varepsilon}(\mathbf{u}) = \varepsilon_{ii}(\mathbf{u})$, $\mathbf{I}$ is the unit tensor in $\mathbb{R}^3$, α is the electric permittivity constant, $\mathcal{E}$ represents the third-order piezoelectric tensor and $\mathcal{E}^*$ is its transpose. We assume that

$$\mathcal{E}\boldsymbol{\varepsilon} = \begin{pmatrix} e(\varepsilon_{13} + \varepsilon_{31}) \\ e(\varepsilon_{23} + \varepsilon_{32}) \\ e\varepsilon_{33} \end{pmatrix} \quad \forall \boldsymbol{\varepsilon} = (\varepsilon_{ij}) \in \mathcal{S}^3, \tag{2.13}$$

where e is a piezoelectric coefficient. We also assume that the coefficients θ, μ, α and e depend on the spatial variables x_1, x_2, but are independent on the spatial variable x_3. Since $\mathcal{E}\boldsymbol{\varepsilon} \cdot \mathbf{v} = \boldsymbol{\varepsilon} \cdot \mathcal{E}^*\mathbf{v}$ for all $\boldsymbol{\varepsilon} \in \mathcal{S}^3$, $\mathbf{v} \in \mathbb{R}^3$, it follows from (2.12) that

$$\mathcal{E}^*\mathbf{v} = \begin{pmatrix} 0 & 0 & ev_1 \\ 0 & 0 & ev_2 \\ ev_1 & ev_2 & ev_3 \end{pmatrix} \quad \forall \mathbf{v} = (v_i) \in \mathbb{R}^3. \tag{2.14}$$

Here and below the dot above represents the derivative with respect to the time variable. The stress field is given by the matrix

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 0 \end{pmatrix}. \tag{2.15}$$

In the antiplane context (2.3), (2.6), using the constitutive equations (2.11), (2.12) and equalities (2.13), (2.14) it follows that the stress field and the electric displacement field are given by

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & \theta\dot{u}_{,1} + \mu u_{,1} + e\varphi_{,1} \\ 0 & 0 & \theta\dot{u}_{,2} + \mu u_{,2} + e\varphi_{,2} \\ \theta\dot{u}_{,1} + \mu u_{,1} + e\varphi_{,1} & \theta\dot{u}_{,2}\, \mu u_{,2} + e\varphi_{,2} & 0 \end{pmatrix}, \tag{2.16}$$

$$\mathbf{D} = \begin{pmatrix} eu_{,1} - \alpha\varphi_{,1} \\ eu_{,2} - \alpha\varphi_{,2} \\ 0 \end{pmatrix}. \tag{2.17}$$

We assume that the process is mechanically quasistatic and electrically static and therefore is governed by the equilibrium equations

$$\mathrm{Div}\,\boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0}, \tag{2.18}$$

$$D_{i,i} - q_0 = 0 \quad \text{in } \mathcal{B} \times (0,T), \tag{2.19}$$

where $\mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j})$ represents the divergence of the tensor field $\boldsymbol{\sigma}$. Thus, keeping in mind (2.16), (2.17), (2.3), (2.6), (2.1) and (2.4), the equilibrium equations above reduce to the following scalar equations

$$\operatorname{div}(\theta\nabla\dot{u}+\mu\nabla u)+\operatorname{div}(e\nabla\varphi)+f_0=0 \quad \text{in } \Omega\times(0,T), \tag{2.20}$$

$$\operatorname{div}(e\nabla u-\alpha\nabla\varphi)=q_0 \quad \text{in } \Omega\times(0,T). \tag{2.21}$$

Here and below we use the notation

$$\operatorname{div}\boldsymbol{\tau}=\tau_{1,1}+\tau_{1,2} \quad \text{for } \boldsymbol{\tau}=(\tau_1(x_1,x_2,t),\tau_2(x_1,x_2,t)),$$

$$\nabla v=(v_{,1},v_{,2}), \quad \partial_\nu v=v_{,1}\ \nu_1+v_{,2}\ \nu_2 \quad \text{for } v=v(x_1,x_2,t).$$

Recall that, since the cylinder is clamped on $\Gamma_1\times(-\infty,+\infty)$, the displacement field vanishes there. Thus (2.3) implies

$$u=0 \quad \text{on } \Gamma_1\times(0,T), \tag{2.22}$$

the electrical potential vanishes too on $\Gamma_a\times(-\infty,+\infty)$; thus (2.6) imply that

$$\varphi=0 \quad \text{on } \Gamma_a\times(0,T). \tag{2.23}$$

Let $\boldsymbol{\nu}$ denote the unit normal on $\Gamma\times(-\infty,+\infty)$. We have

$$\boldsymbol{\nu}=(\nu_1,\nu_2,0), \tag{2.24}$$

with $\nu_i=\nu_i(x_1,x_2):\Gamma\to\mathbb{R}$, $i=1,2$. For a vector $\mathbf{v}$ we denote by v_ν and $\mathbf{v}_\tau$ its *normal* and *tangential* components on the boundary, given by

$$v_\nu=\mathbf{v}\cdot\boldsymbol{\nu}, \quad \mathbf{v}_\tau=\mathbf{v}-v_\nu\boldsymbol{\nu}, \tag{2.25}$$

respectively. In (2.25) and everywhere in this paper "$\cdot$" represents the inner product on the space $\mathbb{R}^3$ ($d=2,3$). Moreover, for a given stress field σ we denote by σ_ν and σ_τ the *normal* and the *tangential* components on the boundary, respectively; i.e.,

$$\sigma_\nu=(\boldsymbol{\sigma\nu})\cdot\boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau=\boldsymbol{\sigma\nu}-\sigma_\nu\boldsymbol{\nu}. \tag{2.26}$$

From (2.24), (2.16) and (2.17) we deduce that the Cauchy stress vector and the normal component of the electric displacement field are given by

$$\boldsymbol{\sigma\nu}=(0,0,\theta\partial_\nu\dot{u}+\mu\partial_\nu u+e\partial_\nu\varphi), \quad \mathbf{D}\cdot\boldsymbol{\nu}=e\partial_\nu u-\alpha\partial_\nu\varphi. \tag{2.27}$$

Here and subsequently we use the notations $\partial_\nu u=u_{,1}\nu_1+u_{,2}\nu_2$ and $\partial_\nu\varphi=\varphi_{,1}\nu_1+\varphi_{,2}\nu_2$. Keeping in mind the traction boundary condition $\boldsymbol{\sigma\nu}=\mathbf{f}_2$ on $\Gamma_2\times(-\infty,\infty)$ and the electric conditions $\mathbf{D}\cdot\boldsymbol{\nu}=q_2$ on $\Gamma_b\times(-\infty,\infty)$, it follows from (2.2), (2.5) and (2.27) that

$$\theta\partial_\nu\dot{u}+\mu\partial_\nu u+e\partial_\nu\varphi=f_2 \quad \text{on } \Gamma_2\times(0,T), \tag{2.28}$$

$$e\partial_\nu u-\alpha\partial_\nu\varphi=q_2 \quad \text{on } \Gamma_b\times(0,T). \tag{2.29}$$

We now describe the frictional contact condition on $\Gamma_3\times(-\infty,+\infty)$. Everywhere in this paper the notation $|\cdot|$ is used to denote the Euclidean norm on $\mathbb{R}$ ($d=1$ or 3). First, we remark that from (2.3), (2.24) and (2.25) we obtain $u_\nu=0$, which shows that the contact is *bilateral*, i.e. there is no loss of contact during the process. Using again (2.3), (2.24) and (2.25), we find

$$\mathbf{u}_\tau=(0,0,u). \tag{2.30}$$

Similarly, from (2.13), (2.24) and (2.26)

$$\boldsymbol{\sigma}_\tau=(0,0,\sigma_\tau). \tag{2.31}$$

where

$$\sigma_\tau=\theta\partial_\nu\dot{u}+\mu\partial_\nu u+e\partial_\nu\varphi. \tag{2.32}$$

We assume that the friction is invariant with respect to the x_3 axis and for all $t \in [0,T]$ it is modelled by the following conditions on Γ_3:

$$\begin{gathered}
|\boldsymbol{\sigma}_\tau(t)| \le g(\int_0^t |\dot{\boldsymbol{u}}_\tau(s)|ds), \\
|\boldsymbol{\sigma}_\tau(t)| < g(\int_0^t |\dot{\boldsymbol{u}}_\tau(s)|ds) \Rightarrow \dot{\boldsymbol{u}}_\tau(t) = \mathbf{0}, \\
|\boldsymbol{\sigma}_\tau(t)| = g(\int_0^t |\dot{\boldsymbol{u}}_\tau(s)|ds) \Rightarrow \exists\, \beta \ge 0 \text{ tel que } \boldsymbol{\sigma}_\tau = -\beta \dot{\boldsymbol{u}}_\tau.
\end{gathered} \tag{2.33}$$

Here $g : \Gamma_3 \to \mathbb{R}_+$ is a given function, the friction bound, and $\dot{\mathbf{u}}_\tau$ represents the tangential velocity on the contact boundary. This is a version of Tresca's friction law where the friction bound g is assumed to depend on the accumulated slip of the surface. In (2.33) the strict inequality holds in the stick zone and the equality in the slip zone.

Using now (2.30)–(2.33) it is straightforward to see that the friction law (2.33) implies

$$\begin{gathered}
|\theta\partial_\nu \dot{u} + \mu\partial_\nu u + e\partial_\nu\varphi| \le g(\int_0^t |\dot{u}_\tau(s)|ds), \\
|\theta\partial_\nu \dot{u} + \mu\partial_\nu u + e\partial_\nu\varphi| < g(\int_0^t |\dot{u}_\tau(s)|ds) \Rightarrow \dot{u} = 0, \\
|\theta\partial_\nu \dot{u} + \mu\partial_\nu u + e\partial_\nu\varphi| = g(\int_0^t |\dot{u}_\tau(s)|ds) \Rightarrow \exists\, \beta \ge 0 \\
\text{such that } \theta\partial_\nu \dot{u} + \mu\partial_\nu u + e\partial_\nu\varphi = -\beta\dot{u}, \quad \text{on } \Gamma_3 \times (0,T).
\end{gathered} \tag{2.34}$$

Next, since the foundation is electrically conductive and the contact is bilateral, we assume that the normal component of the electric displacement field or the free charge is proportional to the difference between the potential on the foundation and the body's surface. Thus,

$$\mathbf{D} \cdot \boldsymbol{\nu} = k\,(\varphi - \varphi_F) \quad \text{on} \quad \Gamma_3 \times (0,T),$$

where φ_F represents the electric potential of the foundation and k is the electric conductivity coefficient. We use (2.27) and the previous equality to obtain

$$e\partial_\nu u - \alpha\partial_\nu\varphi = k\,(\varphi - \varphi_F) \quad \text{on } \Gamma_3 \times (0,T). \tag{2.35}$$

Finally, we prescribe the initial displacement,

$$u(0) = u_0 \quad \text{in } \Omega, \tag{2.36}$$

where u_0 is a given function on Ω.

Now, the mathematical model which describes the antiplane shear of an electro-viscoelastic cylinder in frictional contact with a conductive foundation is completed and can be stated as follows.

Problem $\mathcal{P}$. Find the displacement field $u : \Omega \times [0,T] \to \mathbb{R}$ and the electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\operatorname{div}(\theta\nabla\dot{u} + \mu\nabla u) + \operatorname{div}(e\nabla\varphi) + f_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{2.37}$$

$$\operatorname{div}(e\nabla u - \alpha\nabla\varphi) = q_0 \quad \text{in } \Omega \times (0,T), \tag{2.38}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{2.39}$$

$$\theta\partial_\nu \dot{u} + \mu\partial_\nu u + e\partial_\nu\varphi = f_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{2.40}$$

$$
\begin{cases}
|\theta\partial_\nu \dot{u} + \mu\partial_\nu u + e\partial_\nu\varphi| \le g(\int_0^t |\dot{u}_\tau(s)|ds), \\
|\theta\partial_\nu \dot{u} + \mu\partial_\nu u + e\partial_\nu\varphi| < g(\int_0^t |\dot{u}_\tau(s)|ds) \Rightarrow \dot{u} = 0, \\
|\theta\partial_\nu \dot{u} + \mu\partial_\nu u + e\partial_\nu\varphi| = g(\int_0^t |\dot{u}_\tau(s)|ds) \Rightarrow \exists\,\beta \ge 0 \\
\text{ such that } \theta\partial_\nu \dot{u} + \mu\partial_\nu u + e\partial_\nu\varphi = -\beta\dot{u}, \quad \text{on } \Gamma_3 \times (0,T),
\end{cases} \tag{2.41}
$$

$$
e\partial_\nu u - \alpha\partial_\nu\varphi = q_2 \quad \text{on } \Gamma_b \times (0,T), \tag{2.42}
$$

$$
e\partial_\nu u - \alpha\partial_\nu\varphi = k\,(\varphi - \varphi_F) \quad \text{on } \Gamma_3 \times (0,T), \tag{2.43}
$$

$$
u(0) = u_0 \quad \text{in } \Omega. \tag{2.44}
$$

Note that once the displacement field u and the electric potential φ which solve Problem $\mathcal{P}$ are known, then the stress tensor $\boldsymbol{\sigma}$ and the electric displacement field $\mathbf{D}$ can be obtained by using the constitutive laws (2.16) and (2.17), respectively.

3. Variational formulation and main result

In this section we derive the variational formulation of Problem $\mathcal{P}$ and state our main existence and uniqueness result, Theorem 4.1. To this end, we introduce the subspaces of $H^1(\Omega)$ defined by

$$
V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1\}, \quad W = \{\psi \in H^1(\Omega) : \psi = 0 \text{ on } \Gamma_a\}.
$$

Since $meas\,\Gamma_1 > 0$ and $\operatorname{meas}\Gamma_a > 0$, it is well known that V and W are real Hilbert spaces with the inner products

$$
(u,v)_V = \int_\Omega \nabla u \cdot \nabla v \,dx \quad \forall u,v \in V, \quad (\varphi,\psi)_W = \int_\Omega \nabla\varphi \cdot \nabla\psi \,dx \quad \forall \varphi,\, \psi \in W.
$$

Moreover, the associated norms

$$
\|v\|_V = \|\nabla v\|_{L^2(\Omega)^2} \quad \forall v \in V, \quad \|\psi\|_W = \|\nabla\psi\|_{L^2(\Omega)^2} \quad \forall \psi \in W \tag{3.1}
$$

are equivalent on V and W, respectively, with the usual norm $\|\cdot\|_{H^1(\Omega)}$. By Sobolev's trace Theorem we deduce that there exist two positive constants $c_V > 0$ and $c_W > 0$ such that

$$
\|v\|_{L^2(\Gamma_3)} \le c_V\|v\|_V \quad \forall v \in V, \quad \|\psi\|_{L^2(\Gamma_3)} \le c_W\|\psi\|_W \quad \forall \psi \in W. \tag{3.2}
$$

For a real Banach space $(X, \|\cdot\|_X)$ we use the usual notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$ where $1 \le p \le \infty$, $k = 1,2,\dots$; we also denote by $C([0,T];X)$ and $C^1([0,T];X)$ the spaces of continuous and continuously differentiable functions on $[0,T]$ with values in X, with the respective norms

$$
\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X,
$$

$$
\|x\|_{C^1([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X + \max_{t\in[0,T]} \|\dot{x}(t)\|_X.
$$

Here and subsequently, we still write w for the trace γw of a function w on Γ, for all $w \in V$.

In the study of the Problem $\mathcal{P}$, we assume that the viscosity coefficient and the electric permittivity coefficient satisfy

$$
\theta \in L^\infty(\Omega) \text{ and there exists } \theta^* > 0 \text{ such that } \theta(\mathbf{x}) \ge \theta^* \text{ a.e. } \mathbf{x} \in \Omega, \tag{3.3}
$$

$$
\alpha \in L^\infty(\Omega) \text{ and there exists } \alpha^* > 0 \text{ such that } \alpha(\mathbf{x}) \ge \alpha^* \text{ a.e. } \mathbf{x} \in \Omega. \tag{3.4}
$$

We also assume that the Lamé coefficient and the piezoelectric coefficient satisfy

$$\mu \in L^\infty(\Omega) \quad \text{and} \quad \mu(\mathbf{x}) > 0 \;\text{ a.e. } \mathbf{x} \in \Omega, \tag{3.5}$$

$$e \in L^\infty(\Omega). \tag{3.6}$$

The forces, tractions, volume and surface free charge densities have the regularity

$$f_0 \in W^{1,2}(0,T;L^2(\Omega)), \quad f_2 \in W^{1,2}(0,T;L^2(\Gamma_2)), \tag{3.7}$$

$$q_0 \in W^{1,2}(0,T;L^2(\Omega)), \quad q_2 \in W^{1,2}(0,T;L^2(\Gamma_b)). \tag{3.8}$$

The electric conductivity coefficient satisfy

$$k \in L^\infty(\Gamma_3) \quad \text{and} \quad k(\mathbf{x}) \geq 0 \;\text{ a.e. } \mathbf{x} \in \Gamma_3. \tag{3.9}$$

Finally, we assume that the electric potential of the foundation and the initial displacement are such that

$$\varphi_F \in W^{1,2}(0,T;L^2(\Gamma_3)), \tag{3.10}$$

We suppose that the friction bound function g satisfies the following properties:

$$\begin{aligned}
&(a)\;\; g : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+;\\
&(b)\;\; \exists L_g \geq 0 \text{ such that } |g(x,r_1) - g(x,r_2)| \leq L_g|r_1 - r_2|,\\
&\quad \forall\, r_1, r_2 \in \mathbb{R} \text{ a.e. } x \in \Gamma_3;\\
&(c)\;\; \forall r \in \mathbb{R}, g(.,r) \text{ is Lebesgue measurable on } \Gamma_3;\\
&(d)\;\; g(.,r) \in L^2(\Gamma_3).
\end{aligned} \tag{3.11}$$

The initial data are chosen such that

$$u_0 \in \; V. \tag{3.12}$$

For every $t \in [0,T]$ we need to consider the operator $\mathcal{S}_t$ defined by

$$\begin{gathered}
\mathcal{S}_t : L^\infty(0,T;V) \to L^2(\Gamma),\\
\mathcal{S}_t(v) = \int_0^t |v(s)|\, ds \;\text{ a.e. on } \Gamma.
\end{gathered} \tag{3.13}$$

From (3.2) and (3.13) it follows that the for all $v_1, v_2 \in L^\infty(0,T;V)$ the following inequality holds:

$$\|\mathcal{S}_t(v_1) - \mathcal{S}_t(v_2)\|_{L^2(\Gamma)} \leq C \int_0^t \|v_1(s) - v_2(s)\|_V \, ds. \tag{3.14}$$

Here and bellow C represents a positive constant whose value may change from line to line.

We define now the functional $j : L^2(\Gamma) \times V \to \mathbb{R}_+$ given by

$$j(\xi, v) = \int_{\Gamma_3} g(\xi)|v|\, da \quad \forall \xi \in L^2(\Gamma), \;\; \forall v \in V. \tag{3.15}$$

Using conditions (3.11), we deduce that the integral in (3.15) is well defined.

We also define the mappings $f : [0,T] \to V$ and $q : [0,T] \to W$ respectively, by

$$(f(t), v)_V = \int_\Omega f_0(t) v\, dx + \int_{\Gamma_2} f_2(t) v\, da, \tag{3.16}$$

$$(q(t), \psi)_W = \int_\Omega q_0(t)\psi\, dx - \int_{\Gamma_b} q_2(t)\psi\, da + \int_{\Gamma_3} k\, \varphi_F(t)\psi\, da, \tag{3.17}$$

for all $v \in V$, $\psi \in W$ and $t \in [0,T]$. The definition of f and q are based on Riesz's representation theorem; moreover, it follows from assumptions by 3.7 and 3.8, that the integrals above are well-defined and

$$f \in W^{1,2}(0,T;V), \quad q \in W^{1,2}(0,T;W). \tag{3.18}$$

We define now the bilinear forms :

$$a_\theta : V \times V \to \mathbb{R}, \quad a_\theta(u,v) = \int_\Omega \theta\, \nabla u \cdot \nabla v \, dx, \tag{3.19}$$

$$a_\mu : V \times V \to \mathbb{R}, \quad a_\mu(u,v) = \int_\Omega \mu\, \nabla u \cdot \nabla v \, dx, \tag{3.20}$$

$$a_e : V \times W \to \mathbb{R}, \quad a_e(u,\varphi) = \int_\Omega e\, \nabla u \cdot \nabla \varphi \, dx = a_e^*(\varphi,u), \tag{3.21}$$

$$a_\alpha : W \times W \to \mathbb{R}, \quad a_\alpha(\varphi,\psi) = \int_\Omega \alpha\, \nabla \varphi \cdot \nabla \psi \, dx + \int_{\Gamma_3} k\, \varphi\psi \, dx, \tag{3.22}$$

for all $u, v \in V$ and $\varphi, \psi \in W$. Using the conditions in (3.15)– (3.18), we deduce that the integrals (3.19)– (3.22) are well defined. From (3.1)–(3.2), we can deduce that the bilinear forms $a_\theta(\cdot,\cdot)$, $a_\mu(\cdot,\cdot)$, $a_e(\cdot,\cdot)$, $a_e^*(\cdot,\cdot)$ and $a_\alpha(\cdot,\cdot)$ are continuous; moreover, the forms $a_\theta(\cdot,\cdot)$, $a_\mu(\cdot,\cdot)$ and $a_\alpha(\cdot,\cdot)$ are symmetric and in addition, the form $a_\theta(\cdot,\cdot)$ is V-elliptic and $a_\alpha(\cdot,\cdot)$ is W-elliptic, since

$$a_\theta(v,v) \geq \theta^* \|v\|_V^2 \quad \forall v \in V, \tag{3.23}$$

$$a_\alpha(\psi,\psi) \geq \alpha^* \|\psi\|_W^2 \quad \forall \psi \in W. \tag{3.24}$$

The variational formulation of Problem $\mathcal{P}$ is based of the following result.

Lemma 3.1. *If (u,φ) is a smooth solution to Problem $\mathcal{P}$, then $(u(t),\varphi(t)) \in V \times W$ and we have:*

$$\begin{aligned} &a_\theta(\dot u(t), v - \dot u(t)) + a_\mu(u(t), v - \dot u(t)) + a_e^*(\varphi(t), v - \dot u(t)) \\ &+ j(\mathcal{S}_t(\dot u), v) - j(\mathcal{S}_t(\dot u), \dot u(t)) \\ &\geq (f(t), v - \dot u(t))_V \quad \forall v \in V,\ t \in [0,T], \end{aligned} \tag{3.25}$$

$$a_\alpha(\varphi(t),\psi) - a_e(u(t),\psi) = (q(t),\psi)_W \quad \forall \psi \in W,\ t \in [0,T], \tag{3.26}$$

$$u(0) = u_0. \tag{3.27}$$

Proof. Let (u,φ) denote a smooth solution to Problem $\mathcal{P}$, we have $u(t) \in V$, $\dot u(t) \in V$ and $\varphi(t) \in W$ a.e. $t \in [0,T]$ and, from (2.37), (2.39) and (2.40), we obtain

$$\begin{aligned} &\int_\Omega \theta\, \nabla \dot u(t) \cdot \nabla (v - \dot u(t)) \, dx + \int_\Omega \mu\, \nabla u(t) \cdot \nabla (v - \dot u(t)) \, dx \\ &+ \int_\Omega e\, \nabla \varphi(t) \cdot \nabla (v - \dot u(t)) \, dx \\ &= \int_\Omega f_0(t)\, (v - \dot u(t)) \, dx + \int_{\Gamma_2} f_2(t)\, (v - \dot u(t)) \, da \\ &\quad + \int_{\Gamma_3} (\theta \partial_\nu \dot u(t) + \mu \partial_\nu u(t) + e \partial_\nu \varphi(t))(v - \dot u(t)) \, da, \quad \forall v \in V \ \ t \in (0,T), \end{aligned}$$

and from (2.38) and (2.42)–(2.43) we obtain

$$\begin{aligned}
&\int_\Omega \alpha\,\nabla\varphi(t)\cdot\nabla\psi\,dx - \int_\Omega e\,\nabla u(t)\cdot\nabla\psi\,dx \\
&= \int_\Omega q_0(t)\psi\,dx - \int_{\Gamma_b} q_2(t)\psi\,da + \int_{\Gamma_3} k\,\varphi_F(t)\psi\,da \quad \forall\psi\in W \;\; t\in(0,T).
\end{aligned} \tag{3.28}$$

Using (3.16) and (3.19)–(3.21) we obtain

$$\begin{aligned}
&a_\theta(\dot u(t), v-\dot u(t)) + a_\mu(u(t), v-\dot u(t)) + a_e^*(\varphi(t), v-\dot u(t)) \\
&- \int_{\Gamma_3} (\theta\partial_\nu\dot u(t) + \mu\partial_\nu u(t) + e\partial_\nu\varphi(t))(v-\dot u(t))\,da \\
&= (f(t), v-\dot u(t))_V, \quad \forall v\in V,\; t\in[0,T],
\end{aligned} \tag{3.29}$$

Keeping in mind (3.17) and (3.21)–(3.22), we find the second equality in Lemma (3.1); i.e.,

$$a_\alpha(\varphi(t),\psi) - a_e(u(t),\psi) = (q(t),\psi)_W \quad \forall\psi\in W,\; t\in[0,T], \tag{3.30}$$

Using the frictional contact condition (2.41) and (3.15) on $\Gamma_3\times(0,T)$, we deduce that for all $t\in[0,T]$

$$j(\mathcal{S}_t(\dot u), \dot u(t)) = -\int_{\Gamma_3} (\theta\partial_\nu\dot u(t) + \mu\partial_\nu u(t) + e\partial_\nu\varphi(t))\dot u(t)\,da, \tag{3.31}$$

it is very easy to see that

$$j(\mathcal{S}_t(\dot u), v) \ge -\int_{\Gamma_3} (\theta\partial_\nu\dot u(t) + \mu\partial_\nu u(t) + e\partial_\nu\varphi(t))v\,da, \quad \forall v\in V. \tag{3.32}$$

The first inequality in Lemma (3.1) follows now from (3.29) and (3.31)–(3.32). Now, from Lemma (3.1) and the initial condition (3.27) lead to give the following variational Problem:

Problem $\mathcal{PV}$. Find a displacement field $u:[0,T]\to V$ and an electric potential field $\varphi:[0,T]\to W$ such that

$$\begin{aligned}
&a_\theta(\dot u(t), v-\dot u(t)) + a_\mu(u(t), v-\dot u(t)) + a_e^*(\varphi(t), v-\dot u(t)) \\
&+ j(\mathcal{S}_t(\dot u), v) - j(\mathcal{S}_t(\dot u), \dot u(t)) \\
&\ge (f(t), v-\dot u(t))_V, \quad \forall v\in V,\; t\in[0,T],
\end{aligned} \tag{3.33}$$

$$a_\alpha(\varphi(t),\psi) - a_e(u(t),\psi) = (q(t),\psi)_W, \quad \forall\psi\in W,\; t\in[0,T], \tag{3.34}$$

$$u(0) = u_0. \tag{3.35}$$

□

4. An existence and uniqueness result

Our main existence and uniqueness result, which we state and prove in this section, is the following.

Theorem 4.1. *Assume that* (3.3)–(3.12) *hold. Then the variational problem* $\mathcal{PV}$ *possesses a unique solution* (u,φ) *satisfies*

$$u\in W^{1,2}(0,T;V), \quad \varphi\in W^{1,2}(0,T;W). \tag{4.1}$$

A couple of functions (u,φ) which solves Problem $\mathcal{PV}$ is called a weak solution of the antiplane contact Problem $\mathcal{P}$. We conclude by Theorem (4.1) that the antiplane contact Problem $\mathcal{P}$ has a unique weak solution, provided that (3.3)–(3.12) hold.

We turn to the proof of Theorem (4.1) which will be carried out in several steps. To this end, in the rest of this section we assume that (3.3)–(3.12) hold and let η and ξ be two elements of $W^{1,2}(0,T;V)$. We consider the following variational Problem:

Problem $\mathcal{PV}^1_{\eta\xi}$. Find $v_{\eta\xi}:[0,T]\to V$ such that

$$\begin{aligned} &a_\theta(v_{\eta\xi}(t),v-v_{\eta\xi}(t))+(\eta(t),v-v_{\eta\xi}(t))_V+j(\mathcal{S}_t(\xi),v)-j(\mathcal{S}_t(\xi),v_{\eta\xi}(t))\\ &\geq (f(t),v-v_{\eta\xi}(t))_V,\quad \forall v\in V,\ \ a.e.\ \ t\in[0,T], \end{aligned}\tag{4.2}$$

The unique solvability of the intermediate Problem $\mathcal{PV}^1_{\eta\xi}$ follows from the following result:

Lemma 4.2. *There exists a unique solution $v_{\eta\xi}$ to Problem $\mathcal{PV}^1_{\eta\xi}$. Moreover,*

$$v_{\eta\xi}\in W^{1,2}(0,T;V).\tag{4.3}$$

Proof. It follows from classical results for variational inequalities that there exists a unique solution $v_{\eta\xi}\in V$ that solves (4.2) a.e. $t\in(0,T)$.

Taking $v=0_V$ in (4.2), we deduce that

$$a_\theta(v_{\eta\xi}(t),-v_{\eta\xi}(t))+(\eta(t),-v_{\eta\xi}(t))_V\geq (f(t),-v_{\eta\xi}(t))_V,\quad a.e.\ \ t\in[0,T].$$

From (3.23), we can deduce that

$$\theta\|v_{\eta\xi}(t)\|_V\leq \|f(t)\|_V+\|\eta(t)\|_V,\quad \text{a.e. } t\in[0,T],\tag{4.4}$$

Taking in mind (4.4), (3.18) and the regularity $\eta\in W^{1,2}(0,T;V)$, we obtain $v_{\eta\xi}\in W^{1,2}(0,T;V)$, which conclude the proof. □

In the next step, we use the displacement field $v_{\eta\xi}$ obtained in Lemma (4.2) to define the following variational Problem $\mathcal{PV}^2_{\eta\xi}$ for the electrical potential field:

Problem $\mathcal{PV}^2_{\eta\xi}$. Find an electrical potential field $\varphi_{\eta\xi}:[0,T]\to V$ such that

$$a_\alpha(\varphi_{\eta\xi}(t),\psi)+a_e(u_{\eta\xi}(t),\psi)=(q(t),\psi)_W,\quad \forall\psi\in W,\ \ t\in[0,T],\tag{4.5}$$

The unique solvability of the electrical Problem $\mathcal{PV}^2_{\eta\xi}$ follows from the following result:

Lemma 4.3. *There exists a unique solution $\varphi_{\eta\xi}$ to Problem $\mathcal{PV}^2_{\eta\xi}$ such that*

$$\varphi_{\eta\xi}\in W^{1,2}(0,T;W),\tag{4.6}$$

which satisfies (4.5)*, Moreover, if $\varphi_{\eta\xi_1}$ and $\varphi_{\eta\xi_2}$ are the solutions of* (4.5) *corresponding to $\eta\xi_1$, $\eta\xi_2\in C([0,T];V)$ then, there exists $c>0$, such that*

$$\|\varphi_{\eta\xi_1}(t)-\varphi_{\eta\xi_2}(t)\|_W\leq c\|u_{\eta\xi_1}(t)-u_{\eta\xi_2}(t)\|_V.\tag{4.7}$$

Proof. Let $t\in[0,T]$. We use the properties of the bilinear form a_α and the Lax-Milgram Lemma to see that there exists a unique element $\varphi_{\eta\xi}(t)\in W$ which solves $\mathcal{PV}^2_{\eta\xi}$ at any moment $t\in[0,T]$. Consider now t_1, $t_2\in[0,T]$; using (3.24) and (4.5) we find that

$$\begin{aligned} \alpha^*\|\varphi(t_1)-\varphi(t_2)\|^2_W &\leq \|e\|_{L^\infty(\Omega)}\|u(t_1)-u(t_2)\|_V\|\varphi(t_1)-\varphi(t_2)\|_W\\ &\quad+\|q(t_1)-q(t_2)\|_W\|\varphi(t_1)-\varphi(t_2)\|_W \end{aligned}\tag{4.8}$$

which implies

$$\|\varphi(t_1) - \varphi(t_2)\|_W \le c\,(\|u(t_1) - u(t_2)\|_V + \|q(t_1) - q(t_2)\|_W). \tag{4.9}$$

We note that regularity $u_{\eta\xi} \in C^1([0,T];V)$ combined with (3.18) and (4.9) imply that $\varphi_{\eta\xi} \in W^{1,2}(0,T;W)$, which concludes the proof. □

We consider now the operator $\Lambda_\eta : C([0,T];V) \to C([0,T];V)$ defined for all $\eta \in L^\infty(0,T;V)$ by the equality

$$(\Lambda_\eta \xi(t), w)_V = a_\mu(v_{\eta\xi}(t), w) + a_e^*(\varphi_{\eta\xi}(t), w) \quad \forall w \in V, \ \ t \in [0,T]. \tag{4.10}$$

We have the following result.

Lemma 4.4. *For every element $\eta \in L^\infty(0,T;V)$ the operator Λ_η has a unique fixed point $\xi_\eta \in L^\infty(0,T;V)$.*

Proof. Let $\eta \in L^\infty(0,T;V)$ and $\xi_i \in L^\infty(0,T;V)$, $i = 1,2$. To simplify the notation, we denote by v_i the unique solution to Problem $\mathcal{PV}^1_{\eta\xi_i}$, for $i = 1,2$. Thus, from (4.2) we can write

$$\begin{aligned} &a_\theta(v_i(t), v - v_i(t)) + (\eta(t), v - v_i(t))_V + j(\mathcal{S}_t(\xi_i), v) - j(\mathcal{S}_t(\xi_i), v_i(t)) \\ &\ge (f(t), v - v_i(t))_V, \quad \forall v \in V, \quad \text{a.e. } t \in [0,T]. \end{aligned} \tag{4.11}$$

After some algebra and from (4.11), we find

$$\begin{aligned} \theta\|v_1(t) - v_2(t)\|_V^2 &\le j(\mathcal{S}_t(\xi_1), v_2(t)) - j(\mathcal{S}_t(\xi_2), v_2(t)) \\ &\quad + j(\mathcal{S}_t(\xi_2), v_1(t)) - j(\mathcal{S}_t(\xi_1), v_1(t)), \ \ \textit{a.e. } t \in [0,T]. \end{aligned} \tag{4.12}$$

Using now (3.2), (3.11), (3.14) and (3.15), it follows that

$$\begin{aligned} &j(\mathcal{S}_t(\xi_1), v_2(t)) - j(\mathcal{S}_t(\xi_2), v_2(t)) + j(\mathcal{S}_t(\xi_2), v_1(t)) - j(\mathcal{S}_t(\xi_1), v_1(t)) \\ &\le C \times \|v_1(t) - v_2(t)\|_V \int_0^t \|\xi_1(s) - \xi_2(s)\|_V\, ds \quad \text{a.e. } t \in [0,T]. \end{aligned} \tag{4.13}$$

Using (4.12), (4.13) we deduce that

$$\|v_1(t) - v_2(t)\|_V \le C \int_0^t \|\xi_1(s) - \xi_2(s)\|_V\, ds, \quad \text{a.e. } t \in [0,T]. \tag{4.14}$$

Let $\eta_1,\ \eta_2 \in C([0,T];V)$ and denote by u_i and φ_i the functions $u_{\eta\xi_i}$ and $\varphi_{\eta\xi_i}$ obtained in Lemmas 4.2 and 4.3, for $i = 1,2$. Let $t \in [0,T]$. Using the definition (4.10) of the operator Λ_η we obtain

$$(\Lambda_\eta \xi_1(t), w)_V = a_\mu(v_1(t), w) + a_e^*(\varphi_1(t), w) \quad \forall w \in V, \ \ t \in [0,T], \tag{4.15}$$

and

$$(\Lambda_\eta \xi_2(t), w)_V = a_\mu(v_2(t), w) + a_e^*(\varphi_2(t), w) \quad \forall w \in V, \ \ t \in [0,T]. \tag{4.16}$$

From (3.21), (4.5), (4.15) and (4.16) we deduce

$$\|\Lambda_\eta \xi_1(t) - \Lambda_\eta \xi_2(t)\|_V \le C\,(\int_0^t \|v_1(s) - v_2(s)\|_V\, ds + \|\varphi_1(t) - \varphi_2(t)\|_W),$$

and, keeping in mind (4.7) and (4.14), we find

$$\|\Lambda_\eta \xi_1(t) - \Lambda_\eta \xi_2(t)\|_V \le C' \int_0^t \|\xi_1(s) - \xi_2(s)\|_V\, ds, \quad \text{a.e. } t \in [0,T]. \tag{4.17}$$

We define now the set $\|\cdot\|_\kappa$ as follows

$$\|v\|_\kappa = Inf\left\{M > 0 \ \ e^{-\kappa t}\|v\|_V \le M \quad \text{a.e. } t \in (0,T)\right\}, \quad \forall v \in L^\infty(0,T;V), \tag{4.18}$$

such that $\kappa > 0$ to be determined later. The norm $\|\cdot\|_\kappa$ is equivalent to the standard norm $\|\cdot\|_{L^\infty(0,T;V)}$. Using now (4.18) and the definition of the norm $\|\cdot\|_\kappa$, we can obtain :

$$e^{-\kappa t}\|\Lambda_\eta\xi_1(t) - \Lambda_\eta\xi_2(t)\|_V \le C'\, e^{-\kappa t}\int_0^t e^{\kappa s}e^{-\kappa s}\|\xi_1(s) - \xi_2(s)\|_V\, ds, \tag{4.19}$$

then

$$e^{-\kappa t}\|\Lambda_\eta\xi_1(t) - \Lambda_\eta\xi_2(t)\|_V \le C'\, e^{-\kappa t}\|\xi_1 - \xi_2\|_\kappa \int_0^t e^{\kappa s}\, ds, \tag{4.20}$$

we deduce

$$e^{-\kappa t}\|\Lambda_\eta\xi_1(t) - \Lambda_\eta\xi_2(t)\|_V \le \frac{C'}{\kappa}\, e^{-\kappa t}\|\xi_1 - \xi_2\|_\kappa, \quad a.e.\ t \in [0,T]. \tag{4.21}$$

Consequently, we deduce that

$$\|\Lambda_\eta\xi_1(t) - \Lambda_\eta\xi_2(t)\|_\kappa \le \frac{C'}{\kappa}\|\xi_1 - \xi_2\|_\kappa. \tag{4.22}$$

Taking κ such that $\kappa > C'$, we conclude that the operator Λ_η is a contraction on the space $(L^\infty(0,T;V), \|\cdot\|_\kappa)$. From Banach fixed point Theorem, we deduce that the operator Λ_η has a unique fixed-point $\xi_\eta \in L^\infty(0,T;V)$. □

In what follows, we continue to write

$$v_\eta = v_{\eta\xi_\eta}, \quad \forall \eta \in L^\infty(0,T;V), \tag{4.23}$$

where ξ_η is the unique fixed point of the operator Λ_η, then from (4.10) and (4.23) we obtain

$$v_\eta = \xi_\eta. \tag{4.24}$$

Let $u_\eta : [0,T] \to V$ be the function defined by

$$u_\eta = \int_0^t v_\eta(s)\, ds + u_0 \quad \forall t \in [0,T]. \tag{4.25}$$

We also define the operator $\Lambda : L^\infty(0,T;V) \to L^\infty(0,T;V)$ by

$$(\Lambda\eta(t), w)_V = a_\mu(u_\eta(t), w) + a_e^*(\varphi_\eta(t), w) \quad \forall w \in V,\ \ t \in [0,T]. \tag{4.26}$$

We have the following result.

Lemma 4.5. *The operator Λ has a unique fixed point $\eta^* \in L^\infty(0,T;V)$.*

Proof. Using (4.24) and the fact that v_η is the unique solution of Problem $\mathcal{PV}^1_{\eta\xi}$, we can obtain

$$\begin{gathered}
a_\theta(v_\eta(t), v - v_\eta(t)) + j(\mathcal{S}_t(v_\eta), v) - j(\mathcal{S}_t(v_\eta), v_\eta(t)) \\
\ge (f(t), v - v_\eta(t))_V - (\eta(t), v - v_\eta(t))_V, \quad \forall v \in V, \quad \text{a.e. } t \in [0,T].
\end{gathered} \tag{4.27}$$

Let $\eta_1, \eta_2 \in L^\infty(0,T;V)$ and denote by u_i and φ_i the functions u_{η_i} and φ_{η_i} obtained in Lemmas 4.2 and 4.3, for $i = 1,2$. Let $t \in [0,T]$. Using (4.27) we obtain

$$\begin{aligned}
&\|v_1(s) - v_2(s)\|_V \\
&\le C\,(\|\eta_1(s) - \eta_2(s)\|_V + \int_0^t \|v_1(r) - v_2(r)\|_V\, dr), \quad \text{a.e. } s \in [0,T].
\end{aligned} \tag{4.28}$$

We integrate the (4.28) on the $[0,T]$ with a fixed t and using a Gronwall-type argument, we obtain

$$\int_0^t \|v_1(s)-v_2(s)\|_V\,ds \le C\int_0^t \|\eta_1(s)-\eta_2(s)\|_V ds,\ \forall t\in[0,T]. \tag{4.29}$$

Using the definition (4.27) for η_1 and η_2 we obtain

$$(\Lambda\eta_1(t),w)_V = a_\mu(u_1(t),w)+a_e^*(\varphi_1(t),w)\quad \forall w\in V,\ t\in[0,T], \tag{4.30}$$

$$(\Lambda\eta_2(t),w)_V = a_\mu(u_2(t),w)+a_e^*(\varphi_2(t),w)\quad \forall w\in V,\ t\in[0,T]. \tag{4.31}$$

From (4.30)–(4.31) we can write

$$\|\Lambda\eta_1(t)-\Lambda\eta_2(t)\|_V \le C(\|u_1(t)-u_2(t)\|_V+\|\varphi_1(t)-\varphi_2(t)\|_W). \tag{4.32}$$

On the other hand, (4.5) and arguments similar as those used in the proof of (4.10) yield

$$\|\varphi_1(t)-\varphi_2(t)\|_W \le c\|u_1(t)-u_2(t)\|_V. \tag{4.33}$$

Using (4.25), (4.26), (4.32) and (4.33) we obtain

$$\|\Lambda\eta_1(t)-\Lambda\eta_2(t)\|_V \le C\int_0^t \|\eta_1(s)-\eta_2(s)\|_V\,ds,\quad \forall t\in[0,T]. \tag{4.34}$$

Keeping in mind the defintion of $\|\cdot\|_\kappa$, Lemma 4.5 follows from the previous inequality, after using a fixed-point argument similar to that presented Lemma 4.4. $\square$

Now we have all the ingredients to prove Theorem 4.1.

Existence. Let $\eta^*\in L^\infty(0,T;V)$ be the unique fixed point of the operator Λ and let u and φ be the solutions of Problems $\mathcal{PV}^1_{\eta\xi}$ and $\mathcal{PV}^2_{\eta\xi}$ respectively with $\eta=\eta^*$, i.e. $u=u_{\eta^*}$ and $\varphi=\varphi_{\eta^*}$. Clearly, equalities (3.33)–(3.35) hold from $\mathcal{PV}^1_{\eta\xi}$ and $\mathcal{PV}^2_{\eta\xi}$. Let $u_{\eta^*}\in W^{1,2}(0,T;V)$ be the function defined by the relation (4.25) for $\eta=\eta^*$. We have $\dot u_{\eta^*}=v_{\eta^*}$ and from (4.27), it follows that

$$\begin{aligned}&a_\theta(\dot u_{\eta^*}(t),v-\dot u_{\eta^*}(t))+j(\mathcal{S}_t(\dot u_{\eta^*}),v)-j(\mathcal{S}_t(\dot u_{\eta^*}),\dot u_{\eta^*}(t))\\ &\ge (f(t),v-\dot u_{\eta^*}(t))_V-(\eta^*(t),v-\dot u_{\eta^*}(t))_V,\quad \forall v\in V,\quad \text{a.e. } t\in[0,T].\end{aligned} \tag{4.35}$$

The inequality (3.33) follows now from (4.35) and (4.26), using the fact that η^* is the fixed point of the operator Λ. From the definition (4.25) implies $u_{\eta^*}(0)=u_0$ so that (3.35) is fulfilled. We conclude now that u_{η^*} is a solution to Problem $\mathcal{PV}$. The regularity of the solution expressed in (4.1) follows from Lemmas 4.2 and 4.3. We conclude that (u,φ) is a solution of Problem $\mathcal{PV}$ and it satisfies (4.1).

Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of Λ combined with the unique solvability of previous Problems, guaranteed by Lemmas 4.2–4.4.

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