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UPPER AND LOWER SOLUTIONS FOR A SECOND-ORDER THREE-POINT SINGULAR BOUNDARY-VALUE PROBLEM

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Abstract. We study the singular boundary-value problem

$$\begin{gathered} u'' + q(t)g(t,u) = 0, \quad t \in (0,1),\ \eta \in (0,1),\ \gamma > 0 \\ u(0) = 0, \quad u(1) = \gamma u(\eta)\,. \end{gathered}$$

The singularity may appear at $t = 0$ and the function g may be superlinear at infinity and may change sign. The existence of solutions is obtained via an upper and lower solutions method.

1. Introduction

Motivated by the study of multi-point boundary-value problems for linear second order ordinary differential equations, Gupta [7] studied certain three point boundary-value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multi-point boundary-value problems have been studied by several authors using the Leray-Schauder theorem, nonlinear alternative of Leray-Schauder or coincidence degree theory. We refer the reader to [3, 4, 5, 9, 12, 13, 14, 15] for some existence results of nonlinear multi-point boundary-value problems. Recently, Ma [14] proved the existence of positive solutions for the three point boundary-value problem

$$\begin{gathered} u'' + b(t)g(u) = 0, \quad t \in (0,1) \\ u(0) = 0, \quad u(1) = \alpha u(\eta), \end{gathered}$$

where $\eta \in (0,1)$, $0 < \alpha < 1/\eta$, $b \geq 0$ and $g \geq 0$ is either superlinear or sublinear. He applied a fixed point theorem in cones.

In this paper, we study the singular three-point boundary-value problem

$$\begin{gathered} u'' + q(t)g(t,u) = 0, \quad t \in (0,1),\ \eta \in (0,1),\ \gamma > 0 \\ u(0) = 0, \quad u(1) = \gamma u(\eta). \end{gathered} \tag{1.1}$$

The singularity may appear at $t = 0$, and the function g may be superlinear at $u = \infty$ and may change sign.

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Some basic results on the singular two point boundary-value problems were obtain in [1, 11, 17], in all these papers the arguments rely on the assumption that $g(t,u)$ is positive. This implies that the solutions are concave. Recently, some authors have studied the case when g is allowed to change sign by applying the modified upper and lower solutions method; see for example [11].

The present work is a direct extension of some results on the singular two-point boundary-value problems. As in [11], our technique relies essentially on a modified method of upper and lower solutions method for singular three-point boundary-value problems which we believe is well adapted to this type of problems.

2. Upper and lower solutions

Consider the three-point boundary-value problem

$$\begin{gathered} u'' + f(t,u) = 0, \quad t \in (0,1),\ \eta \in (0,1),\ \gamma \in (0,1/\eta) \\ u(0) = A, \quad u(1) - \gamma u(\eta) = B. \end{gathered} \tag{2.1}$$

We use the following assumption:

(A1) $f : (0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, there exist two functions $\alpha, \beta \in C([0,1],\mathbb{R})$ and $\alpha(t) \le \beta(t)$, for all $t \in [0,1]$, if there exist a function $h \in C(\,(0,1],(0,\infty))$, such that

$$|f(t,u)| \le h(t) \quad \text{for } \alpha(t) \le u \le \beta(t), \tag{2.2}$$

$$\lim_{t\to 0^+} t^2 h(t) = 0, \quad \int_0^1 t h(t)dt < \infty. \tag{2.3}$$

We call a function $\alpha(t)$ a lower solution for (2.1), if $\alpha \in C([0,1],\mathbb{R}) \cap C^2((0,1),\mathbb{R})$, and

$$\begin{gathered} \alpha'' + f(t,\alpha) \ge 0, \quad \text{for } t \in (0,1), \\ \alpha(0) \le A, \quad \alpha(1) - \gamma\alpha(\eta) \le B. \end{gathered}$$

Similarly, we call a function $\beta(t)$ an upper solution for (2.1), if $\beta \in C([0,1],\mathbb{R}) \cap C^2((0,1),\mathbb{R})$, and

$$\begin{gathered} \beta'' + f(t,\beta) \le 0, \quad \text{for } t \in (0,1), \\ \beta(0) \ge A, \quad \beta(1) - \gamma\beta(\eta) \ge B. \end{gathered}$$

A function $u(t)$ is said to be a solution to (2.1), if it is both a lower and an upper solution to (2.1).

Our first result reads as follows.

Theorem 2.1. *Assume* (A1) *and let α, β be, respectively, a lower solution and an upper solution for* (2.1) *such that $\alpha(t) \le \beta(t)$ on* $[0,1]$. *Then* (2.1) *has at least one solution $u(t)$ such that*

$$\alpha(t) \le u(t) \le \beta(t), \quad \textit{for } t \in [0,1].$$

Consider now the modified boundary-value problem

$$\begin{gathered} u'' + f_1(t,u) = 0, \quad \text{for } t \in (0,1), \\ u(0) = A, \quad u(1) - \gamma u(\eta) = B, \end{gathered} \tag{2.4}$$

where

$$f_1(t,u)=\begin{cases} f(t,\alpha(t)), & \text{if } u<\alpha(t),\\ f(t,u), & \text{if } \alpha(t)\le u\le\beta(t),\\ f(t,\beta(t)), & \text{if } u>\beta(t).\end{cases}$$

Lemma 2.2. *Assume that* (2.3) *holds. Then the boundary-value problem*

$$\begin{gathered} y''=-h(t),\quad 0<t<1,\\ y(0)=A,\quad y(1)-\gamma y(\eta)=B \end{gathered}\tag{2.5}$$

has a unique solution $y(t)$ *in* $C([0,1],[0,\infty))\cap C^2((0,1),\mathbb{R})$, *which can be written as*

$$y(t)=A+\frac{B-A(1-\gamma)}{1-\gamma\eta}t+\int_0^1 G(t,s)h(s)ds,\quad 0\le t\le 1,$$

where $G(t,s)$ *is Green's function of the boundary-value problem* $-y''=0$, $y(0)=0$, $y(1)=\gamma y(\eta)$. *The function* G *is explicitly given by: when* $0\le s\le\eta$,

$$G(t,s)=\begin{cases}\frac{s[1-t-\gamma(\eta-t)]}{1-\gamma\eta}, & s\le t,\\ \frac{t[1-s-\gamma(\eta-s)]}{1-\gamma\eta}, & s>t;\end{cases}$$

when $\eta<s\le 1$,

$$G(t,s)=\begin{cases}\frac{s(1-t)+\gamma\eta(t-s)}{1-\gamma\eta}, & s\le t,\\ \frac{t(1-s)}{1-\gamma\eta}, & s>t.\end{cases}$$

Proof. Uniqueness. The proof of the uniqueness of a solution is standard and hence omitted. Existence. Let

$$y(t):=A+\frac{B-A(1-\gamma)}{1-\gamma\eta}t+\int_0^1 G(t,s)h(s)ds,\quad 0\le t\le 1;$$

i.e.,

$$y(t)=\begin{cases} A+\frac{B-A(1-\gamma)}{1-\gamma\eta}t+\int_0^t\frac{s[1-t-\gamma(\eta-t)]}{1-\gamma\eta}h(s)ds\\ +\int_t^\eta\frac{t[1-s-\gamma(\eta-s)]}{1-\gamma\eta}h(s)ds+\int_\eta^1\frac{t(1-s)}{1-\gamma\eta}h(s)ds, & 0\le t\le\eta,\\ A+\frac{B-A(1-\gamma)}{1-\gamma\eta}t+\int_0^\eta\frac{s[1-t-\gamma(\eta-t)]}{1-\gamma\eta}h(s)ds\\ +\int_\eta^t\frac{s(1-t)+\gamma\eta(t-s)}{1-\gamma\eta}h(s)ds+\int_t^1\frac{t(1-s)}{1-\gamma\eta}h(s)ds, & \eta<t\le 1.\end{cases}$$

Then we have

$$y'(t)=\begin{cases}\frac{B-A(1-\gamma)}{1-\gamma\eta}+\int_0^t\frac{s(\gamma-1)}{1-\gamma\eta}h(s)ds\\ +\int_t^\eta\frac{1-s-\gamma(\eta-s)}{1-\gamma\eta}h(s)ds\\ +\int_\eta^1\frac{1-s}{1-\gamma\eta}h(s)ds, & 0<t\le\eta,\\ \frac{B-A(1-\gamma)}{1-\gamma\eta}+\int_0^\eta\frac{s(\gamma-1)}{1-\gamma\eta}h(s)ds\\ +\int_\eta^t\frac{\gamma\eta-s}{1-\gamma\eta}h(s)ds+\int_t^1\frac{1-s}{1-\gamma\eta}h(s)ds, & \eta<t\le 1.\end{cases}$$

and $y''(t)=-h(t)$ for all $t\in(0,1)$. Since $\int_0^1 th(t)dt<\infty$, $\lim_{t\to0^+}\int_0^t sh(s)ds=0$; so we have

$$y(0)=A+\lim_{t\to0^+}t\int_t^\eta\frac{1-s-\gamma(\eta-s)}{1-\gamma\eta}h(s)ds.$$

If $\int_0^1 \frac{1-s-\gamma(\eta-s)}{1-\gamma\eta}h(s)ds<\infty$, then $y(0)=A$. If $\int_0^1 \frac{1-s-\gamma(\eta-s)}{1-\gamma\eta}h(s)ds=\infty$, then by (2.3) we obtain

$$y(0)=A+\lim_{t\to 0^+}\frac{\int_t^\eta \frac{1-s-\gamma(\eta-s)}{1-\gamma\eta}h(s)ds}{1/t}=A+\lim_{t\downarrow 0^+}t^2h(t)\frac{1-\gamma\eta+t(\gamma-1)}{1-\gamma\eta}=A.$$

We have also

$$\begin{aligned}
&y(1)-\gamma y(\eta)\\
&=\frac{B-A(1-\gamma)}{1-\gamma\eta}+\int_0^\eta \frac{s\gamma(1-\eta)}{1-\gamma\eta}h(s)ds+\int_\eta^1 \frac{\gamma\eta(1-s)}{1-\gamma\eta}h(s)ds\\
&\quad-\gamma\Big(\frac{B-A(1-\gamma)}{1-\gamma\eta}\eta+\int_0^\eta \frac{s(1-\eta)}{1-\gamma\eta}h(s)ds+\int_\eta^1 \frac{\eta(1-s)}{1-\gamma\eta}h(s)ds\Big)=B.
\end{aligned}$$

This shows that $y(t)$ is a positive solution of (2.5), and $y\in C([0,1],[0,\infty))\cap C^2((0,1),\mathbb{R})$. ☐

Let us define an operator $\Phi: X\to X$ by

$$(\Phi u)(t)=A+\frac{B-A(1-\gamma)}{1-\gamma\eta}t+\int_0^1 G(t,s)f_1(s,u(s))ds, \tag{2.6}$$

where $X=\{u\in C([0,1],\mathbb{R})$ with the norm $\|u\|\}$ is a Banach space, with

$$\|u\|:=\sup\{|u(t)|:0\le t\le 1\}.$$

Without loss of generality, we assume that $A=B=0$.

To prove the existence of a solution to (2.4), we need the following Lemma.

Lemma 2.3. *The function Φ is continuous from X to X and $\Phi(X)$ is a compact subset of X.*

Proof. As in the proof of Lemma 2.2, from the definition of f_1 and from (2.6), we have

$$|(\Phi u)(t)|\le \int_0^1 G(t,s)|f_1(s,u(s))|ds\le \int_0^1 G(t,s)h(s)ds=y(t),\quad t\in[0,1]. \tag{2.7}$$

So we have $\Phi u\in C([0,1],\mathbb{R})\cap C^2((0,1),\mathbb{R})$, and

$$\|\Phi u\|\le\|y\|. \tag{2.8}$$

This shows that $\Phi(X)$ is a bounded subset of X.

Noting the facts that $y(0)=0$ and the continuity of $y(t)$ on $[0,1]$, we have from (2.7) that for any $\epsilon>0$, one can find a $\delta_1>0$ (independent with u) such that $0<\delta_1<1/8$ and

$$(\Phi u)(t)<\frac{\epsilon}{2},\quad t\in[0,2\delta_1]. \tag{2.9}$$

On the other hand, from (2.6), since $|f_1(s,u(s))|\le h(s)$, $s\in(0,1)$, we can obtain

$$|(\Phi u)'(t)|\le L,\quad t\in[\delta_1,1].$$

Let $\delta_2=\frac{\epsilon}{2L}$, then for $t_1,t_2\in[\delta_1,1]$, $|t_2-t_1|<\delta_2$, we have

$$|(\Phi u)(t_1)-(\Phi u)(t_2)|\le L|t_1-t_2|<\frac{\epsilon}{2}. \tag{2.10}$$

Define $\delta=\min\{\delta_1,\delta_2\}$, then using (2.9), (2.10), we obtain

$$|(\Phi u)(t_1)-(\Phi u)(t_2)|<\epsilon, \tag{2.11}$$

for $t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta$. This shows that $\{(\Phi u)(t) : u \in X\}$ is equicontinuous on $[0,1]$.

We can obtain the continuity of Φ in a similar way as above. In fact, if $u_n, u \in X$ and $\|u_n - u\| \to 0$ as $n \to \infty$, then we have

$$|(\Phi u_n)(t) - (\Phi u)(t)| \le 2\int_0^1 G(t,s)h(s)ds = 2y(t), \quad t \in [0,1], \tag{2.12}$$

Noting the facts that $y(0) = 0$ and the continuity of $y(t)$ on $[0,1]$, then for any $\epsilon > 0$, one can find a $\delta_1 > 0$ (independent of u_n) such that $0 < \delta_1 < 1/8$ and

$$|(\Phi u_n)(t) - (\Phi u)(t)| < \epsilon, \quad t \in [0, \delta_1]. \tag{2.13}$$

On the other hand, from the continuity of f_1, one has

$$|(\Phi u_n)(t) - (\Phi u)(t)| \to 0, \quad t \in [\delta_1, 1], \tag{2.14}$$

as $n \to \infty$. This together with (2.13) implies that $\|\Phi u_n - \Phi u\| \to 0$ as $n \to \infty$. Therefore, $\Phi : X \to X$ is completely continuous. The proof is complete. □

Lemma 2.4. *Let $u(t)$ be a solution to* (2.4). *Then $\alpha(t) \le u(t) \le \beta(t)$ for all $t \in [0,1]$; i.e., $u(t)$ is a solution to* (2.1).

Proof. We first prove that $u(t) \le \beta(t)$ on $[0,1]$. Let $x(t) := u(t) - \beta(t)$. Assume that $u(t) > \beta(t)$ for some $t \in [0,1]$. Since $u(0) = 0 \le \beta(0)$, it follows that

$$x(0) \le 0, \quad x(1) = u(1) - \beta(1) \le \gamma u(\eta) - \gamma\beta(\eta) = \gamma x(\eta).$$

Let $\sigma \in (0,1]$ be such that $x(\sigma) = \max_{t\in[0,1]} x(t)$. Then $x(\sigma) > 0$.

Case(i): $\sigma \in (0,1)$. So there exists an interval $(a,\sigma] \subset (0,1)$ such that $x(t) > 0$ in $(a,\sigma]$, and

$$x(a) = 0, \quad x(\sigma) = \max_{t\in[0,1]} x(t) > 0, \quad x'(\sigma) = 0.$$

For $t \in (a,\sigma]$ we have that $f_1(t,u(t)) = f(t,\beta(t))$ and therefore

$$u''(t) + f_1(t,u(t)) = u''(t) + f(t,\beta(t)) = 0 \quad \text{for all } t \in (a,\sigma].$$

On the other hand, as β is an upper solution for (2.1), we have

$$\beta''(t) + f(t,\beta(t)) \le 0 \quad \text{for all } t \in (a,\sigma].$$

Thus, we obtain $u''(t) \ge \beta''(t)$ for all $t \in (a,\sigma]$, and hence, $x''(t) \ge 0$. Then $x'(t) \le 0$ on $(a,1]$ which is a contradiction.

Case(ii): $\sigma = 1$. So there exists $(a,1] \subset (0,1]$ such that

$$x(a) = 0, \quad x(1) = \max_{t\in[0,1]} x(t), x(1) - \gamma x(\eta) \le 0.$$

In the same way as in Case(i), we can obtain that $x(t) > 0, x''(t) \ge 0, t \in (a,1]$. Since $x(\eta) \ge \frac{1}{\gamma}x(1) > 0$, then $\eta > a$. □

Consider the three-point boundary-value problem

$$\begin{gathered} x'' = h(t) > 0, \quad a < t < 1, \\ x(a) = 0, \quad x(1) - \gamma x(\eta) = b_1 \le 0. \end{gathered} \tag{2.15}$$

Then this equation has a unique solution $x(t) \in C([a,\sigma],[0,\infty)) \cap C^2((a,1),\mathbb{R})$, which can be represented as

$$x(t) = \frac{b_1(t-a)}{1-a-\gamma(\eta-a)} - \int_a^1 G_{[a,1]}(t,s)h(s)ds, \quad a \le t \le 1,$$

where $G_{[a,1]}(t,s)$ is the Green's function of the boundary-value problem $-y''=0$, $y(a)=0$, $y(1)=\gamma y(\eta)$, which is explicitly given by: when $a\le s\le \eta$,

$$G_{[a,1]}(t,s)=\begin{cases}\frac{(s-a)[1-t-\gamma(\eta-t)]}{1-a-\gamma(\eta-a)}, & s\le t,\\ \frac{(t-a)[1-s-\gamma(\eta-s)]}{1-a-\gamma(\eta-a)}, & s>t;\end{cases}$$

when $\eta<s\le 1$,

$$G_{[a,1]}(t,s)=\begin{cases}\frac{(s-a)(1-t)+\gamma(t-s)(\eta-a)}{1-a-\gamma(\eta-a)}, & s\le t,\\ \frac{(t-a)(1-s)}{1-a-\gamma(\eta-a)}; s>t.\end{cases}$$

Since $0<\gamma<\frac{1}{\eta}<\frac{1-a}{\eta-a}$, then $G_{[a,1]}(t,s)\ge 0$, and hence $x(t)\le 0$ on $[a,1]$, which is a contradiction. In very much the same way, we can prove that $u(t)\ge\alpha(t)$ on $[0,1]$.

3. Main results

Let $g:[0,1]\times(0,\infty)\to\mathbb{R}$ be a continuous function and $q\in C((0,1],\mathbb{R}_0^+)$. Consider the three-point boundary-value problem

$$\begin{gathered}u''+q(t)g(t,u)=0,\quad t\in(0,1),\ \eta\in(0,1),\ \gamma\in(0,1]\\ u(0)=0,\quad u(1)=\gamma u(\eta).\end{gathered}\tag{3.1}$$

Theorem 3.1. *Assume that*

- (H1) $|g(t,x)|\le F(x)+Q(x)$ *on* $[0,1]\times(0,\infty)$ *with* $F>0$ *continuous and nonincreasing on* $(0,\infty)$, $Q\ge 0$ *continuous on* $[0,\infty)$, *and* $\frac{Q}{F}$ *nondecreasing on* $(0,\infty)$;
- (H2) *there exist constants* $L>0$ *and* $\varepsilon>0$ *such that* $g(t,x)>L$ *for all* $(t,x)\in[0,1]\times(0,\varepsilon]$, *and* $F(x)>L$, $x\in(0,\varepsilon]$;
- (H3)

$$\lim_{t\to 0^+}t^2q(t)=0,\quad \int_0^1 tq(t)dt<\infty,\tag{3.2}$$

$$\sup_{c\in(0,\infty)}\Big(\frac{1}{1+\frac{Q(c)}{F(c)}}\int_0^c\frac{du}{F(u)}\Big)>b_0,\tag{3.3}$$

where $b_0=\int_0^1 rq(r)dr$.

Then (3.1) *has at least one solution* $u\in C([0,1],[0,\infty))\cap C^2((0,1),\mathbb{R})$ *with* $u(t)>0$ *on* $(0,1]$.

From Lemma 2.2, we obtain the following result.

Lemma 3.2. *There exists an unique solution* $W\in C([0,1],[0,\infty))\cap C^2((0,1),\mathbb{R})$, *with* $W(t)>0$ *on* $(0,1]$ *to the problem*

$$\begin{gathered}W''+q(t)=0,\quad 0<t<1,\\ W(0)=0,\quad W(1)=\gamma W(\eta).\end{gathered}\tag{3.4}$$

Choose $M>0$, $\delta>0$ ($\delta<M$) such that

$$\frac{1}{1+\frac{Q(M)}{F(M)}}\int_\delta^M\frac{du}{F(u)}>b_0.\tag{3.5}$$

Let $n_0 \in \{1, 2, \dots\}$ be chosen so that $1/n_0 < \min\{\varepsilon - m\|W\|, \delta\}$, where W is the solution of (3.4), and $0 < m < \min\{L, \varepsilon/\|W\|, 1\}$ is chosen and fixed. Let $N^+ = \{n_0, n_0 + 1, \dots\}$.

We first show that the boundary-value problem

$$\begin{gathered} u'' + q(t)g(t, u) = 0, \quad 0 < t < 1, \\ u(0) = \frac{1}{n}, \quad u(1) - \gamma u(\eta) = \frac{1-\gamma}{n}, \quad n \in N^+ \end{gathered} \tag{3.6}$$

has a solution u_n for each $n \in N^+$ with $u_n(t) \geq \frac{1}{n}$ for $t \in [0, 1]$ and $\|u_n\| < M$.

We have the following Claim

Claim: Let $\alpha_n(t) = mW(t) + \frac{1}{n}$, $t \in [0, 1]$, then $\alpha_n(t)$ is a (strict) lower solution for problem (3.6).

Proof. For the choice of m and n, we have $mW(t) + \frac{1}{n} \leq m\|W\| + \frac{1}{n_0} < \varepsilon$, then from (H2),

$$g(t, mW(t) + \frac{1}{n}) > L > m \quad \text{for all } t \in [0, 1].$$

Then we obtain

$$\begin{aligned} \alpha_n''(t) + q(t)g(t, \alpha_n(t)) &= (mW(t) + \frac{1}{n})'' + q(t)g(t, mW(t) + \frac{1}{n}) \\ &= mW''(t) + q(t)g(t, mW(t) + \frac{1}{n}) \\ &= q(t)(g(t, mW(t) + \frac{1}{n}) - m) > 0, \quad 0 < t < 1. \end{aligned}$$

We obtain $\alpha_n(0) = mW(0) + \frac{1}{n} = \frac{1}{n}$, and

$$\begin{aligned} \alpha_n(1) - \gamma\alpha_n(\eta) &= mW(1) + \frac{1}{n} - \gamma(mW(\eta) + \frac{1}{n}) \\ &= m(W(1) - \gamma W(\eta)) + \frac{1-\gamma}{n} = \frac{1-\gamma}{n}. \end{aligned}$$

Thus the proof of Claim is complete. □

To find the upper solution of (3.6), we consider the problem

$$\begin{gathered} u'' + q(t)F(u)(1 + \frac{Q(M)}{F(M)}) = 0, \quad 0 < t < 1, \\ u(0) = \frac{1}{n}, u(1) - \gamma u(\eta) = \frac{1-\gamma}{n}. \end{gathered} \tag{3.7}$$

To show that this problem has a solution we study

$$\begin{gathered} u'' + q(t)F^*(u)(1 + \frac{Q(M)}{F(M)}) = 0, \quad 0 < t < 1, \\ u(0) = \frac{1}{n}, u(1) - \gamma u(\eta) = \frac{1-\gamma}{n}, \end{gathered} \tag{3.8}$$

where

$$F^*(u) = \begin{cases} F(u), & u \geq 1/n, \\ F(\frac{1}{n}), & u < 1/n. \end{cases}$$

Then $F^*(u) \leq F(u)$ for $u > 0$.

In the same way as in the Claim, we can easily prove $\alpha_n(t) = \frac{1}{n} + mW(t)$ is also a (strict) lower solution of (3.8).

By Lemma 2.2, let $\beta_n^0 \in C([0,1],\mathbb{R}) \cap C^2((0,1),\mathbb{R})$ be the unique solution of the boundary-value problem

$$\begin{gathered} u'' + q(t)F(\alpha_n(t))(1+\frac{Q(M)}{F(M)}) = 0, \quad 0<t<1, \\ u(0) = \frac{1}{n}, \quad u(1) - \gamma u(\eta) = \frac{1-\gamma}{n}. \end{gathered} \tag{3.9}$$

Since β_n^0 is a solution of this equation,

$$\begin{gathered} {\beta_n^0}'' + q(t)F(\alpha_n(t))(1+\frac{Q(M)}{F(M)}) = 0, \quad 0<t<1, \\ \beta_n^0(0) = \frac{1}{n}, \quad \beta_n^0(1) - \gamma\beta_n^0(\eta) = \frac{1-\gamma}{n}. \end{gathered}$$

On the other hand, as α_n is a lower solution of (3.8), and $\alpha_n \ge 1/n$, we have

$$\begin{gathered} {\alpha_n}'' + q(t)F(\alpha_n(t))(1+\frac{Q(M)}{F(M)}) \ge 0, \quad 0<t<1, \\ \alpha_n(0) = \frac{1}{n}, \quad \alpha_n(1) - \gamma\alpha_n(\eta) = \frac{1-\gamma}{n}. \end{gathered}$$

So we obtain $\alpha_n(t) \le \beta_n^0(t)$ for $t \in [0,1]$. Thus

$$\begin{aligned} &{\beta_n^0}'' + q(t)F^*(\beta_n^0)(1+\frac{Q(M)}{F(M)}) \\ &= -q(t)F(\alpha_n)(1+\frac{Q(M)}{F(M)}) + q(t)F(\beta_n^0)(1+\frac{Q(M)}{F(M)}) \\ &= q(t)(1+\frac{Q(M)}{F(M)})(F(\beta_n^0) - F(\alpha_n)) \le 0, \end{aligned}$$

so that β_n^0 is an upper solution for problem (3.8).

If we now take $\alpha_n^0 \equiv \alpha_n$, we have that α_n^0 and β_n^0 are, respectively, a lower and an upper solution of (3.8) with $\alpha_n^0(t) \le \beta_n^0(t)$, for all $t \in [0,1]$. So by the Lemma 2.4, we know that there exists a solution $\beta_n \in C([0,1],\mathbb{R}) \cap C^2((0,1),\mathbb{R})$ of (3.8) such that

$$\alpha_n(t) = \alpha_n^0(t) \le \beta_n(t) \le \beta_n^0(t), \quad \forall t \in [0,1].$$

Now we claim that $\|\beta_n\| < M$. Suppose this is false; i.e., suppose $\|\beta_n\| \ge M$. Since $\beta_n(1) - \frac{1}{n} = \gamma(\beta_n(\eta) - \frac{1}{n}) \le \beta_n(\eta) - \frac{1}{n}$, $\beta_n''(t) \le 0$ on (0,1) and $\beta_n \ge \frac{1}{n}$ on $[0,1]$, there exists $\sigma \in (0,1)$ with $\beta_n'(t) \ge 0$ on $(0,\sigma)$, $\beta_n'(t) \le 0$ on $(\sigma,1)$ and $\beta_n(\sigma) = \|\beta_n\|$.

Then for $z \in (0,1)$, we have

$$-\beta_n''(z) \le F(\beta_n(z))(1+\frac{Q(M)}{F(M)})q(z). \tag{3.10}$$

Integrate from $t (0 < t \le \sigma)$ to σ to obtain

$$\beta_n'(t) \le (1+\frac{Q(M)}{F(M)}) \int_t^\sigma F(\beta_n(z))q(z)dz;$$

so we have

$$\frac{\beta_n'(t)}{F(\beta_n(t))} \le (1+\frac{Q(M)}{F(M)}) \int_t^\sigma q(z)dz\,.$$

Then integrate from 0 to σ to obtain

$$\int_{\frac{1}{n}}^{\beta_n(\sigma)} \frac{dy}{F(y)} \le \big(1+\frac{Q(M)}{F(M)}\big)\int_0^\sigma \Big(\int_t^\sigma q(z)dz\Big)dt = \big(1+\frac{Q(M)}{F(M)}\big)\int_0^\sigma tq(t)dt.$$

Consequently

$$\int_\delta^M \frac{dy}{F(y)} \le \big(1+\frac{Q(M)}{F(M)}\big)\int_0^1 tq(t)dt. \tag{3.11}$$

This contradicts (3.5) and consequently $\|\beta_n\| < M$.

It follows from the fact $\beta_n \ge 1/n$, we can obtain β_n is a solution of (3.7) also. Since F is nonincreasing on $(0,\infty)$, similar to the proof of Lemma 2.4, we can obtain the uniqueness of solutions to (3.7).

Next we show that β_n is an upper solution of (3.6). Observe that

$$|g(t,x)| \le F(x) + Q(x) \quad \text{on } [0,1]\times(0,\infty).$$

We have

$$\begin{aligned}
\beta_n''(t) + q(t)g(t,\beta_n(t)) &\le -q(t)F(\beta_n(t))\Big(1+\frac{Q(M)}{F(M)}\Big) + q(t)|g(t,\beta_n(t))|\\
&\le q(t)F(\beta_n(t))\Big(\frac{Q(\beta_n(t))}{F(\beta_n(t))} - \frac{Q(M)}{F(M)}\Big) \le 0, \quad t\in(0,1).
\end{aligned}$$

Thus β_n is an upper solution for problem (3.6).

This together with the Claim yields that α_n and β_n are, respectively, a lower and an upper solution for (3.6) with $\alpha_n \le \beta_n$ for all $t\in[0,1]$. So we conclude (3.6) has a solution $u_n \in C([0,1],\mathbb{R})\cap C^2((0,1),\mathbb{R})$ such that

$$mW(t) + \frac{1}{n} = \alpha_n(t) \le u_n(t) \le \beta_n(t) \le M, \forall t\in[0,1].$$

Consider now the pointwise limit

$$z(t) := \lim_{n\to+\infty} u_n(t), \quad \forall t\in[0,1]. \tag{3.12}$$

Let $e=[a,1]\subset(0,1]$, Let $t_n\in(a,1)$ such that $u_n'(t_n) = \big(u_n(1)-u_n(a)\big)/(1-a)$. We obtain

$$u_n'(t) = \frac{u_n(1)-u_n(a)}{1-a} + \int_t^{t_n} q(s)g(s,u_n(s)ds, \quad t\in e.$$

Since $mW(t)\le u_n(t)\le M$, then we have

$$|u_n'(t)| \le \frac{2M}{1-a} + \Big(1+\frac{Q(M)}{F(M)}\Big)\int_a^1 q(t)F(mW(t))dt := C(a,1), \quad t\in e. \tag{3.13}$$

Set $v_n = \max_{t\in e}|u_n'(t)|$, which implies v_n is bounded. That means $u_n'(t)$ is bounded on e.

Then, by the Ascoli-Arzela theorem, it is standard to conclude that $z(t)$ is a solution of (3.1) on the interval $e=[a,1]$. Since e is arbitrary, we find that

$$z\in C((0,1],[0,\infty))\cap C^2((0,1),\mathbb{R}), \quad \text{and} \quad z''(t)+q(t)g(t,z(t))=0, \quad t\in(0,1).$$

Also, we have

$$z(0) = \lim_{n\to+\infty}\frac{1}{n} = 0, \quad z(1)-\gamma z(\eta) = \lim_{n\to+\infty}\frac{1-\gamma}{n} = 0\,.$$

The same argument as in [11] works, we can prove the continuity of $z(t)$ at $t=0$ and $t=1$. The proof is complete.

By essentially the same argument as in Theorem 3.1 and [2, Theorem 4.2], we have the following result.

Theorem 3.3. *Assume that*

(H1*) *for any* $r > 0$ *there is* $h_r \in C((0,1],(0,\infty))$*:* $|q(t)g(t,x)| \leq h_r(t)$ *for all* $(t,x) \in (0,1] \times [r,\infty)$*, such that*

$$\lim_{t\to 0^+} t^2 h_r(t) = 0, \quad \int_0^1 t h_r(t)dt < +\infty;$$

(H2*) *there exist constants* $L > 0$ *and* $\varepsilon > 0$ *such that* $g(t,x) > L$ *for all* $(t,x) \in [0,1] \times (0,\varepsilon]$*.*

Then (3.1) *has at least one solution* $u \in C([0,1],[0,\infty) \cap C^2((0,1),\mathbb{R})$*. Moreover, if* $g(t,x)$ *is non-increasing in* $x > 0$*, then the solution is unique.*

4. An example

Consider the singular boundary-value problem

$$\begin{gathered} u'' + \sigma t^{-m}(u^{-\alpha} + u^{\beta} - T\sin(8\pi t)) = 0, \quad t \in (0,1) \\ u(0) = 0, \quad zu(1) = \gamma u(\eta), \quad \eta \in (0,1),\ \gamma \in (0,1] \end{gathered} \tag{4.1}$$

with $0 \leq m < 2$, $\sigma > 0$, $\alpha > 0$, $\beta \geq 0$. Set

$$F(u) = u^{-\alpha}, \quad Q(u) = u^{\beta} + 1, \quad q(t) = \sigma t^{-m},$$

$$b_0 = \int_0^1 rq(r)dr = \frac{\sigma}{2-m}.$$

Applying Theorem 3.1, we find that (4.1) has a positive solutions if

$$\sigma < (2-m) \sup_{x\in(0,\infty)} \frac{x^{\alpha+1}}{(\alpha+1)(1 + x^{\alpha} + x^{\alpha+\beta})}. \tag{4.2}$$

Obviously, (H1)-(H3) in Theorem 3.1 are satisfied. Thus, (4.1) has a solution $u \in C([0,1],[0,\infty) \cap C^2((0,1),\mathbb{R})$ with $u > 0$ on $(0,1]$.

We remark that if $0 \leq \beta < 1$, then (4.1) has at least one positive solution for all $\sigma > 0$, since the right-hand side of (4.2) is infinity.

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