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EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS FOR A BVP WITH A P-LAPLACIAN ON THE HALF-LINE

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Abstract. In this work, we consider the second order multi-point boundary-value problem with a p-Laplacian

$$(\rho(t)\Phi_p(x'(t)))' + f(t, x(t), x'(t)) = 0, \quad t \in [0, +\infty),$$
$$x(0) = \sum_{i=1}^{m} \alpha_i x(\xi_i), \quad \lim_{t\to\infty} x(t) = 0\,.$$

By applying a nonlinear alternative theorem, we establish existence and uniqueness of solutions on the half-line. Also a uniqueness result for positive solutions is discussed when f depends on the first-order derivative. The emphasis here is on the one dimensional p-Laplacian operator.

1. Introduction

In recent years, a great deal of work has been done in the study of multi-point boundary-value problems which arise in different areas of applied mathematics and physics. The study of multi-point boundary-value problems for linear second order differential equations was initiated by Il'in and Moiseev [6]. Since then, more general nonlinear multi-point boundary-value problems were studied by several authors, see [4, 5, 7, 8, 10] and the references cited therein.

For a finite interval, He and Ge [5] used the Leggett-Williams fixed point theorem to the following second-order three-point boundary-value problem

$$\begin{gathered} u''(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \\ u(0) = 0, \quad u(1) = \xi u(\eta), \end{gathered} \tag{1.1}$$

where $\xi > 0$, $0 < \eta < 1$ and $\xi\eta < 1$. Du, Xue and Ge [4] applied Leray-Schauder degree theory and lower and upper solutions method to (1.1) when f does not depend on the first-order derivative explicitly, and obtained the existence of at least three solutions. Nonlinear differential equation on finite interval

$$(\Phi_p(u'))' + f(t, u, u') = 0, \quad t \in (0, 1) \tag{1.2}$$

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with different boundary conditions has been studied extensively. Ma, Du and Ge [7] obtained some criteria for the existence of monotone positive solutions to the equation (1.2) with boundary condition $u'(0) = \sum_{i=1}^{n} \alpha_i u'(\xi_i), \quad u(1) = \sum_{i=1}^{n} \beta_i u(\xi_i)$.

For a infinite interval, in a monograph [1] Agarwal and O'Regan studied two-point boundary-value problems on the half-line and obtained a series of interesting results. Inspired by [1], many authors devoted the study of two-point and multi-point boundary-value problems on the half-line, see [2, 10, 11, 12, 13]. Tian and Ge [10] established the existence of at least three positive solutions for the problem

$$\begin{gathered}(\rho(t)x'(t))' + f(t, x(t), x'(t)) = 0, \quad t \in I = [0, +\infty), \\ x(0) = \alpha x(\xi), \quad \lim_{t\to\infty} x(t) = 0,\end{gathered} \tag{1.3}$$

where $\rho \in C[0, +\infty) \cap C^1(0, +\infty), \rho(t) > 0$ for $t \in [0, +\infty)$, $\int_0^\infty \frac{1}{\rho(t)} dt < \infty$, $\alpha \geq 0$, $0 \leq \xi < \infty$, $f : I \times I \times R \to I$.

However, in [4, 5, 8, 10], the one dimensional p-Laplacian operator is not involved. Ma, Du and Ge [7] studied only boundary-value problem on finite interval and the nonlinear term does not depend on the first order derivative explicitly. Moreover, only existence results were established in the above literature. By so far, very few existence and uniqueness results were established for multi-point boundary-value problem with a p-Laplacian on the half-line.

Motivated by the above results, we consider the existence of positive solutions for multi-point boundary-value problem

$$\begin{gathered}(\rho(t)\Phi_p(x'(t)))' + f(t, x(t), x'(t)) = 0, \quad t \in I = [0, +\infty), \\ x(0) = \sum_{i=1}^{m} \alpha_i x(\xi_i), \quad \lim_{t\to\infty} x(t) = 0.\end{gathered} \tag{1.4}$$

where $\Phi_p(s) = |s|^{p-2}s$, $p > 1$, $\xi_i \in (0, \infty)$, $i = 1, 2, \dots, m$, and α_i, ρ, f satisfy

- (H1) $0 \leq \alpha_i < 1, (i = 1, 2, \dots, m)$ satisfies $0 \leq \sum_{i=1}^{m} \alpha_i < 1$,
- (H2) $\rho \in C[0, +\infty) \cap C^1(0, +\infty), \rho(t) > 0$ for $t \in [0, +\infty)$, and non-decreasing on $[0, +\infty)$, $\int_0^\infty \Phi_p^{-1}(1/\rho(t))dt < \infty$,
- (H3) $f : [0, +\infty) \times [0, +\infty) \times R \to [0, +\infty)$ is an L^1-Carathédory function, that is
  (i) $t \to f(t, x, y)$ is measurable for any $(x, y) \in [0, +\infty) \times R$,
  (ii) $(x, y) \to f(t, x, y)$ is continuous for a.e. $t \in I$,
  (iii) for each $r_1, r_2 > 0$ there exists $l_{r_1,r_2} \in L^1[0, \infty)$ such that $|x| \leq r_1, |y| \leq r_2$ imply $|f(t, x, y)| \leq l_{r_1,r_2}(t)$ for almost all $t \in I$.

Furthermore, when f does not depend on the first-order derivative explicitly, we establish the uniqueness result of positive solutions. We note that when $p = 2$, $\xi_1 = \xi_2 = \dots = \xi_m$, problem (1.4) reduces to (1.3).

Definition 1.1. A function x is said to be a positive solution of boundary-value problem (1.4), if $x \in C^1(I, I), (\Phi_p(x'(t)))' \in L^1(I), x(t) \geq 0$, and x satisfies (1.4) for $t \in I$.

By using fixed point theorem on cone, we establish the existence of positive solutions for problem (1.4). In order to apply fixed point theory, it is very important to transform BVP into an equivalent integral equation. For $p = 2$, the process is easy to be realized since the Green's function exists, however, for $p \neq 2$, it is impossible since the differential operator $(\Phi_p(x'))'$ is nonlinear. Besides, nonlinearity

f depends on the first-order derivative, which brings about much trouble, such as, the verification of the compactness and continuity of the operator.

In this paper, we will need the following lemmas.

Lemma 1.2 (Nonlinear alternative [9]). *Let C be a convex subset of a normed linear space E, and U be an open subset of C, with $p^* \in U$. Then every compact, continuous map $N : \overline{U} \to C$ has at least one of the following two properties:*

(a) *N has a fixed point;*
(b) *there is an $x \in \partial U$, with $x = (1 - \overline{\lambda})p^* + \overline{\lambda} Nx$ for some $0 < \overline{\lambda} < 1$.*

Lemma 1.3 ([3, 9]). *Let $C_l([0,\infty), R) = \{x \in C([0,\infty)) : \lim_{t\to\infty} x(t)$ exists$\}$, then subset M of C_L is precompact if the following conditions hold:*

(a) *M is bounded in C_l;*
(b) *the functions belonging to M are locally equicontinuous on any interval of $[0,\infty)$;*
(c) *the functions from M are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|x(t) - x(\infty)| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $x \in M$.*

2. Related Lemmas

We consider the Banach space $E = \{x \in C^1(I) : \lim_{t\to\infty} x(t) = 0\}$ equipped with the norm

$$\|x\| = \max\{\|x\|_0, \|x'\|_0\}, \quad \|x\|_0 = \sup_{t\in I} |x(t)|.$$

Let $P = \{x \in E : x(t) \geq 0, t \in I\}$.

Let $x \in P$. Suppose that x is a solution of BVP

$$\begin{gathered}(\rho(t)\Phi_p(x'(t)))' + f(t, x(t), x'(t)) = 0, \quad t \in I = [0, +\infty), \\ x(0) = \sum_{i=1}^{m} \alpha_i x(\xi_i), \quad \lim_{t\to\infty} x(t) = 0.\end{gathered} \tag{2.1}$$

Then

$$\begin{gathered}\Phi_p(x'(t)) = \frac{1}{\rho(t)}\Big(\rho(0)A_x - \int_0^t f(r, x(r), x'(r))dr\Big), \\ x(t) = x(0) + \int_0^t \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)A_x - \int_0^s f(r, x(r), x'(r))dr\Big)\Big]ds.\end{gathered}$$

Since x satisfies $x(0) = \sum_{i=1}^m \alpha_i x(\xi_i)$, by computing, one has

$$x(0) = \frac{1}{1 - \sum_{i=1}^m \alpha_i} \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)A_x - \int_0^s f(r, x(r), x'(r))dr\Big)\Big]ds.$$

Thus

$$\begin{aligned}x(t) = &\frac{1}{1 - \sum_{i=1}^m \alpha_i} \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)A_x - \int_0^s f(r, x(r), x'(r))dr\Big)\Big]ds \\ &+ \int_0^t \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)A_x - \int_0^s f(r, x(r), x'(r))dr\Big)\Big]ds.\end{aligned}$$

The second boundary condition $\lim_{t\to\infty} x(t) = 0$ means that A_x satisfies

$$\begin{aligned}
&\frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)A_x - \int_0^s f(r,x(r),x'(r))dr\Big)\Big]ds \\
&+\int_0^\infty \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)A_x - \int_0^s f(r,x(r),x'(r))dr\Big)\Big]ds = 0.
\end{aligned} \tag{2.2}$$

Lemma 2.1. *For $x \in P$, there exists a unique $A_x \in \big(0, \frac{1}{\rho(0)}\int_0^\infty f(r,x(r),x'(r))dr\big)$ satisfying* (2.2).

Proof. Let $x \in P$. Define

$$\begin{aligned}
H_x(c) = &\frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)c - \int_0^s f(r,x(r),x'(r))dr\Big)\Big]ds \\
&+\int_0^\infty \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)c - \int_0^s f(r,x(r),x'(r))dr\Big)\Big]ds.
\end{aligned}$$

Then $H_x \in C(R,R)$ is increasing and $H_x(0) < 0$. Let

$$\bar{c} = \frac{1}{\rho(0)}\int_0^\infty f(r,x(r),x'(r))dr,$$

then $H_x(\bar{c}) > 0$. By mean value theorem, there exists $A_x \in (0,\bar{c})$ satisfying $H_x(A_x) = 0$. Since $H_x(c)$ is increasing about c, there exists a unique A_x satisfying $H_x(A_x) = 0$. □

Lemma 2.2. *The function $A_x : P \to [0,+\infty)$ is continuous on x.*

Proof. Let $\{x_n\} \in P$ with $x_n \to x_0 \in P$ as $n \to \infty$ in P. Let $\{A_{x_n}\}(n = 1,2,\dots,m)$ be constants decided by equation (2.2) corresponding to $x_n(n = 1,2,\dots,m)$. Since $x_n \to x_0$ in P as $n \to \infty$, there exists an $M > 0$ such that $\|x_n\| \le M$. The fact f is an L^1-Carathédory function means

$$\int_0^\infty |f(r,x_n(r),x_n'(r)) - f(r,x_0(r),x_0'(r))|dr \le 2\int_0^\infty l_{M,M}(r)dr < \infty;$$

that is,

$$\int_0^\infty f(r,x_n(r),x_n'(r))dr \le \int_0^\infty f(r,x_0(r),x_0'(r))dr + 2\int_0^\infty l_{M,M}(r)dr < \infty.$$

So

$$\begin{aligned}
A_{x_n} &\in \Big(0, \frac{1}{\rho(0)}\int_0^\infty f(r,x_n(r),x_n'(r))dr\Big) \\
&\subseteq \Big(0, \frac{1}{\rho(0)}\Big(\int_0^\infty f(r,x_0(r),x_0'(r))dr + 2\int_0^\infty l_{M,M}(r)dr\Big)\Big),
\end{aligned}$$

which means that $\{A_{x_n}\}$ is bounded.

Suppose that $\{A_{x_n}\}$ does not converge to A_{x_0}. Then there exist two subsequences $\{A^{(1)}_{x_{n_k}}\}$ and $\{A^{(2)}_{x_{n_k}}\}$ of $\{A_{x_{n_k}}\}$ with $A^{(1)}_{x_{n_k}} \to c_1$ and $A^{(2)}_{x_{n_k}} \to c_2$ since $\{A_{x_n}\}$ is bounded, but $c_1 \neq c_2$. By the construction of A_{x_n}, $(n = 1, 2, \dots)$, we have

$$\begin{aligned}&\frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)A^{(1)}_{x_{n_k}} - \int_0^s f(r, x^{(1)}_{n_k}(r), x^{(1)\prime}_{n_k}(r))dr\Big)\Big]ds\\ &+\int_0^\infty \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)A^{(1)}_{x_{n_k}} - \int_0^s f(r, x^{(1)}_{n_k}(r), x^{(1)\prime}_{n_k}(r))dr\Big)\Big]ds = 0.\end{aligned}$$

Let $n_k \to \infty$, using Lebesgue's dominated convergence theorem, the above equality implies

$$\begin{aligned}&\frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)c_1 - \int_0^s f(r, x_0(r), x_0'(r))dr\Big)\Big]ds\\ &+\int_0^\infty \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)c_1 - \int_0^s f(r, x_0(r), x_0'(r))dr\Big)\Big]ds = 0.\end{aligned}$$

Since $\{A_{x_n}\}$ $(n = 1, 2, \dots)$ is unique with respect to x_n, we get $c_1 = A_{x_0}$. Similarly, $c_2 = A_{x_0}$. Thus $c_1 = c_2$, a contradiction. So, for any $x_n \to x_0$, one has $A_{x_n} \to A_{x_0}$, which means $A_x : P \to R$ is continuous. □

Define the operator T on P as

$$\begin{aligned}&Tx(t)\\ &= \frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)A_x - \int_0^s f(r, x(r), x'(r))dr\Big)\Big]ds\\ &\quad+\int_0^t \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)A_x - \int_0^s f(r, x(r), x'(r))dr\Big)\Big]ds,\end{aligned} \tag{2.3}$$

where A_x is defined in (2.2) corresponding to x. By Lemma 2.2, we know that T is well defined. The fixed point $x \in P$ of the operator T is just a positive solution of (1.4).

Lemma 2.3. *The operator $T : P \to P$ is completely continuous.*

Proof. (1) First we show that the operator T maps P to P. By the construction of T, there exists $\tau \in (0, \infty)$ such that

$$Tx(t) \text{ is increasing for } t \in [0, \tau] \text{ and decreasing for } t \in [\tau, \infty). \tag{2.4}$$

If we show $Tx(0) \geq 0$, then $Tx(t) \geq 0, t \in I$ since (2.4) and $\lim_{t\to\infty} Tx(t) = 0$ hold. For this, we assume that $Tx(0) < 0$. Since $\lim_{t\to\infty} Tx(t) = 0$ and (2.4) holds, there exists $t_0 > 0$ such that $Tx(t) \geq 0, t \in [t_0, \infty)$. Without loss of generality, we assume there exists $i_0 \in \{1, 2, \dots, m\}$ such that $Tx(\xi_i) < 0, i = 1, \dots, i_0$ and $Tx(\xi_i) \geq 0, i = i_0 + 1, \dots, m$. Then

$$Tx(0) = \sum_{i=1}^m \alpha_i Tx(\xi_i) \geq \sum_{i=1}^{i_0} \alpha_i Tx(\xi_i) \geq \sum_{i=1}^{i_0} \alpha_i Tx(0).$$

So $\sum_{i=1}^{i_0} \alpha_i \geq 1$, a contradiction. Thus, $Tx(0) \geq 0$ and so $Tx(t) \geq 0, t \in I$.

(2) Next we show that T is continuous on P. From the continuity of f and A_x, the result follows.

(3) Next we show that T is relatively compact. Given a bounded set $D \subseteq P$. Then, there exists $M > 0$ such that $D \subseteq \{x \in P : \|x\| \le M\}$. For any $x \in D$, we have

$$\int_0^\infty f(t, x(t), x'(t))dt \le \int_0^\infty l_{M,M}(t)dt := L.$$

Thus $|A_x| \le \frac{L}{\rho(0)}$. Therefore,

$$\|Tx\|_0 \le \Phi_p^{-1}(2L)\frac{\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \Phi_p^{-1}\big(\frac{1}{\rho(s)}\big)ds}{1-\sum_{i=1}^m \alpha_i} + \Phi_p^{-1}(2L)\int_0^\infty \Phi_p^{-1}\big(\frac{1}{\rho(s)}\big)ds < \infty.$$
$$\|(Tx)'\|_0 \le \Phi_p^{-1}(2L)\sup_{t\in I}\Phi_p^{-1}\big(\frac{1}{\rho(t)}\big).$$

Since the condition (H2) holds, one has $\sup_{t\in I}\Phi_p^{-1}\big(\frac{1}{\rho(t)}\big) < \infty$, which means that $\|(Tx)'\|_0 < \infty$. So, $\{TD(t)\}$, $\{(TD)'(t)\}$ are bounded. Besides, $\{TD(t)\}$ is equicontinuous. Now we shall show that $\{(TD)'(t)\}$ is local equi-continuous on I. For any $K > 0$, $t_1, t_2 \in [0, K]$ and $x \in D$, then

$$\begin{aligned}
&|\Phi_p((Tx)'(t_1)) - \Phi_p((Tx)'(t_2))|\\
&= \Big|\frac{1}{\rho(t_1)}\Big(\rho(0)A_x - \int_0^{t_1} f(r, x(r), x'(r))dr\Big)\\
&\quad - \frac{1}{\rho(t_2)}\Big(\rho(0)A_x - \int_0^{t_2} f(r, x(r), x'(r))dr\Big)\Big|\\
&\le \Big|\frac{1}{\rho(t_1)} - \frac{1}{\rho(t_2)}\Big| \times \Big|\rho(0)A_x - \int_0^{t_1} f(s, x(s), x'(s))ds\Big|\\
&\quad + \frac{1}{\rho(t_2)}\Big|\int_{t_1}^{t_2} f(r, x(r), x'(r))dr\Big|\\
&\le \Big|\frac{1}{\rho(t_1)} - \frac{1}{\rho(t_2)}\Big| \times 2L + \frac{1}{\rho(t_2)}\Big|\int_{t_1}^{t_2} l_{M,M}(r)dr\Big|.
\end{aligned}$$

Since $\int_0^\infty \frac{1}{\rho(s)}ds < \infty, \int_0^\infty l_{M,M}(r)dr < \infty$, for any $\varepsilon > 0$, there exists $\delta > 0$, such that $|\Phi_p(Tx)'(t_1) - \Phi_p(Tx)'(t_2)| < \varepsilon$ for any $|t_1 - t_2| < \delta$. Noticing $\Phi_p(x)$ is continuous about x, $|(Tx)'(t_1) - (Tx)'(t_2)| < \varepsilon'$. Therefore, $\{(TD)'(t)\}$ is equicontinuous.

(4) At last we will show that T is equiconvergent at ∞. Since $\lim_{t\to\infty} Tx(t) = 0$, one has

$$\begin{aligned}
&\lim_{t\to\infty} |(Tx)(t) - (Tx)(\infty)| \\
&= \lim_{t\to\infty} |(Tx)(t)| \\
&= \lim_{t\to\infty} \Big| \frac{1}{1-\sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)A_x - \int_0^s f(r,x(r),x'(r))dr\Big)\Big]ds \\
&\quad + \int_0^t \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)A_x - \int_0^s f(r,x(r),x'(r))dr\Big)\Big]ds\Big| \\
&= \Big| \frac{1}{1-\sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)A_x - \int_0^s f(r,x(r),x'(r))dr\Big)\Big]ds \\
&\quad + \int_0^\infty \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)A_x - \int_0^s f(r,x(r),x'(r))dr\Big)\Big]ds\Big|.
\end{aligned}$$

Since A_x satisfies (2.2), one has

$$\lim_{t\to\infty} |(Tx)(t) - (Tx)(\infty)| = 0.$$

Therefore, $T : P \to P$ is equiconvergent at ∞.

By Lemma 1.3, the operator $T : P \to P$ is completely continuous. □

3. Existence of positive solutions

For convenience, we denote

$$\Delta_1 = \max\Big\{\frac{1}{1-\sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \Phi_p^{-1}\big(\frac{1}{\rho(s)}\big)ds, \int_0^\infty \Phi_p^{-1}\big(\frac{1}{\rho(s)}\big)ds\Big\} \tag{3.1}$$

$$\Delta_2 = \sup_{t\in I} \Phi_p^{-1}\big(\frac{1}{\rho(t)}\big). \tag{3.2}$$

Theorem 3.1. *Suppose that* (H1)–(H3) *hold and* $f(t,0,0) \not\equiv 0$ *for* $t \in I$. *Also assume there exist functions* $a, b, c \in L^1([0,\infty),[0,\infty))$ *satisfying*

$$\Phi_p^{-1}(\|b\|_{L^1}) + \Phi_p^{-1}(\|c\|_{L^1}) < \min\Big\{\frac{1}{3^{q-1}\Delta_1}, \frac{1}{3^{q-1}\Delta_2}\Big\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $\|b\|_{L^1} = \int_0^\infty |b(t)|dt$, *such that*

$$f(t,x,y) \le a(t) + b(t)\Phi_p(x) + c(t)\Phi_p(|y|).$$

Then problem (1.4) *has at least one nontrivial positive solution.*

Proof. We will apply Lemma 1.2 to show this theorem. From Lemma 2.3, $T : P \to P$ is a completely continuous operator. Let

$$\begin{aligned}
M > \max\Big\{&\frac{3^{q-1}\Delta_1\Phi_p^{-1}(\|a\|_{L^1})}{1-3^{q-1}\Delta_1(\Phi_p^{-1}(\|b\|_{L^1}) + \Phi_p^{-1}(\|c\|_{L^1}))}, \\
&\frac{3^{q-1}\Delta_2\Phi_p^{-1}(\|a\|_{L^1})}{1-3^{q-1}\Delta_2(\Phi_p^{-1}(\|b\|_{L^1}) + \Phi_p^{-1}(\|c\|_{L^1}))}\Big\},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Now we define $\Omega = \{x \in P : \|x\| < M\}$. For any $x \in \partial\Omega$, $\|x\| = M$, so $\|x\|_0 \leq M, \|x'\|_0 \leq M$, by assumption of theorem and Lemma 2.1,

$$\begin{aligned}
|Tx(t)| &= \frac{1}{1-\sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)A_x - \int_0^s f(r,x(r),x'(r))dr\Big)\Big]ds \\
&\quad + \int_0^t \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)A_x - \int_0^s f(r,x(r),x'(r))dr\Big)\Big]ds \\
&\leq \frac{1}{1-\sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \Phi_p^{-1}\Big[\frac{1}{\rho(s)} \int_0^\infty f(r,x(r),x'(r))dr\Big]ds \\
&\quad + \int_0^\infty \Phi_p^{-1}\Big[\frac{1}{\rho(s)} \int_0^\infty f(r,x(r),x'(r))dr\Big]ds \\
&\leq \frac{1}{1-\sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \Phi_p^{-1}(\frac{1}{\rho(s)})ds\Phi_p^{-1}\Big(\int_0^\infty f(r,x(r),x'(r))dr\Big) \\
&\quad + \int_0^\infty \Phi_p^{-1}(\frac{1}{\rho(s)})ds\Phi_p^{-1}\Big(\int_0^\infty f(r,x(r),x'(r))dr\Big) \\
&\leq \Delta_1\Phi_p^{-1}(\|a\|_{L^1} + \|b\|_{L^1}\Phi_p(\|x\|_0) + \|c\|_{L^1}\Phi_p(\|x'\|_0)).
\end{aligned}$$

$$\begin{aligned}
|(Tx)'(t)| &= \Big|\Phi_p^{-1}\Big[\frac{1}{\rho(t)}\Big(\rho(0)A_x - \int_0^t f(r,x(r),x'(r))dr\Big)\Big]\Big| \\
&\leq \sup_{t\in I} \Phi_p^{-1}(\frac{1}{\rho(t)})\Phi_p^{-1}\Big(\int_0^\infty f(r,x(r),x'(r))dr\Big) \\
&\leq \Delta_2\Phi_p^{-1}(\|a\|_{L^1} + \|b\|_{L^1}\Phi_p(\|x\|_0) + \|c\|_{L^1}\Phi_p(\|x'\|_0)).
\end{aligned}$$

By primary inequality

$$|a_1 + a_2 + \dots + a_n|^r \leq C_r(|a_1|^r + \dots + |a_n|^r), \quad C_r = \begin{cases} 1, & 0 < r \leq 1, \\ n^{r-1}, & r > 1, \end{cases}$$

we have

$$\begin{aligned}
\|Tx\|_0 &\leq 3^{q-1}\Delta_1\left[\Phi_p^{-1}(\|a\|_{L^1}) + \Phi_p^{-1}(\|b\|_{L^1})\|x\|_0 + \Phi_p^{-1}(\|c\|_{L^1})\|x'\|_0\right] \\
&\leq 3^{q-1}\Delta_1\left[\Phi_p^{-1}(\|a\|_{L^1}) + (\Phi_p^{-1}(\|b\|_{L^1}) + \Phi_p^{-1}(\|c\|_{L^1}))M\right] \\
&< M = \|x\|,
\end{aligned}$$

and

$$\begin{aligned}
\|(Tx)'\|_0 &\leq 3^{q-1}\Delta_2\left[\Phi_p^{-1}(\|a\|_{L^1}) + \Phi_p^{-1}(\|b\|_{L^1})\|x\|_0 + \Phi_p^{-1}(\|c\|_{L^1})\|x'\|_0\right] \\
&\leq 3^{q-1}\Delta_2\left[\Phi_p^{-1}(\|a\|_{L^1}) + (\Phi_p^{-1}(\|b\|_{L^1}) + \Phi_p^{-1}(\|c\|_{L^1}))M\right] \\
&< M = \|x\|.
\end{aligned}$$

So $\|Tx\| < \|x\|$, i.e. taking $p^* = 0$ in Lemma 1.2, for any $x \in \partial\Omega$, $x = \overline{\lambda}Tx$ $(0 < \overline{\lambda} < 1)$ does not hold. Thus Lemma 1.2 implies that the operator T has at least one fixed point. So problem (1.4) has at least one positive solution. Besides, by $f(t,0,0) \neq 0$ for $t \in [0,\infty)$, problem (1.4) has at least one nontrivial positive solution. □

Corollary 3.2. *If $f(t,0,0) \not\equiv 0$ and there exists $r > 0$ such that*
$$\int_0^\infty f(t,x,y)dt < \min\big\{\Phi_p(\frac{r}{\Delta_1}), \Phi_p(\frac{r}{\Delta_2})\big\}, \tag{3.3}$$
where $x \in [0,r], y \in [-r,r]$. Then (1.4) has at least one nontrivial positive solution.

Proof. From Lemma 2.3, $T : P \to P$ is a completely continuous operator. Now we define $\Omega = \{x \in P : \|x\| < r\}$. For any $x \in \partial\Omega$, $\|x\| = r$. So $\|x\|_0 \le r, \|x'\|_0 \le r$. By assumption of theorem and Lemma 2.1,
$$\begin{aligned}
|Tx(t)| &= \frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)A_x - \int_0^s f(r,x(r),x'(r))dr\Big)\Big]ds \\
&\quad + \int_0^\infty \Phi_p^{-1}\Big[\frac{1}{\rho(s)}\Big(\rho(0)A_x - \int_0^s f(r,x(r),x'(r))dr\Big)\Big]ds \\
&\le \Delta_1\Phi_p^{-1}\Big(\int_0^\infty f(r,x(r),x'(r))dr\Big) \\
&< \Delta_1\Phi_p^{-1}\Big(\min\Big\{\Phi_p(\frac{r}{\Delta_1}), \Phi_p(\frac{r}{\Delta_2})\Big\}\Big) \\
&\le r = \|x\|.
\end{aligned}$$
$$\begin{aligned}
|(Tx)'(t)| &= \Big|\Phi_p^{-1}\Big[\frac{1}{\rho(t)}\Big(\rho(0)A_x - \int_0^t f(r,x(r),x'(r))dr\Big)\Big]\Big| \\
&\le \Delta_2\Phi_p^{-1}\Big(\int_0^\infty f(r,x(r),x'(r))dr\Big) \\
&< \Delta_2\Phi_p^{-1}\Big(\min\Big\{\Phi_p(\frac{r}{\Delta_1}), \Phi_p(\frac{r}{\Delta_2})\Big\}\Big) \\
&\le r = \|x\|.
\end{aligned}$$
So $\|Tx\| < \|x\|$. Similar to the process in Theorem 3.1, the result follows. □

Corollary 3.3. *If $f(t,0,0) \not\equiv 0$ and*
$$\lim_{d\to 0}\frac{\max_{x\in[0,d],y\in[-d,d]}\int_0^\infty f(t,x,y)dt}{d^{p-1}} = 0, \tag{3.4}$$
then (1.4) has at least one nontrivial positive solution.

Proof. Let $\varepsilon^* = \min\{\Phi_p(\frac{1}{\Delta_1}), \Phi_p(\frac{1}{\Delta_2})\}$. By (3.4), there exists $r > 0$, such that
$$\max_{x\in[0,d],y\in[-d,d]}\int_0^\infty f(t,x,y)dt \le \varepsilon^* d^{p-1} = \min\big\{\Phi_p(\frac{d}{\Delta_1}), \Phi_p(\frac{d}{\Delta_2})\big\}, \quad \forall\, d \le r,$$
which implies (3.3). By Corollary 3.2, BVP (1.4) has at least one nontrivial positive solution. □

4. Uniqueness of positive solutions

In this section, we establish the uniqueness of positive solutions for the problem
$$\begin{gathered}
(\rho(t)\Phi_p(x'(t)))' + f(t,x(t)) = 0, \quad t \in I, \\
x(0) = \sum_{i=1}^m \alpha_i x(\xi_i), \quad \lim_{t\to\infty} x(t) = 0.
\end{gathered} \tag{4.1}$$

Lemma 4.1. *Suppose that $f(t,x)$ is non-increasing in x for all $t \in I$. Then* (4.1) *has at most one positive solution.*

Proof. Assume to the contrary, that (4.1) has two positive solutions x_1, x_2. Let $y = x_2 - x_1$. Since x_1, x_2 are two positive solutions of (4.1),

$$(\rho(t)\Phi_p(x_2'(t)))' - (\rho(t)\Phi_p(x_1'(t)))' = f(t, x_1(t)) - f(t, x_2(t)), \quad t \in I,$$

$$y(0) = \sum_{i=1}^{m} \alpha_i y(\xi_i), \quad \lim_{t\to\infty} y(t) = 0.$$

Let $z(t) = \rho(t)\big(\Phi_p(x_2'(t)) - \Phi_p(x_1'(t))\big)$. Now we will complete the proof in three cases.

Case 1. If $y(t) \geq 0$, $y(t) \not\equiv 0$, $t \in I$. Since $f(t,x)$ is non-increasing in x, $z'(t) \geq 0$, $t \in I$. We claim that there exists a unique $\eta \in I$ satisfying $z(\eta) = 0$. If not, we get a contradiction by the following two cases.

(i) $z(t) > 0$, $t \in I$. Thus $\Phi_p(x_2'(t)) > \Phi_p(x_1'(t))$, $t \in I$; i.e., $y'(t) = (x_2 - x_1)'(t) > 0$, $t \in I$. So $\lim_{t\to+\infty} y(t) > 0$, a contradiction.

(ii) $z(t) < 0$, $t \in I$. Thus $\Phi_p(x_2'(t)) < \Phi_p(x_1'(t))$, $t \in I$; i.e., $y'(t) = (x_2 - x_1)'(t) < 0$, $t \in I$. So

$$y(0) = \sum_{i=1}^{m} \alpha_i y(\xi_i) \leq \sum_{i=1}^{m} \alpha_i y(0) < y(0), \quad \text{if } \sum_{i=1}^{m} \alpha_i \in (0,1),$$

$$\lim_{t\to\infty} y(t) < y(0) = 0, \quad \text{if } \sum_{i=1}^{m} \alpha_i = 0,$$

a contradiction. Our claim is proved.

So $z(t) < 0$ for $t \in [0,\eta)$ and $z(t) > 0$ for $t \in (\eta,\infty)$. Thus

$$y'(t) < 0 \text{ for } t \in [0,\eta) \text{ and } y'(t) > 0 \text{ for } t \in (\eta,\infty). \tag{4.2}$$

If $\sum_{i=1}^{m} \alpha_i \in (0,1)$, we have by the first boundary condition,

$$y(0) = \sum_{i=1}^{m} \alpha_i y(\xi_i) \leq \sum_{i=1}^{m} \alpha_i y(\xi_j) < y(\xi_j),$$

where $y(\xi_j) = \max\{y(\xi_i) : i = 1,2,\dots,m\}$. So $\xi_j > \eta$. By (4.2), we have

$$\lim_{t\to+\infty} y(t) \geq y(\xi_j) > y(0) \geq 0,$$

which contradicts the second boundary condition.

If $\sum_{i=1}^{m} \alpha_i = 0$, we have $0 = \lim_{t\to\infty} y(t) > y(0)$, which contradicts the first boundary condition.

Case 2. There exists $0 < a < b$, $b \in (0,\infty]$ satisfying $y(t) > 0$ for $t \in (a,b)$, $y(a) = y(b) = 0$, $y'(a) \geq 0$. By the definition of $z(t)$, we have $z'(t) > 0$, $t \in (a,b)$ and $z(a) \geq 0$. So $z(t) > 0$, $t \in (a,b)$; i.e., $y'(t) > 0$, $t \in (a,b)$. By $y(a) = 0$, we have $y(b) > 0$, a contradiction.

Case 3. There exists $b \in (0,\infty)$ satisfying $y(t) > 0$, $t \in [0,b)$, $y(b) = 0$, $y'(b) \leq 0$. By the definition of $z(t)$, we have $z'(t) > 0$, $t \in [0,b)$ and $z(b) \leq 0$. So $z(t) < 0$, $t \in [0,b)$. Then $y'(t) = (x_2 - x_1)'(t) < 0$, $t \in [0,b)$.

If $\sum_{i=1}^{m} \alpha_i \in (0,1)$, we have by the first boundary condition,

$$0 < y(0) = \sum_{i=1}^{m} \alpha_i y(\xi_i) < \sum_{i=1}^{m} \alpha_i y(\xi_j) < y(\xi_j),$$

where $y(\xi_j) = \max\{y(\xi_i) : i = 1, 2, \dots, m\}$. So $\xi_j > b$ and $y(\xi_j) > y(0) > 0$. So there exist c, d satisfying $b < c < \xi_j < d < \infty$ such that $y(t) > 0$ for $t \in (c, d)$, $y(c) = y(d) = 0, y'(c) > 0$. By Case 2, there is a contradiction.

If $\sum_{i=1}^{m} \alpha_i = 0$, then $y(0) = 0$, which contradicts $y(t) > 0, t \in [0, b)$. □

By Corollary 3.2 and Lemma 4.1, we have the following result.

Theorem 4.2. *Suppose that $f(t,x)$ is nonincreasing in x for all $t \in I$. Also assume that there exists $r > 0$ such that*

$$\int_0^\infty f(t,0)dt < \Phi_p\big(\frac{r}{\Delta_1}\big).$$

Then problem (4.1) has a unique positive solution.

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