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# EXISTENCE AND UNIQUENESS OF SOLUTIONS TO IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we establish sufficient conditions for the existence of solutions for a class of initial value problem for impulsive fractional differential equations involving the Caputo fractional derivative.

## 1. Introduction

This article studies the existence and uniqueness of solutions for the initial value problems (IVP for short), for fractional order differential equations

$$^cD^\alpha y(t) = f(t, y), \quad t \in J = [0, T],\ t \neq t_k, \tag{1.1}$$

$$\Delta y\big|_{t=t_k} = I_k(y(t_k^-)), \tag{1.2}$$

$$y(0) = y_0, \tag{1.3}$$

where $k = 1, \dots, m$, $0 < \alpha \leq 1$, $^cD^\alpha$ is the Caputo fractional derivative, $f : J\times\mathbb{R} \to \mathbb{R}$ is a given function, $I_k : \mathbb{R} \to \mathbb{R}$, and $y_0 \in \mathbb{R}$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

Differential equations of fractional order have proved to be valuable tools in the modelling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [16, 24, 25, 28, 33, 34, 38]). There has been a significant development in fractional differential and partial differential equations in recent years; see the monographs of Kilbas *et al* [31], Miller and Ross [35], Samko *et al* [42] and the papers of Agarwal *et al* [1], Babakhani and Daftardar-Gejji [2, 3], Belmekki *et al* [6], Benchohra *et al* [5, 7, 8, 10], Daftardar-Gejji and Jafari [14], Delbosco and Rodino [15], Diethelm *et al* [16, 17, 18], El-Sayed [19, 20, 21], Furati and Tatar [22, 23], Kaufmann and Mboumi [29], Kilbas and Marzan [30], Mainardi [33], Momani and Hadid [36], Momani *et al* [37], Podlubny *et al* [41], Yu and Gao [44] and Zhang [45] and the references therein.

Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain $y(0)$, $y'(0)$, etc., the same requirements of boundary conditions. Caputo's fractional derivative satisfies

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these demands. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types see [27, 40].

Impulsive differential equations (for $\alpha \in \mathbb{N}$) have become important in recent years as mathematical models of phenomena in both the physical and social sciences. There has a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments; see for instance the monographs by Bainov and Simeonov [4], Benchohra *et al* [9], Lakshmikantham *et al* [32], and Samoilenko and Perestyuk [43] and the references therein. To the best knowledge of the authors, no papers exist in the literature devoted to differential equations with fractional order and impulses. Thus the results of the present paper initiate this study. This paper is organized as follows. In Section 2 we present some preliminary results about fractional derivation and integration needed in the following sections. Section 3 will be concerned with existence and uniqueness results for the IVP (1.1)-(1.3). We give three results, the first one is based on Banach fixed point theorem (Theorem 3.5), the second one is based on Schaefer's fixed point theorem (Theorem 3.6) and the third one on the nonlinear alternative of Leray-Schauder type (Theorem 3.7). In Section 4 we indicate some generalizations to nonlocal initial value problems. The last section is devoted to an example illustrating the applicability of the imposed conditions. These results can be considered as a contribution to this emerging field.

## 2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts which are used throughout this paper. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from $J$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

**Definition 2.1** ([31, 39])**.** The fractional (arbitrary) order integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where $\Gamma$ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \leq 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where $\delta$ is the delta function.

**Definition 2.2** ([31, 39])**.** For a function $h$ given on the interval $[a,b]$, the $\alpha th$ Riemann-Liouville fractional-order derivative of $h$, is defined by

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \big(\frac{d}{dt}\big)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds.$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of $\alpha$.

**Definition 2.3** ([30])**.** For a function $h$ given on the interval $[a,b]$, the Caputo fractional-order derivative of order $\alpha$ of $h$, is defined by

$$({}^cD_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$.

## 3. Existence of Solutions

Consider the set of functions

$$PC(J,\mathbb{R}) = \{y : J \to \mathbb{R} : y \in C((t_k, t_{k+1}], \mathbb{R}),\ k = 0, \dots, m \text{ and there exist } y(t_k^-) \text{ and } y(t_k^+),\ k = 1, \dots, m \text{ with } y(t_k^-) = y(t_k)\}.$$

This set is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t\in J} |y(t)|.$$

Set $J' := [0, T]\backslash\{t_1, \dots, t_m\}$.

**Definition 3.1.** A function $y \in PC(J,\mathbb{R})$ whose $\alpha$-derivative exists on $J'$ is said to be a solution of (1.1)–(1.3) if $y$ satisfies the equation ${}^cD^\alpha y(t) = f(t, y(t))$ on $J'$, and satisfy the conditions

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1, \dots, m,$$
$$y(0) = y_0$$

To prove the existence of solutions to (1.1)–(1.3), we need the following auxiliary lemmas.

**Lemma 3.2** ([45])**.** *Let $\alpha > 0$, then the differential equation*

$${}^cD^\alpha h(t) = 0$$

*has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [\alpha] + 1$.*

**Lemma 3.3** ([45])**.** *Let $\alpha > 0$, then*

$$I^{\alpha c}D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$$

*for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [\alpha] + 1$.*

As a consequence of Lemma 3.2 and Lemma 3.3 we have the following result which is useful in what follows.

**Lemma 3.4.** *Let $0 < \alpha \le 1$ and let $h : J \to \mathbb{R}$ be continuous. A function $y$ is a solution of the fractional integral equation*

$$y(t) = \begin{cases} y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds & \text{if } t \in [0, t_1], \\ y_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h(s) ds & \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} h(s) ds + \sum_{i=1}^k I_i(y(t_i^-)), & \text{if } t \in (t_k, t_{k+1}], \end{cases} \tag{3.1}$$

*where $k = 1, \dots, m$, if and only if $y$ is a solution of the fractional IVP*

$${}^cD^\alpha y(t) = h(t), \quad t \in J', \tag{3.2}$$
$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{3.3}$$
$$y(0) = y_0. \tag{3.4}$$

*Proof.* Assume $y$ satisfies (3.2)-(3.4). If $t \in [0, t_1]$ then

$${}^cD^\alpha y(t) = h(t).$$

Lemma 3.3 implies

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds.$$

If $t \in (t_1, t_2]$ then Lemma 3.3 implies

$$\begin{aligned} y(t) &= y(t_1^+) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} h(s) ds \\ &= \Delta y|_{t=t_1} + y(t_1^-) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} h(s) ds \\ &= I_1(y(t_1^-)) + y_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} h(s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} h(s) ds. \end{aligned}$$

If $t \in (t_2, t_3]$ then from Lemma 3.3 we get

$$\begin{aligned} y(t) &= y(t_2^+) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} h(s) ds \\ &= \Delta y|_{t=t_2} + y(t_2^-) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} h(s) ds \\ &= I_2(y(t_2^-)) + I_1(y(t_1^-)) + y_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} h(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} h(s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} h(s) ds. \end{aligned}$$

If $t \in (t_k, t_{k+1}]$ then again from Lemma 3.3 we get (3.1).

Conversely, assume that $y$ satisfies the impulsive fractional integral equation (3.1). If $t \in [0, t_1]$ then $y(0) = y_0$ and using the fact that ${}^cD^\alpha$ is the left inverse of $I^\alpha$ we get

$${}^cD^\alpha y(t) = h(t), \quad \text{for each } t \in [0, t_1].$$

If $t \in [t_k, t_{k+1})$, $k = 1, \dots, m$ and using the fact that ${}^cD^\alpha C = 0$, where $C$ is a constant, we get

$${}^cD^\alpha y(t) = h(t), \text{for each } t \in [t_k, t_{k+1}).$$

Also, we can easily show that

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m.$$

□

Our first result is based on Banach fixed point theorem.

**Theorem 3.5.** *Assume that*

- (H1) *There exists a constant $l > 0$ such that $|f(t,u) - f(t,\overline{u})| \le l|u - \overline{u}|$, for each $t \in J$, and each $u, \overline{u} \in \mathbb{R}$.*
- (H2) *There exists a constant $l^* > 0$ such that $|I_k(u) - I_k(\overline{u})| \le l^*|u - \overline{u}|$, for each $u, \overline{u} \in \mathbb{R}$ and $k = 1, \dots, m$.*

*If*

$$\Big[\frac{T^\alpha l(m+1)}{\Gamma(\alpha+1)} + ml^*\Big] < 1, \tag{3.5}$$

*then* (1.1)-(1.3) *has a unique solution on $J$.*

*Proof.* We transform the problem (1.1)–(1.3) into a fixed point problem. Consider the operator $F : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ defined by

$$F(y)(t) = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1} f(s, y(s)) ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,y(s))ds+\sum_{0<t_k<t}I_k(y(t_k^-)).$$

Clearly, the fixed points of the operator $F$ are solution of the problem (1.1)-(1.3). We shall use the Banach contraction principle to prove that $F$ has a fixed point. We shall show that $F$ *is a contraction.* Let $x,y\in PC(J,\mathbb{R})$. Then, for each $t\in J$ we have

$$\begin{aligned}
&|F(x)(t)-F(y)(t)|\\
&\le \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|f(s,x(s))-f(s,y(s))|ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|f(s,x(s))-f(s,y(s))|ds+\sum_{0<t_k<t}|I_k(x(t_k^-))-I_k(y(t_k^-))|\\
&\le \frac{l}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|x(s)-y(s)|ds\\
&\quad+\frac{l}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|x(s)-y(s)|ds+\sum_{k=1}^{m}l^*|x(t_k^-)-y(t_k^-)|\\
&\le \frac{mlT^\alpha}{\Gamma(\alpha+1)}\|x-y\|_\infty+\frac{T^\alpha l}{\Gamma(\alpha+1)}\|x-y\|_\infty+ml^*\|x-y\|_\infty.
\end{aligned}$$

Therefore,

$$\|F(x)-F(y)\|_\infty\le\Big[\frac{T^\alpha l(m+1)}{\Gamma(\alpha+1)}+ml^*\Big]\|x-y\|_\infty.$$

Consequently by (3.5), $F$ is a contraction. As a consequence of Banach fixed point theorem, we deduce that $F$ has a fixed point which is a solution of the problem $(1.1)-(1.3)$. □

Our second result is based on Schaefer's fixed point theorem.

**Theorem 3.6.** *Assume that:*

- (H3) *The function $f:J\times\mathbb{R}\to\mathbb{R}$ is continuous.*
- (H4) *There exists a constant $M>0$ such that $|f(t,u)|\le M$ for each $t\in J$ and each $u\in\mathbb{R}$.*
- (H5) *The functions $I_k:\mathbb{R}\to\mathbb{R}$ are continuous and there exists a constant $M^*>0$ such that $|I_k(u)|\le M^*$ for each $u\in\mathbb{R}$, $k=1,\dots,m$.*

*Then* (1.1)-(1.3) *has at least one solution on $J$.*

*Proof.* We shall use Schaefer's fixed point theorem to prove that $F$ has a fixed point. The proof will be given in several steps.

**Step 1:** $F$ is continuous. Let $\{y_n\}$ be a sequence such that $y_n\to y$ in $PC(J,\mathbb{R})$. Then for each $t\in J$

$$\begin{aligned}
|F(y_n)(t)-F(y)(t)|&\le\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|f(s,y_n(s))-f(s,y(s))|ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|f(s,y_n(s))-f(s,y(s))|ds\\
&\quad+\sum_{0<t_k<t}|I_k(y_n(t_k^-))-I_k(y(t_k^-))|.
\end{aligned}$$

Since $f$ and $I_k$, $k = 1, \dots, m$ are continuous functions, we have

$$\|F(y_n) - F(y)\|_\infty \to 0 \quad \text{as } n \to \infty.$$

**Step 2:** $F$ maps bounded sets into bounded sets in $PC(J, \mathbb{R})$. Indeed, it is enough to show that for any $\eta^* > 0$, there exists a positive constant $\ell$ such that for each $y \in B_{\eta^*} = \{y \in PC(J, \mathbb{R}) : \|y\|_\infty \le \eta^*\}$, we have $\|F(y)\|_\infty \le \ell$. By (H4) and (H5) we have for each $t \in J$,

$$\begin{aligned} |F(y)(t)| &\le |y_0| + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |f(s, y(s))| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |f(s, y(s))| ds + \sum_{0<t_k<t} |I_k(y(t_k^-))| \\ &\le |y_0| + \frac{mMT^\alpha}{\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)} + mM^*. \end{aligned}$$

Thus

$$\|F(y)\|_\infty \le |y_0| + \frac{MT^\alpha(m+1)}{\Gamma(\alpha+1)} + mM^* := \ell.$$

**Step 3:** $F$ maps bounded sets into equicontinuous sets of $PC(J, \mathbb{R})$. Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$, $B_{\eta^*}$ be a bounded set of $PC(J, \mathbb{R})$ as in Step 2, and let $y \in B_{\eta^*}$. Then

$$\begin{aligned} &|F(y)(\tau_2) - F(y)(\tau_1)| \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} |(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}| |f(s, y(s))| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{\alpha-1}| |f(s, y(s))| ds + \sum_{0<t_k<\tau_2-\tau_1} |I_k(y(t_k^-))| \\ &\le \frac{M}{\Gamma(\alpha+1)} [2(\tau_2 - \tau_1)^\alpha + \tau_2^\alpha - \tau_1^\alpha] + \sum_{0<t_k<\tau_2-\tau_1} |I_k(y(t_k^-))|. \end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $F : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is completely continuous.

**Step 4:** A priori bounds. Now it remains to show that the set

$$\mathcal{E} = \{y \in PC(J, \mathbb{R}) : y = \lambda F(y) \ \text{ for some } \ 0 < \lambda < 1\}$$

is bounded. Let $y \in \mathcal{E}$, then $y = \lambda F(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$\begin{aligned} y(t) &= \lambda y_0 + \frac{\lambda}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, y(s)) ds \\ &\quad + \frac{\lambda}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} f(s, y(s)) ds + \lambda \sum_{0<t_k<t} I_k(y(t_k^-)). \end{aligned}$$

This implies by (H4) and (H5) (as in Step 2) that for each $t \in J$ we have

$$|y(t)| \le |y_0| + \frac{mMT^\alpha}{\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)} + mM^*.$$

Thus for every $t \in J$, we have

$$\|y\|_\infty \leq |y_0| + \frac{mMT^\alpha}{\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)} + mM^* := R.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that $F$ has a fixed point which is a solution of the problem (1.1)-(1.3). $\square$

In the following theorem we give an existence result for the problem (1.1)-(1.3) by applying the nonlinear alternative of Leray-Schauder type and which the conditions (H4) and (H5) are weakened.

**Theorem 3.7.** *Assume that* (H2) *and the following conditions hold:*

(H6) *There exists $\phi_f \in C(J, \mathbb{R}^+)$ and $\psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that*

$$|f(t,u)| \leq \phi_f(t)\psi(|u|) \quad \text{for all } t \in J,\ u \in \mathbb{R}.$$

(H7) *There exists $\psi^* : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that*

$$|I_k(u)| \leq \psi^*(|u|) \quad \text{for allu} \in \mathbb{R}.$$

(H8) *There exists an number $\overline{M} > 0$ such that*

$$\frac{\overline{M}}{|y_0| + \psi(\overline{M})\frac{mT^\alpha\phi_f^0}{\Gamma(\alpha+1)} + \psi(\overline{M})\frac{T^\alpha\phi_f^0}{\Gamma(\alpha+1)} + m\psi^*(\overline{M})} > 1,$$

*where $\phi_f^0 = \sup\{\phi_f(t) :\ t \in J\}$.*

*Then* (1.1)-(1.3) *has at least one solution on $J$.*

*Proof.* Consider the operator $F$ defined in Theorems 3.5 and 3.6. It can be easily shown that $F$ is continuous and completely continuous. For $\lambda \in [0,1]$, let $y$ be such that for each $t \in J$ we have $y(t) = \lambda(Fy)(t)$. Then from (H6)-(H7) we have for each $t \in J$,

$$\begin{aligned}
|y(t)| &\leq |y_0| + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}\phi_f(s)\psi(|y(s)|)ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}\phi_f(s)\psi(|y(s)|)ds + \sum_{0<t_k<t} \psi^*(|y(s)|) \\
&\leq |y_0| + \psi(\|y\|_\infty)\frac{mT^\alpha\phi_f^0}{\Gamma(\alpha+1)} + \psi(\|y\|_\infty)\frac{T^\alpha\phi_f^0}{\Gamma(\alpha+1)} + m\psi^*(\|y\|_\infty).
\end{aligned}$$

Thus

$$\frac{\|y\|_\infty}{|y_0| + \psi(\|y\|_\infty)\frac{mT^\alpha\phi_f^0}{\Gamma(\alpha+1)} + \psi(\|y\|_\infty)\frac{T^\alpha\phi_f^0}{\Gamma(\alpha+1)} + m\psi^*(\|y\|_\infty)} \leq 1.$$

Then by condition (H8), there exists $\overline{M}$ such that $\|y\|_\infty \neq \overline{M}$. Let

$$U = \{y \in PC(J,\mathbb{R}) : \|y\|_\infty < \overline{M}\}.$$

The operator $F : \overline{U} \to PC(J,\mathbb{R})$ is continuous and completely continuous. From the choice of $U$, there is no $y \in \partial U$ such that $y = \lambda F(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [26], we deduce that $F$ has a fixed point $y$ in $\overline{U}$ which is a solution of the problem (1.1)–(1.3). This completes the proof. $\square$

## 4. Nonlocal impulsive differential equations

This section is concerned with a generalization of the results presented in the previous section to nonlocal impulsive fractional differential equations. More precisely we shall present some existence and uniqueness results for the following nonlocal problem

$$
{}^cD^\alpha y(t) = f(t,y), \quad \text{for each } t \in J = [0,T],\ t \neq t_k, \tag{4.1}
$$

$$
\Delta y\big|_{t=t_k} = I_k(y(t_k^-)), \tag{4.2}
$$

$$
y(0) + g(y) = y_0, \tag{4.3}
$$

where $k = 1,\dots,m$, $0 < \alpha \le 1$, $f, I_k$, are as in Section 3 and $g : PC(J,\mathbb{R}) \to \mathbb{R}$ is a continuous function. Nonlocal conditions were initiated by Byszewski [13] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [11, 12], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. For example, $g(y)$ may be given by

$$
g(y) = \sum_{i=1}^{p} c_i y(\tau_i)
$$

where $c_i$, $i = 1,\dots,p$, are given constants and $0 < \tau_1 < \dots < \tau_p \le T$. Let us introduce the following set of conditions.

- (H9) There exists a constant $M^{**} > 0$ such that $|g(u)| \le M^{**}$ for each $u \in PC(J,\mathbb{R})$.
- (H10) There exists a constant $k > 0$ such that $|g(u) - g(\overline{u})| \le l^{**}|u - \overline{u}|$ for each $u, \overline{u} \in PC(J,\mathbb{R})$.
- (H11) There exists $\psi^{**} : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that $|g(u)| \le \psi^{**}(|u|)$ for each $u \in PC(J,\mathbb{R})$.
- (H12) There exists an number $\overline{M}^* > 0$ such that

$$
\frac{\overline{M}^*}{|y_0| + \psi^{**}(\overline{M}^*) + \psi(\overline{M}^*)\frac{mT^\alpha \phi_f^0}{\Gamma(\alpha+1)} + \psi(\overline{M}^*)\frac{T^\alpha \phi_f^0}{\Gamma(\alpha+1)} + m\psi^*(\overline{M}^*)} > 1,
$$

**Theorem 4.1.** *Assume that* (H1), (H2), (H10) *hold. If*

$$
\Big[\frac{T^\alpha l(m+1)}{\Gamma(\alpha+1)} + ml^* + l^{**}\Big] < 1, \tag{4.4}
$$

*then the nonlocal problem* (4.1)-(4.3) *has a unique solution on* $J$.

*Proof.* We transform the problem (4.1)–(4.3) into a fixed point problem. Consider the operator $\tilde{F} : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ defined by

$$
\begin{aligned}
\tilde{F}(y)(t) = y_0 - g(y) + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, y(s))ds \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s,y(s))ds + \sum_{0<t_k<t} I_k(y(t_k^-)).
\end{aligned}
$$

Clearly, the fixed points of the operator $\tilde{F}$ are solution of the problem (4.1)-(4.3). We can easily show the $\tilde{F}$ is a contraction. $\square$

**Theorem 4.2.** *Assume that* (H3)-(H5), (H9) *hold. Then the nonlocal problem* (4.1)-(4.3) *has at least one solution on* $J$.

**Theorem 4.3.** *Assume that* (H6)-(H7), (H11)-(H12) *hold. Then the nonlocal problem* (4.1)-(4.3) *has at least one solution on* $J$.

## 5. An Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the impulsive fractional initial-value problem,

$$^cD^{\alpha}y(t) = \frac{e^{-t}|y(t)|}{(9+e^t)(1+|y(t)|)}, \quad t \in J := [0,1],\ t \neq \frac{1}{2},\ 0 < \alpha \leq 1, \tag{5.1}$$

$$\Delta y|_{t=\frac{1}{2}} = \frac{|y(\frac{1}{2}^-)|}{3+|y(\frac{1}{2}^-)|}, \tag{5.2}$$

$$y(0) = 0. \tag{5.3}$$

Set

$$f(t,x) = \frac{e^{-t}x}{(9+e^t)(1+x)}, \quad (t,x) \in J \times [0,\infty),$$

and

$$I_k(x) = \frac{x}{3+x}, \quad x \in [0,\infty).$$

Let $x, y \in [0,\infty)$ and $t \in J$. Then we have

$$\begin{aligned} |f(t,x) - f(t,y)| &= \frac{e^{-t}}{(9+e^t)}\Big|\frac{x}{1+x} - \frac{y}{1+y}\Big| \\ &= \frac{e^{-t}|x-y|}{(9+e^t)(1+x)(1+y)} \\ &\leq \frac{e^{-t}}{(9+e^t)}|x-y| \\ &\leq \frac{1}{10}|x-y|. \end{aligned}$$

Hence the condition $(H1)$ holds with $l = 1/10$. Let $x, y \in [0,\infty)$. Then we have

$$|I_k(x) - I_k(y)| = \Big|\frac{x}{3+x} - \frac{y}{3+y}\Big| = \frac{3|x-y|}{(3+x)(3+y)} \leq \frac{1}{3}|x-y|.$$

Hence the condition $(H2)$ holds with $l^* = 1/3$. We shall check that condition (3.5) is satisfied with $T = 1$ and $m = 1$. Indeed

$$\Big[\frac{T^{\alpha}l(m+1)}{\Gamma(\alpha+1)} + ml^*\Big] < 1 \Longleftrightarrow \Gamma(\alpha+1) > \frac{3}{10}, \tag{5.4}$$

which is satisfied for some $\alpha \in (0,1]$. Then by Theorem 3.5 the problem (5.1)-(5.3) has a unique solution on $[0,1]$ for values of $\alpha$ satisfying (5.4).

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