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EXPONENTIAL ATTRACTORS FOR A NONCLASSICAL DIFFUSION EQUATION

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Abstract. In this article, we prove the existence of exponential attractors for a nonclassical diffusion equation in $H^2(\Omega) \cap H^1_0(\Omega)$ when the space dimension is less than 4.

1. Introduction

Let Ω be an open bounded set of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. We consider the equation

$$u_t - \Delta u_t - \Delta u + f(u) = g(x), \quad \text{in } \Omega \times \mathbb{R}_+, \tag{1.1}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{1.2}$$

$$u(x,0) = u_0, \quad x \in \Omega. \tag{1.3}$$

This equation is a special form of the nonclassical diffusion equation used in fluid mechanics, solid mechanics and heat conduction theory [1, 4]. Existence of the global attractors for problem (1.1)-(1.3) was studied originally by Kalantarov in [3] in the Hilbert space $H^1_0(\Omega)$. In recent years, many authors have proved the existence of global attractors under different assumptions, [3, 6, 7, 9] in the Hilbert space $H^1_0(\Omega)$, and [5, 8] in the Hilbert space $H^2(\Omega) \cap H^1_0(\Omega)$. In this paper, we study the existence of exponential attractors in the Hilbert space $H^2(\Omega) \cap H^1_0(\Omega)$.

In this article the nonlinear function satisfies the following conditions:

- (G1) There exists $l > 0$ such that $f'(s) \geq -l$ for all $s \in \mathbb{R}$;
- (G2) there exists $\kappa_1 > 0$ such that $f'(s) \leq \kappa_1(1 + |s|^2)$ for all $s \in \mathbb{R}$;
- (G3) $\liminf_{|s|\to\infty} F(s)/s^2 \geq 0$, where
$$F(s) = \int_0^s f(r)\,dr;$$
- (G4) there exists $\kappa_2 > 0$ such that
$$\liminf_{|s|\to\infty} \frac{sf(s) - \kappa_2 F(s)}{s^2} \geq 0.$$

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The main results of this paper will be stated as Theorem 3.10 below.

2. Preliminaries

Let $H = L^2(\Omega)$, $V_1 = H_0^1(\Omega)$, $V_2 = H^2(\Omega) \cap H_0^1(\Omega)$. We denote by $(\cdot, \cdot)$ denote the scalar product, and $\|\cdot\|$ the norm of H. The scalar product in V_1 and V_2 are denoted by

$$((u, v)) = \int_\Omega \nabla u \nabla v \, dx, \quad \forall u, v \in V_1,$$
$$[u, v] = \int_\Omega \Delta u \Delta v \, dx, \quad \forall u, v \in V_2.$$

The corresponding norms are denoted by $\|\cdot\|_1$, $\|\cdot\|_2$. It is well known that the norm $\|\cdot\|_s$ is equivalent to the usual norm of V_s for $s = 1, 2$. Let X be a separable Hilbert space and $\mathscr{B}$ be a compact subset of X, $\{S(t)\}_{t\geq 0}$ be a nonlinear continuous semigroup that leaves the set $\mathscr{B}$ invariant and $\mathscr{A} = \cap_{t>0} S(t)\mathscr{B}$, that is, $\mathscr{A}$ is a global attractor for $\{S(t)\}_{t\geq 0}$ on $\mathscr{B}$.

Definition 2.1 ([2])**.** A compact set $\mathscr{A} \subseteq \mathscr{M} \subseteq \mathscr{B}$ is called an exponential attractor for $(S(t), \mathscr{B})$ if:

(1) $\mathscr{M}$ has finite fractal dimension;
(2) $\mathscr{M}$ is a positive invariant set of $S(t) : S(t)\mathscr{M} \subseteq \mathscr{M}$, for all $t > 0$;
(3) $\mathscr{M}$ is an exponentially attracting set for the semigroup $\{S(t)\}_{t\geq 0}$; i.e. there exist universal constants α,$\beta > 0$ such that

$$\text{dist}_X(S(t)u, \mathscr{M}) \leq \alpha e^{-\beta t}, \quad \forall u \in \mathscr{B},\ t > 0,$$

where dist denotes the nonsymmetric Hausdorff distance between sets.

A sufficient condition for the existence of an exponential attractor depends on a dichotomy principle called the squeezing property; we recall this property as follows.

Definition 2.2 ([2])**.** A continuous semigroup of operators $\{S(t)\}_{t\geq 0}$ is said to satisfy the squeezing property on $\mathscr{B}$ if there exists $t_* > 0$ such that $S_* = S(t_*)$ satisfies that there exists an orthogonal projection operator P of rank N_0 such that, for every u and v in $\mathscr{B}$, either

$$\|(I - P)(S(t_*)u_1 - S(t_*)u_2)\|_X \leq \|P(S(t_*)u_1 - S(t_*)u_2)\|_X, \quad \text{or}$$
$$\|S(t_*)u_1 - S(t_*)u_2\|_X \leq \frac{1}{8}\|u_1 - u_2\|_X.$$

Definition 2.3 ([2])**.** For every u, v in the compact set $\mathscr{B}$, if there exists a local bounded function $l(t)$ such that

$$\|S(t)u - S(t)v\|_X \leq l(t)|u - v\|_X,$$

then $S(t)$ is Lipschitz continuous in $\mathscr{B}$. Here $l(t)$ does not depend on u or v.

3. Exponential Attractor in V_2

Lemma 3.1 ([8])**.** *Assume that $g \in V_s^{'}$ $(s = 1, 2)$. Then for each $u_0 \in V_s$ the problem* (1.1)-(1.3) *has a unique solution $u = u(t) = u(t; u_0)$ with $u \in C^1([0, \tau), V_s)$ on some interval $[0, \tau)$. Also for each t fixed, u is continuous in u_0.*

Lemma 3.2 ([3]). *Assume that $g \in H$, then for any $R > 0$, there exist positive constants $E_1(R)$, ρ_1 and $t_1(R)$ such that for every solution u of problem (1.1)-(1.3),*

$$\|u\|_1 \le E_1(R), \quad t \ge 0,$$
$$\|u\|_1 \le \rho_1, \quad t \ge t_1(R),$$

provided $\|u_0\|_1 \le R$.

Lemma 3.3 ([8]). *Assume $g \in V_1$, then for any $R > 0$, there exist positive constants $E_2(R)$, ρ_2 and $t_2(R)$ such that for every solution u of problem (1.1)-(1.3),*

$$\|u\|_2 \le E_2(R), \quad t \ge 0,$$
$$\|u\|_2 \le \rho_2, \quad t \ge t_2(R),$$

provided $\|u_0\|_2 \le R$.

Remark 3.4. From the proof of Lemma 3.3 [8, Theorem 3.2], we obtain

$$\int_t^{t+1} (\|u_t\|_1^2 + \|u_t\|_2^2) \le m,$$

where m is a positive constant.

According to Lemmas 3.2 and 3.3, we have

$$\mathscr{B}_0 = \{u \in V_2 : \|\nabla u\| \le \rho_1, \|\Delta u\| \le \rho_2\} \tag{3.1}$$

is a compact absorbing set of a semigroup of operators $\{S(t)\}_{t\ge 0}$ generated by (1.1)-(1.3). Namely, for any given $u_0 \in V_2$, there exists $T = T(u_0) > 0$ such that $\|S(t)u_0\| \le \rho$, for all $t \ge T$. Hence

$$\mathscr{B} = \overline{\cup_{0\le t\le T} S(t)\mathscr{B}_0}$$

is a compact positive invariant set in V_2 under $S(t)$.

Lemma 3.5 ([8]). *Assume that $f \in C^2(\mathbb{R};\mathbb{R})$ and satisfies (G1)–(G4) with $f(0) = 0$, $g \in V_1$. Then the semigroup $S(t)$ generated by (1.1)–(1.3) possesses a global attractor $\mathscr{A}$ in V_2.*

Lemma 3.6. *Assume that f satisfies (G1)–(G4), $u(t), v(t)$ are two solutions of (1.1)–(1.3) with initial values $u_0, v_0 \in \mathscr{B}$, then*

$$\|u(t) - v(t)\|_2 \le e^{c_1 t}\|u(0) - v(0)\|_2 \tag{3.2}$$

Proof. Setting $w(t) = u(t) - v(t)$, we see that $w(t)$ satisfies

$$w_t - \Delta w_t - \Delta w + f(u) - f(v) = 0. \tag{3.3}$$

Taking the inner product with $-\Delta w$ of (3.3), we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\Delta w\|^2 + \|\nabla w\|^2) + \|\Delta w\|^2 + (f(u) - f(v), -\Delta w) = 0. \tag{3.4}$$

Using $H_0^1(\Omega) \subset L^6(\Omega)$ and (G2), it follows that

$$\begin{aligned}
&\Big|\int_\Omega (f(u)-f(v))\Delta w\,dx\Big| \\
&\le \int_\Omega |f'(\theta u+(1-\theta)v)||w||\Delta w|\,dx \quad (0<\theta<1) \\
&\le c\int_\Omega (1+|u|^2+|v|^2)|w||\Delta w|\,dx \\
&\le c\int_\Omega |w||\Delta w|dx + c\int_\Omega |u|^2|w||\Delta w|dx + c\int_\Omega |v|^2|w||\Delta w|\,dx \\
&\le c\|w\|\|\Delta w\| + c\|u\|_6^2\|w\|_6\|\Delta w\| + c\|v\|_6^2\|w\|_6\|\Delta w\|.
\end{aligned} \tag{3.5}$$

Since $\mathscr{B}$ is a bounded absorbing set given by (3.1), $u_0, v_0 \in \mathscr{B}$, from (3.5) we get

$$\Big|\int_\Omega (f(u)-f(v))\Delta w\,dx\Big| \le c\|\nabla w\|\|\Delta w\| \le \frac{\|\Delta w\|^2}{2} + \frac{c_1}{2}\|\nabla w\|^2, \tag{3.6}$$

where c_1 is dependent on ρ_1 and ρ_2. Combining (3.4) with (3.6), we deduce that

$$\frac{d}{dt}(\|\Delta w\|^2 + \|\nabla w\|^2) + \|\Delta w\|^2 \le c_1\|\nabla w\|^2. \tag{3.7}$$

This yields

$$\frac{d}{dt}(\|\Delta w\|^2 + \|\nabla w\|^2) \le c_1(\|\nabla w\|^2 + \|\Delta w\|^2). \tag{3.8}$$

By the Gronwall Lemma, we get

$$\|\Delta w(t)\|^2 + \|\nabla w(t)\|^2 \le e^{c_1 t}(\|\Delta w(0)\|^2 + \|\nabla w(0)\|^2).$$

□

Lemma 3.7. *Under the assumptions of Lemma 3.5, there exists $L > 0$ such that*

$$\sup_{u_0\in\mathscr{B}} \|u_t(t)\|_2 \le L, \quad \forall t \ge 0.$$

Proof. Differentiating (1.1) with respect to time and denoting $v = u_t$, we have

$$v_t - \Delta v_t - \Delta v = -f'(u)v \tag{3.9}$$

Multiplying the above equality by $-\Delta v$ and using (G1),

$$\frac{1}{2}\frac{d}{dt}(\|\nabla v\|^2 + \|\Delta v\|^2) + \|\Delta v\|^2 \le l\|\nabla v\|^2. \tag{3.10}$$

This inequality and Remark 3.4, by the uniform Gronwall lemma, complete the proof. □

Lemma 3.8. *Under the assumptions of lemma 3.5, for every $T > 0$, the mapping $(t, u) \mapsto S(t)u$ is Lipschitz continuous on $[0, T] \times \mathscr{B}$.*

Proof. For u_1, $u_2 \in \mathscr{B}$ and t_1, $t_2 \in [0, T]$ we have

$$\|S(t_1)u_1 - S(t_2)u_2\|_2 \le \|S(t_1)u_1 - S(t_1)u_2\|_2 + \|S(t_1)u_2 - S(t_2)u_2\|_2 \tag{3.11}$$

The fist term of the above inequality is handled by estimate (3.2). For the second term, we have

$$\|u(t_1) - u(t_2)\|_2 \le \Big|\int_{t_1}^{t_2} \|u_t(y)\|_2 dy\Big| \le L|t_1 - t_2|. \tag{3.12}$$

Hence

$$\|S(t_1)u_1 - S(t_2)u_2\|_2 \le L[|t_1 - t_2| + \|u_1 - u_2\|_2]. \tag{3.13}$$

for some $L = L(T) \ge 0$. □

Lemma 3.9. *Assume that f satisfies* (G1)–(G4)*, $u(t), v(t)$ are two solutions of problem* (1.1)–(1.3) *with initial values $u_0, v_0 \in \mathscr{B}$, then the semigroup $S(t)$ generated from* (1.1)–(1.3) *satisfies the squeezing property; i.e., there exist t_* and $N = N_0 = N(t_*)$ such that*

$$\|(I - P)(S(t_*)u_0 - S(t_*)v_0)\|_2 > \|P(S(t_*)u_0 - S(t_*)v_0)\|_2$$

then

$$\|S(t_*)u_0 - S(t_*)v_0\|_2 \le \frac{1}{8}\|u_0 - v_0\|_2.$$

Proof. We consider the operator $A = -\Delta$. Since A is self-adjoint, positive operator and has a compact inverse, there exists a complete set of eigenvectors $\{\omega_i\}_{i=1}^{\infty}$ in H, the corresponding eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ satisfy

$$A\omega_i = \lambda_i \omega_i, \quad 0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_i \le \dots \to +\infty, \quad i \to +\infty.$$

We set $H_N = \operatorname{span}\{\omega_1, \omega_2, \dots, \omega_N\}$. P_N is the orthogonal projection onto H_N, and $Q_N = I - P_N$ is the orthogonal projection onto the orthogonal complement of H_N, $w = P_N w + Q_N w = p + q$. Assume that $\|P_N w(t)\| \le \|Q_N w(t)\|$, taking the inner product of (3.3) with $-\Delta q$, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\Delta q\|^2 + \|\nabla q\|^2) + \|\Delta q\|^2 + (f(u) - f(v), -\Delta q) = 0. \tag{3.14}$$

Similar to (3.5), it leads to

$$\begin{aligned} \Big|\int_\Omega (f(u) - f(v))\Delta q \, dx\Big| &\le c\int_\Omega |w||\Delta q| \, dx + c\int_\Omega |u|^2|w||\Delta q| \, dx \\ &\quad + c\int_\Omega |v|^2|w||\Delta q| \, dx. \end{aligned} \tag{3.15}$$

Since

$$\int_\Omega |u|^2|w||\Delta q| \, dx \le \|u\|_\infty^2 \|w\|\|\Delta q\|$$

and by the Agmon inequality: $\|u\|_\infty \le c\|\nabla u\|^{1/2}\|\Delta u\|^{1/2}$, and (3.1), from (3.15) we obtain

$$\Big|\int_\Omega (f(u) - f(v))\Delta q \, dx\Big| \le c\|w\|\|\Delta q\| \le \frac{\|\Delta q\|^2}{2} + \frac{c_2}{2}\|w\|^2, \tag{3.16}$$

where c_2 depends on ρ_1 and ρ_2. Combining (3.14) and (3.16), we deduce that

$$\frac{d}{dt}(\|\Delta q\|^2 + \|\nabla q\|^2) + \|\Delta q\|^2 \le c_2\|w\|^2. \tag{3.17}$$

Furthermore, by lemma 3.6 and the Poincaré inequality, we have

$$\begin{aligned} \frac{d}{dt}(\|\Delta q\|^2 + \|\nabla q\|^2) + \frac{\|\Delta q\|^2}{2} + \frac{\lambda_{N+1}}{2}\|\nabla q\|^2 &\le c_2\|w\|^2 \le c_2\|p + q\|^2 \\ &\le 2c_2\|q\|^2 \le 2c_2\lambda_{N+1}^{-1}\|\nabla q\|^2 \\ &\le c_3\lambda_{N+1}^{-2}\|\Delta w\|^2 \\ &\le c_3\lambda_{N+1}^{-2}e^{c_1 t}\|\Delta w(0)\|^2. \end{aligned} \tag{3.18}$$

Since $\lambda_1 \le \lambda_{N+1}$,

$$\frac{d}{dt}(\|\Delta q\|^2 + \|\nabla q\|^2) + \frac{\|\Delta q\|^2}{2} + \frac{\lambda_1}{2}\|\nabla q\|^2 \le c_3\lambda_{N+1}^{-2}e^{c_1 t}\|\Delta w(0)\|^2. \tag{3.19}$$

Let $c_4 = \min\{\frac{1}{2}, \frac{\lambda_1}{2}\}$. Then

$$\frac{d}{dt}(\|\Delta q\|^2 + \|\nabla q\|^2) + c_4(\|\Delta q\|^2 + \|\nabla q\|^2) \le c_3\lambda_{N+1}^{-2}e^{c_1 t}\|\Delta w(0)\|^2. \tag{3.20}$$

By the Gronwall Lemma, we conclude that

$$\begin{aligned}\|\Delta q(t)\|^2 + \|\nabla q(t)\|^2 &\le e^{-c_4 t}(\|\Delta q(0)\|^2 + \|\nabla q(0)\|^2) + c_5\lambda_{N+1}^{-2}e^{c_1 t}\|\Delta w(0)\|^2\\ &\le c_6(e^{-c_4 t} + c_7\lambda_{N+1}^{-2}e^{c_1 t})\|\Delta w(0)\|^2.\end{aligned}$$

Hence

$$\|\Delta w(t)\|^2 \le 2\|\Delta q(t)\|^2 \le c_8(e^{-c_4 t} + c_9\lambda_{N+1}^{-2}e^{c_1 t})\|\Delta w(0)\|^2. \tag{3.21}$$

Choose $t_* > 0$, such that $c_8 e^{-c_4 t_*} \le 1/128$, and then let t_* be fixed, and N large enough , such that $c_8 c_9 \lambda_{N+1}^{-2} e^{c_1 t_*} \le 1/128$. We obtain

$$\|\Delta w(t_*)\| \le \frac{1}{8}\|\Delta w(0)\|.$$

□

Theorem 3.10. *Assume that* $f \in C^2(\mathbb{R};\mathbb{R})$ *and satisfies* (G1)–(G4) *with* $f(0) = 0$, $g \in V_1$. *Then there exists an exponential attractor* $\mathscr{M} \subset V_2$ *for the semigroup of operators* $\{S(t)\}_{t\ge 0}$ *generated by* (1.1)–(1.3).

Proof. From Lemma 3.9, $S(t_*)$ satisfies the squeezing property for some $t_* > 0$. According to [2, Theorem 2.1], there exists an exponential attractor $\mathscr{M}_*$ for $(S(t_*), \mathscr{B})$ and we set

$$\mathscr{M} = \bigcup_{0\le t\le t_*} S(t)\mathscr{M}_*.$$

By Lemma 3.8, $(t, u) \mapsto S(t)u$ is Lipschitz continuous from $[0, T] \times \mathscr{B}$ to $\mathscr{B}$. Then as in the proof of [2, Theorem 3.1], $\mathscr{M}$ is an exponential attractor for $(\{S(t)\}_{t\ge 0}, \mathscr{B})$. □

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References

[1] Aifantis. E. C; *On the problem of diffusion in solids*, Acta Mech, **37** (1980): 265-296.

[2] A. Eden, C. Foias, B. Nicolaenko, R. Temam; *Exponential Attractors for Dissipative Evolution Equations*, New York:Masson. Paris, Wiely, (1994).

[3] V. K. Kalantarov; *On the attractors for some non-linear problems of mathematical physics*, Zap. Nauch. Sem. LOMI **152** (1986): 50-54.

[4] J. L. Lions, E. Magenes; *Non-homogeneous boundary value problems and applications*, Springverlag,Berlin, (1972).

[5] Q. Z. Ma, C. K. Zhong; *Global attractors of strong solutions to nonclassical diffusion equations*, Journal of Lanzhou University. **40**(2004): 7-9.

[6] C. Y. Sun, S. Y. Wang, C. K. Zhong; *Global attractors for a nonclassical diffusion equation*, Acta math sinica, English series. **23**(2007): 1271-1280.

[7] R. Temam; *Infinite-dimensional dynamical systems in mechanics and physic*, Springer, New York, (1997).

[8] S. Y. Wang, D. S. Li, C. K. Zhong; *On the dynamics of a class of nonclassical parabolic equations*, J. Math. Anal. Appl. **317**(2006): 565-582.

[9] Y. L. Xiao; *Attractors for a nonclassical diffusion equation*, Acta Math Appl Sinica. **18** (2002): 273-276.

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