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STABILIZATION OF SOLUTIONS FOR SEMILINEAR PARABOLIC SYSTEMS AS $|x| \to \infty$

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ABSTRACT. We prove that solutions of the Cauchy problem for semilinear parabolic systems converge to solutions of the Cauchy problem for a corresponding systems of ordinary differential equations, as $|x| \to \infty$.

1. INTRODUCTION

In this paper we consider the Cauchy problem for the system of semilinear parabolic equations

$$\begin{gathered} u_{1t} = a_1^2 \Delta u_1 + f_1(x,t,u_1,\dots,u_k), \\ \dots \\ u_{kt} = a_k^2 \Delta u_k + f_k(x,t,u_1,\dots,u_k), \end{gathered} \tag{1.1}$$

subject to the initial conditions

$$u_1(x,0) = \varphi_1(x), \dots, u_k(x,0) = \varphi_k(x), \tag{1.2}$$

where $x \in \mathbb{R}^n$, $n \geq 1$, $0 < t < T_0$, $T_0 \leq \infty$. Put $S_T = \mathbb{R}^n \times [0,T)$, $\mathbb{R}^k_+ = \{x \in \mathbb{R}^k : x_i \geq 0, i = 1,\dots,k\}$. We assume that the data of problem (1.1)-(1.2) satisfy the following conditions:

$f_i(x,t,u_1,\dots,u_k)$, $i = 1,\dots,k$ are defined and locally Hölder continuous functions in $\mathbb{R}^n \times [0,T_0) \times \mathbb{R}^k_+$ and $\varphi_i(x)$, $i = 1,\dots,k$ are continuous functions in $\mathbb{R}^n$; (1.3)

$$f_i(x,t,u_1,\dots,u_k), \quad i = 1,\dots,k \text{ do not decrease in } u_1,\dots,u_k; \tag{1.4}$$

$f_i(x,t,u_1,\dots,u_k) \to \bar{f}_i(t,u_1,\dots,u_k)$, $i = 1,\dots,k$, as $|x| \to \infty$ uniformly on any bounded subset of $[0,T_0) \times \mathbb{R}^k_+$; (1.5)

$$0 \leq f_i(x,t,u_1,\dots,u_k) \leq \bar{f}_i(t,u_1,\dots,u_k), \quad i = 1,\dots,k; \tag{1.6}$$

$$0 \leq \varphi_i(x) \leq c_i, \quad \lim_{|x|\to\infty} \varphi_i(x) = c_i, \quad c_i \geq 0, \quad i = 1,\dots,k. \tag{1.7}$$

The above assumptions are satisfied, in particular, for large class problems (1.1)-(1.2), whose solutions exist only on a finite time interval. Note also that the solution of (1.1)-(1.2) may not be unique.

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Let us consider the Cauchy problem for the system of ordinary differential equations

$$\begin{aligned} g_1' &= \bar f_1(t, g_1, \dots, g_k), \\ &\dots \\ g_k' &= \bar f_k(t, g_1, \dots, g_k), \end{aligned} \tag{1.8}$$

subject to the initial conditions

$$g_1(0) = c_1, \dots, g_k(0) = c_k. \tag{1.9}$$

We suppose that the minimal nonnegative solution $g_i(t)$, $i = 1, \dots, k$, of (1.8)-(1.9) exists on $[0, T_0)$. The main result of the paper is the following theorem.

Theorem 1.1. *Let $u_i(x,t)$ be the minimal nonnegative solution of the problem (1.1)-(1.2). Then*

$$u_i(x,t) \to g_i(t), \quad i = 1, \dots, k, \quad \text{as } |x| \to \infty \tag{1.10}$$

uniformly for $t \in [0, T]$, $(T < T_0)$.

The behavior of solutions of parabolic equations as $|x| \to \infty$ has been investigated by several authors. The case of one semilinear parabolic equation on half line has been considered in [1, 5] for nonlinearities $f(x,t,u) = u^p$ and $f(x,t,u) = \exp u$. The same problem with general nonlinearity $f(x,t,u)$ has been investigated in [4]. The behavior of solutions of nonlinear parabolic equations for the Cauchy problem as $|x| \to \infty$ has been analyzed in [2, 3, 6, 7].

The plan of this paper is as follows. In the next section, the existence of a minimal solution for the problem (1.1)-(1.2) is proved. The proof of Theorem 1.1 is given in Section 3.

2. Existence of a minimal solution

We prove the existence of a minimal solution for (1.1)-(1.2). It is well known that (1.1)-(1.2) is equivalent to the system

$$\begin{aligned} u_1(x,t) &= \int_{\mathbb{R}^n} E_1(x-y,t)\varphi_1(y)\,dy \\ &\quad + \int_0^t \int_{\mathbb{R}^n} E_1(x-y,t-\tau) f_1(y,\tau,u_1,\dots,u_k)\,dy\,d\tau, \\ &\dots \\ u_k(x,t) &= \int_{\mathbb{R}^n} E_k(x-y,t)\varphi_k(y)\,dy \\ &\quad + \int_0^t \int_{\mathbb{R}^n} E_k(x-y,t-\tau) f_k(y,\tau,u_1,\dots,u_k)\,dy\,d\tau, \end{aligned} \tag{2.1}$$

where $E_i(x,t) = (2a_i\sqrt{\pi t})^{-n} \exp(-|x|^2/[4a_i^2 t])$, $i = 1, \dots, k$, are the fundamental solutions of the correspondent heat equations.

Let $u_{i0}(x,t) \equiv 0$, $i = 1, \dots, k$. We define sequences of functions $u_{im}(x,t)$, $i = 1, \dots, k$, $m \in \mathbb{N}$, the following way

$$\begin{aligned} u_{im}(x,t) &= \int_{\mathbb{R}^n} E_i(x-y,t)\varphi_i(y)\,dy \\ &\quad + \int_0^t \int_{\mathbb{R}^n} E_i(x-y,t-\tau) f_i(y,\tau,u_{1(m-1)},\dots,u_{k(m-1)})\,dy\,d\tau. \end{aligned} \tag{2.2}$$

Obviously, the functions $g_i(t)$, $i = 1,\dots,k$, satisfy the integral equations

$$g_i(t) = \int_{\mathbb{R}^n} E_i(x-y,t)c_i\,dy + \int_0^t \int_{\mathbb{R}^n} E_i(x-y,t-\tau)\bar{f}_i(\tau, g_1,\dots,g_k)\,dy\,d\tau. \quad (2.3)$$

Using (1.4), (1.6), (2.2) and (2.3), we have

$$0 \le u_{i(m-1)}(x,t) \le u_{im}(x,t) \le g_i(t), \quad i = 1,\dots,k,\ m \in \mathbb{N}. \quad (2.4)$$

By the Lebesgue theorem, and from (2.2) and (2.4), we obtain that the sequences $u_{im}(x,t)$ converge to functions $u_i(x,t)$ that satisfy (2.1), which means, ones satisfy the problem (1.1)-(1.2). Let $v_i(x,t)$, $i = 1,\dots,k$, be any other solution of (1.1)-(1.2). By induction on m it is easy to prove that $u_{im}(x,t) \le v_i(x,t)$, $i = 1,\dots,k$, $m \in \mathbb{N}$. Therefore, $u_i(x,t)$, $i = 1,\dots,k$, is the minimal nonnegative solution of this problem. We have proved the following statement.

Theorem 2.1. *There exists a minimal nonnegative solution $u_i(x,t), i = 1,\dots,k$, of the problem* (1.1)-(1.2) *in S_{T_0} that satisfies the inequalities*

$$0 \le u_{im}(x,t) \le u_i(x,t) \le g_i(t), \quad (x,t) \in S_{T_0},\ i = 1,\dots,k,\ m \in \mathbb{N}. \quad (2.5)$$

3. Behavior of a minimal solution as $|x| \to \infty$

We show that for the minimal nonnegative solution of (1.1)-(1.2), property (1.10) is satisfied. We define sequences of functions $g_{im}(t)$, $i = 1,\dots,k$, $m = 0,1,...$, as follows

$$g_{i0}(t) \equiv 0, \quad g_{im}(t) = \int_0^t \bar{f}_i(\tau, g_{1(m-1)},\dots,g_{k(m-1)})\,d\tau + c_i, \quad i = 1,\dots,k,\ m \in \mathbb{N}. \quad (3.1)$$

Obviously, the sequences $g_{im}(t)$ are monotonically nondecreasing, converging to the minimal nonnegative solution $g_i(t)$, $i = 1,\dots,k$, of problem (1.8)-(1.9) on any interval $[0,T]$, $(T < T_0)$, and

$$g_{im}(t) \le g_i(t), \quad i = 1,\dots,k,\ m \in \mathbb{N}. \quad (3.2)$$

According to the Dini criterion on uniform convergence of functional sequences, we have

$$g_{im}(t) \to g_i(t), \quad i = 1,\dots,k,\ \text{as } m \to \infty \text{ uniformly on } [0,T]. \quad (3.3)$$

It is easy to prove that $g_{im}(t)$, $i = 1,\dots,k$, $m \in \mathbb{N}$, satisfy the following equations

$$g_{im}(t) = \int_0^t \int_{\mathbb{R}^n} E_i(x-y,t-\tau)\bar{f}_i(\tau, g_{1(m-1)},\dots,g_{k(m-1)})\,dy\,d\tau + c_i. \quad (3.4)$$

Now we prove an auxiliary lemma.

Lemma 3.1. *For any $\delta > 0$, $0 < T < T_0$, $i = 1,\dots,k$, and $m \ge 0$ there exists a constant p such that if $|x| > p$ and $0 \le t \le T$, then*

$$|u_{im}(x,t) - g_{im}(x,t)| < \delta. \quad (3.5)$$

Proof. We use induction on m. It is obviously that $u_{i0}(x,t) - g_{i0}(t) = 0$, $i = 1,\dots,k$. We assume that (3.5) holds for $m = l$, and we shall prove the inequality for $m = l+1$. By the induction assumption, for any $\varepsilon_1 > 0$ and $0 < T < T_0$ there exists p_1 such that

$$|u_{il}(x,t) - g_{il}(t)| < \varepsilon_1, \quad i = 1,\dots,k, \quad (3.6)$$

if $|x| > p_1$ and $0 \le t \le T$. Put $B(q) = \{x \in \mathbb{R}^n : |x| \le q\}$. From (2.2) and (3.4), we have

$$
\begin{aligned}
&|u_{i(l+1)} - g_{i(l+1)}| \\
&\le \Big| \int_0^t \int_{B(q)} E_i(x-y,t-\tau)(f_i(y,\tau,u_{1l},\dots,u_{kl}) - \bar{f}_i(\tau,g_{1l},\dots,g_{kl}))\,dy\,d\tau \Big| \\
&\quad + \Big| \int_0^t \int_{\mathbb{R}^n\setminus B(q)} E_i(x-y,t-\tau)(f_i(y,\tau,u_{1l},\dots,u_{kl}) - \bar{f}_i(\tau,u_{1l},\dots,u_{kl}))\,dy\,d\tau \Big| \\
&\quad + \Big| \int_0^t \int_{\mathbb{R}^n\setminus B(q)} E_i(x-y,t-\tau)(\bar{f}_i(\tau,u_{1l},\dots,u_{kl}) - \bar{f}_i(\tau,g_{1l},\dots,g_{kl}))\,dy\,d\tau \Big| \\
&\quad + \Big| \int_{B(q)} E_i(x-y,t)(\varphi_i(y)-c_i)\,dy \Big| + \Big| \int_{\mathbb{R}^n\setminus B(q)} E_i(x-y,t)(\varphi_i(y)-c_i)\,dy \Big|,
\end{aligned}
\tag{3.7}
$$

where q will be choose later. We denote by I_j, $j = 1,\dots,5$ the integrals from the right-hand side of (3.7), respectively. Obviously, $\bar{f}_i(t,u_1,\dots,u_k)$, $i = 1,\dots,k$, are uniformly continuous on any compact subset of $[0,T] \times \mathbb{R}^k_+$. Using this and (1.5), (1.7), (2.4), (3.2), (3.6) for suitable ε_1 and q, we get

$$
|I_2| + |I_3| + |I_5| < \delta/2 \quad \text{if } |x| > p_2 \tag{3.8}
$$

for some p_2. Since $E_i(x-y,t) \to 0$ as $|x| \to \infty$ uniformly on $[0,T] \times B(q)$, we have

$$
|I_1| + |I_4| < \delta/2 \quad \text{if } |x| > p_3 \tag{3.9}
$$

for some p_3. Now (3.5) follows from (3.8), (3.9). □

Proof of Theorem 1.1. We fix a positive ε. From Lemma 3.1 and (3.3), for suitable m and q, we have

$$
|u_{im}(x,t) - g_i(t)| \le |u_{im}(x,t) - g_{im}(t)| + |g_{im}(t) - g_i(t)| < \varepsilon, \quad i = 1,\dots,k, \tag{3.10}
$$

if $|x| > q$ and $0 \le t \le T$. From (2.5) and (3.10) we obtain

$$
g_i(t) - \varepsilon \le u_{im}(x,t) \le u_i(x,t) \le g_i(t), \quad i = 1,\dots,k,
$$

for $|x| > q$ and $0 \le t \le T$. The statement of the theorem follows immediately from these arguments. □

References

[1] J. Bebernes, W. Fulks; The small heat-loss problem, *J. Diff. Equat.* **57** (1985), 324-332.

[2] Y. Giga, N. Umeda; Blow-up directions at space infinity for solutions of semilinear heat equations, *Bol. Soc. Parana Mat.* **23** (2005), 9-28.

[3] Y. Giga, N. Umeda; On blow-up at space infinity for semilinear heat equations, *J. Math. Anal. Appl.* **316** (2006), 538-555.

[4] A. L. Gladkov; Behavior of solutions of semilinear parabolic equations, *Matematicheskie Zametki* **51** (1992), 29-34 (in Russian).

[5] A. A. Lacey; The form of blow-up for nonlinear parabolic equations, *Proc. Roy. Soc. Edinburgh Sect. A* **98** (1-2) (1984), 183-202.

[6] Y. Seki; On directional blow-up for quasilinear parabolic equations with fast diffusion, *J. Math. Anal. Appl.* **338** (2008), 572-587.

[7] Y. Seki, R. Suzuki, N. Umeda; Blow-up directions for quasilinear parabolic equations, *Proc. Roy. Soc. Edinburgh Sect. A*, **138A** (2008), 379-405.

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