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ANNULUS OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR ELLIPTIC DIFFERENTIAL EQUATIONS WITH DAMPING

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Abstract. We establish oscillation criteria for the second-order elliptic differential equation

$$\nabla \cdot (A(x)\nabla y) + B^T(x)\nabla y + q(x)f(y) = e(x), \quad x \in \Omega,$$

where Ω is an exterior domain in $\mathbb{R}^N$. These criteria are different from most known ones in the sense that they are based on the information only on a sequence of annulus of Ω, rather than on the whole exterior domain Ω. Both the cases when $\frac{\partial b_i}{\partial x_i}$ exists for all i and when it does not exist for some i are considered.

1. Introduction

In this paper, we consider the oscillation of solutions to the second-order elliptic differential equation

$$\nabla \cdot (A(x)\nabla y) + B^T(x)\nabla y + q(x)f(y) = e(x), \tag{1.1}$$

where $x \in \Omega$, an exterior domain in $\mathbb{R}^N$, $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_N})$. The following notation will be adopted in this article: $\mathbb{R}$ and $\mathbb{R}^+$ denote the intervals $(-\infty, +\infty), (0, +\infty)$, respectively. The norm of x is denoted by $|x| = [\sum_{i=1}^N x_i^2]^{1/2}$. For a positive constant $a > 0$, let

$$S_a = \{x \in \mathbb{R}^N : |x| = a\}, \quad G(a, +\infty) = \{x \in \mathbb{R}^N : |x| > a\},$$
$$G[a, b] = \{x \in \mathbb{R}^N : a \le |x| \le b\}, \quad G(a, b) = \{x \in \mathbb{R}^N : a < |x| < b\}.$$

For the exterior domain Ω in $\mathbb{R}^N$, there exists a positive number a_0 such that $G(a_0, +\infty) \subset \Omega$.

A function $y \in C_{\rm loc}^{2+\mu}(\Omega, \mathbb{R}), \mu \in (0, 1)$ is said to be a solution of (1.1) in Ω, if $y(x)$ satisfies (1.1) for all $x \in \Omega$. For the existence of solutions of (1.1), we refer the reader to the monograph [3]. We restrict our attention only to the nontrivial solution $y(x)$ of (1.1); i.e., for any $a > a_0$, $\sup\{|y(x)| : |x| > a\} > 0$. A nontrivial solution $y(x)$ of (1.1) is called oscillatory if the zero set $\{x : y(x) = 0\}$ of $y(x)$ is

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unbounded, otherwise it is called nonoscillatory. (1.1) is called oscillatory if all its nontrivial solutions are oscillatory.

In the qualitative theory of nonlinear partial differential equations, one of the important problems is to determine whether or not solutions of the equation under consideration are oscillatory. For the similinear elliptic equation

$$\nabla \cdot (A(x)\nabla y) + q(x)f(y) = 0, \tag{1.2}$$

the oscillation theory is fully developed by many authors. Noussair and Swanson [7] first extended the Wintner theorem by using the following partial Riccati type transformation equation

$$W(x) = -\frac{\alpha(|x|)}{f(y(x))}(A\nabla y)(x), \tag{1.3}$$

where $\alpha \in C^2$ is an arbitrary positive function. Swanson [3] summarized the oscillation results for (1.2) up to 1979. For recent contributions, we refer the reader to [13, 14, 12]. However, as far as we know that the (1.1) has never been the subject of systematic investigations.

When $N = 1$, (1.1) reduces to second-order ordinary differential equations such as:

$$y''(t) + q(t)f(y) = e(t), \tag{1.4}$$

$$(r(t)y'(t))' + q(t)y(t) = e(t), \tag{1.5}$$

$$(r(t)y'(t))' + q(t)f(y) = e(t), \tag{1.6}$$

There is a great number of papers devoted to (1.4)-(1.6) (see, for example, [8, 9, 10] and the references quoted therein). Some of the known oscillation criteria are established by making use of a technique introduced by Kartsatos [5] where it is assumed that there exists a second derivative function "$h(t)$" such that $h''(t) = e(t)$ in order to reduce (1.4) or (1.5) to a second order homogeneous equation. However, these results require the information of "q" on the entire half-line $[t_0, \infty)$.

In 1993, El-Sayed [1] gave an interval oscillation criterion for (1.4) which depends only on the behavior of "q" in certain subintervals of $[t_0, \infty)$. In 1999, Wong [11] and Kong [6] have, respectively, noted that interval criteria which Ei-Sayed [1] established for oscillation of (1.5) are not very sharp, because a comparison with a equation of constant coefficients is used in Ei-Sayed's proof. Therefore, some other interval criteria for oscillation,that is, criteria given by the behavior of (1.5) and (1.5) with $e(t) = 0$ only a sequence of subintervals of $[t_0, \infty)$ are obtained by Wong [11] and Kong [6], respectively.

In 2003, Yang [15] employed the technique in the work of Philos [8] and Kong [6] for (1.4), and presented several Interval oscillation criteria for (1.6). One of the oscillation criteria of Kamenev's type in [15] is as follows.

Theorem 1.1. *Suppose $f(y)/y \geq K|y|^{\nu-1}$ for $y \neq 0$, $K > 0$ and $\nu > 1$. Then (1.4) with $r(t) \equiv 1$ is oscillatory provided that for each $t \geq t_0$ and for some $\lambda > 1$, the following conditions hold*

(1) *For any $T \geq t_0$, there exist $T \leq a_1 < b_1 \leq a_2 < b_2$ such that*

$$e(t) \begin{cases} \leq 0, & t \in [a_1, b_1], \\ \geq 0, & t \in [a_2, b_2] \end{cases}$$

and $q(t) \geq 0$ ($\not\equiv 0$), $t \in [a_1, b_1] \cup [a_2, b_2]$

(2) *there exist* $c_i \in (a_i, b_i)$ *for* $i = 1, 2$, *such that* $T \le a_1 < b_1 \le a_2 < b_2$ *and the following inequalities hold for* $i = 1, 2$,

$$\frac{1}{(c_i - a_i)^{\lambda-1}} \int_{a_i}^{c_i} (s - a_i)^\lambda |e(s)|^{1-(1/\nu)} [Kq(s)]^{1/\nu} ds \ge \frac{\lambda^2}{4(\lambda - 1)} \tag{1.7}$$

$$\frac{1}{(b_i - c_i)^{\lambda-1}} \int_{c_i}^{b_i} (b_i - s)^\lambda |e(s)|^{1-(1/\nu)} [Kq(s)]^{1/\nu} ds \ge \frac{\lambda^2}{4(\lambda - 1)}. \tag{1.8}$$

Motivate by the ideas of Philos [8], Kong [6], and Yang [15]. In this paper, by using generalized Riccati techniques which are introduced by Noussair [7], we obtain several annulus criteria for oscillation, that is, criteria given by the behavior of (1.1) (or of A, q, f and e) only on a sequence of annulus of Ω in $\mathbb{R}^N$. Our results improve and extend the results of Ei-Sayed [1], Kong [6] and Yang [15]. Also information about the distribution of the zero of solutions for(1.1) is obtained.

2. Oscillation results when $\frac{\partial b_i}{\partial x_i}$ exists for all i

To establish oscillation theorems when $\frac{\partial b_i}{\partial x_i}$ exists for all i we shall impose the following conditions:

(C1) $A(x) = (A_{ij}(x))_{N\times N}$ is a real symmetric positive definite matrix function (ellipticity condition) with $A_{ij} \in C^{1+\mu}_{\rm loc}(\Omega(a_0), \mathbb{R}), \mu \in (0, 1), i, j = 1, \dots, N$, $\lambda_{\max}(x)$ denotes the largest (necessarily positive) eigenvalue of the matrix $A(x)$; there exists a function $\lambda \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ such that $\lambda(r) \ge \max_{|x|=r} \lambda_{\max}(x)$ for $r > 0$;
(C2) $B^T = (b_i(x))_{1\times N}, b_i \in C^{1+\mu}_{\rm loc}(\Omega(a_0), \mathbb{R}), i = 1, \dots, N$;
(C3) $q \in C^{\mu}_{\rm loc}(\Omega(a_0), \mathbb{R}), \mu \in (0, 1)$ and $q(x) \not\equiv 0$ for $|x| \ge a_0$;
(C4) $f \in C^1(\mathbb{R}, \mathbb{R}), yf(y) > 0$ and $f'(y) \ge k > 0$ for all $y \ne 0$ and some constant k.

For convenience, we let

$$Q_1(r) = \int_{S_r} \big[q(x) - \frac{1}{4k} B^T A^{-1} B - \frac{1}{2k} \nabla \cdot B\big] d\sigma,$$
$$g_1(r) = \frac{\omega}{k} \lambda(r) r^{N-1},$$

where $S_r = \{x \in \mathbb{R}^N : |x| = r\}, r > 0, d\sigma$ denotes the spherical integral element in $\mathbb{R}^N$, ω is the area of unit sphere in $\mathbb{R}^N$ and k is defined in (C4).

Theorem 2.1. *Let* (C1)–(C4) *hold. Suppose that for any* $T \ge a_0$, *there exist* $T \le a_1 < b_1 \le a_2 < b_2$ *such that*

$$e(x) \begin{cases} \le 0, & x \in G[a_1, b_1], \\ \ge 0, & x \in G[a_2, b_2] \end{cases}$$

and $q(x) \ge 0 (\not\equiv 0)$, $x \in G[a_1, b_1] \cup G[a_2, b_2]$. *Denote by* $\Psi(a_i, b_i)$ *the set*

$$\big\{H \in C^1[a_i, b_i], H(r) \ge 0 (\not\equiv 0), H(a_i) = H(b_i) = 0, H'_r = 2h(r)\sqrt{H(r)}\big\},$$

$i = 1, 2$. *If there exist* $H \in \Psi(a_i, b_i)$ *such that*

$$M_i(H) = \int_{a_i}^{b_i} \Big\{g_1(s) h^2(s) - Q_1(s) H(s)\Big\} ds < 0,$$

for $i = 1, 2$, *then* (1.1) *is oscillatory.*

Proof. Suppose to the contrary that there exists a solution $y(x)$ of (1.1) such that $y(x) > 0$ for $|x| \geq a_1 \geq a_0$. Define

$$W(x) = \frac{1}{f(y)}(A\nabla y)(x) + \frac{1}{2k}B, \quad x \in G[a_1, +\infty), \tag{2.1}$$

$$V(r) = \int_{S_r} W(x) \cdot \gamma(x) d\sigma, \quad x \in G[a_1, +\infty), \tag{2.2}$$

where ∇y denotes the gradient of $y(x)$, $\gamma(x) = \frac{x}{|x|}, |x| \neq 0$ is the outward unit normal to S_r. From (1.1) and (2.1), it follows that

$$\begin{aligned}
\nabla \cdot W(x) &= -\frac{f'(y)}{f^2(y)}(\nabla y)^T A \nabla y - \frac{1}{f(y)}[q(x)f(y) + B^T \nabla y - e(x)] + \frac{1}{2k}\nabla \cdot B \\
&\leq -k[W - \frac{1}{2k}B]^T A^{-1}[W - \frac{1}{2k}B] - q(x) - B^T A^{-1}[W - \frac{1}{2k}B] \\
&\quad + \frac{1}{2k}\nabla \cdot B + \frac{e(x)}{f(y)} \\
&= -kW^T A^{-1} W - q(x) + \frac{1}{4k}B^T A^{-1} B + \frac{1}{2k}\nabla \cdot B + \frac{e(x)}{f(y)}.
\end{aligned} \tag{2.3}$$

where W^T denotes the transpose of W. Using Green's formula in (2.2), we obtain

$$\begin{aligned}
V'(r) &= \int_{S_r} \nabla \cdot W(x) d\sigma \\
&\leq -\int_{S_r} q(x) d\sigma + \int_{S_r} \big[\frac{1}{4k}B^T A^{-1} B + \frac{1}{2k}\nabla \cdot B\big] d\sigma \\
&\quad - k\int_{S_r} (W^T A^{-1} W)(x) d\sigma + \int_{S_r} \frac{e(x)}{f(y)} d\sigma.
\end{aligned} \tag{2.4}$$

In view of (C1), we have $(W^T A^{-1} W)(x) \geq \lambda_{\max}^{-1}(x)|W(x)|^2$. Then, by Cauchy-Schwartz inequality, we obtain

$$\int_{S_r} |W(x)|^2 d\sigma \geq \frac{r^{1-N}}{\omega}\Big[\int_{S_r} W(x) \cdot \gamma(x) d\sigma\Big]^2.$$

Moreover, by (2.4) and (2.2), we get

$$\begin{aligned}
V'(r) &\leq -\int_{S_r} \Big[q(x) - \frac{1}{4k}B^T A^{-1} B - \frac{1}{2k}\nabla \cdot B\Big] d\sigma - \frac{1}{g_1(r)}V^2(r) + \int_{S_r} \frac{e(x)}{f(y)} d\sigma \\
&= -Q_1(r) - \frac{1}{g_1(r)}V^2(r) + \int_{S_r} \frac{e(x)}{f(y)} d\sigma.
\end{aligned} \tag{2.5}$$

By the assumption, we can choose $a_1, b_1 \geq T_0 (a_1 < b_1)$ such that $e(x) \leq 0, x \in G[a_1, b_1]$, then we have for $x \in G[a_1, b_1]$,

$$V'(r) \leq -Q_1(r) - \frac{1}{g_1(r)}V^2(r). \tag{2.6}$$

Let $H(r) \in \Psi(a_1, b_1)$ be given as in the hypothesis, Multiplying $H(r)$ throughout (2.6) and integrating from a_1 to b_1, we obtain

$$\int_{a_1}^{b_1} H(s)V'(s) ds \leq -\int_{a_1}^{b_1} Q_1(s)H(s) ds - \int_{a_1}^{b_1} H(s)\frac{1}{g_1(s)}V^2(s) ds. \tag{2.7}$$

Integrating by parts and using the fact $H(a_1) = H(b_1) = 0$, we find

$$-\int_{a_1}^{b_1} 2h(s)\sqrt{H(s)}V(s)ds \le -\int_{a_1}^{b_1} Q_1(s)H(s)ds - \int_{a_1}^{b_1} H(s)\frac{1}{g_1(s)}V^2(s)ds. \quad (2.8)$$

which is equivalent to

$$\begin{aligned} 0 &\le -\int_{a_1}^{b_1} Q_1(s)H(s)ds + \int_{a_1}^{b_1} \Big[2h(s)\sqrt{H(s)}V(s) - \frac{H(s)}{g_1(s)}V^2(s)\Big]ds \\ &= \int_{a_1}^{b_1} [g_1(s)h^2(s) - Q_1(s)H(s)]ds - \int_{a_1}^{b_1} \Big[\sqrt{\frac{H(s)}{g_1(s)}}V(s) - \sqrt{g_1(s)}h(s)\Big]^2 ds \quad (2.9) \\ &= M_1(H) - \int_{a_1}^{b_1} \Big[\sqrt{\frac{H(s)}{g_1(s)}}V(s) - \sqrt{g_1(s)}h(s)\Big]^2 ds \end{aligned}$$

Because $M_1(H) < 0$, (2.9) is incompatible. This contradiction proves that $y(x)$ must be oscillatory.

When $y(x)$ is eventually negative, we use $H(r) \in \Psi(a_2, b_2)$ and $e(x) \ge 0, x \in G[a_2, b_2]$ to reach a similar contradiction. the proof is complete. □

Following Philos [8] and Kong [6], we introduce the class of function $\Re$ which will be extensively and use in the sequel.

Let $D = \{(r, s) : -\infty < s \le r < \infty\}$, a function $H = H(r, s)$ is said to belong to $\Re$, if $H \in C(D, \mathbb{R})$ and satisfies

- (H1) $H(r, r) = 0, r \ge a_0; H(r, s) > 0$ for all $r > s \ge a_0$;
- (H2) H has partial derivatives $\partial H/\partial r$ and $\partial H/\partial s$ on D such that:

$$\frac{\partial H}{\partial r} = 2h_1(r, s)\sqrt{H(r, s)} \qquad \frac{\partial H}{\partial s} = -2h_2(r, s)\sqrt{H(r, s)},$$

  where $h_1, h_2 \in L_{\rm loc}(D, \mathbb{R})$.

Lemma 2.2. *Let* (C1)–(C4) *hold. Assume that there exist $c_1 < b_1 < c_2 < b_2$ such that $q(x) \ge 0$ for $x \in G[c_1, b_1] \cup G[c_2, b_2]$ and*

$$e(x) \begin{cases} \le 0, & x \in G[c_1, b_1], \\ \ge 0, & x \in G[c_2, b_2], \end{cases}$$

$y(x)$ is a solution of (1.1) *such that $y(x) > 0$ for $x \in G[c_1, b_1]$ and $y(x) < 0$ for $x \in G[c_2, b_2]$. Then for any $H \in \Re$ and $i = 1, 2$,*

$$\frac{1}{H(b_i, c_i)} \int_{c_i}^{b_i} H(b_i, s)Q_1(s)ds \le V(c_i) + \frac{1}{H(b_i, c_i)} \int_{c_i}^{b_i} g_1(s)h_2^2(b_i, s)ds. \quad (2.10)$$

Proof. Suppose that $y(x)$ is a solution of (1.1) such that $y(x) > 0$ for $x \in G[c_1, b_1]$ and $y(x) < 0$ for $x \in G[c_2, b_2]$. Then, similar to the proof of Theorem 2.1, we multiply (2.6) by $H(r, s)$, integrate it with respect to s from r to c_i, we get for

$s \in [c_i, r)$

$$
\begin{aligned}
&\int_{c_i}^{r} H(r,s)Q_1(s)ds \\
&\le -\int_{c_i}^{r} H(r,s)V'(s)ds - \int_{c_i}^{r} H(r,s)\frac{1}{g_1(s)}V^2(s)ds \\
&= H(r,c_i)V(c_i) - \int_{c_i}^{r} 2h_2(r,s)\sqrt{H(r,s)}V(s)ds - \int_{c_i}^{r} H(r,s)\frac{1}{g_1(s)}V^2(s)ds \\
&= H(r,c_i)V(c_i) + \int_{c_i}^{r} g_1(s)h_2^2(r,s)ds - \int_{c_i}^{r} \Big[\sqrt{\frac{H(r,s)}{g_1(s)}}V(s) + \sqrt{g_1(s)h_2^2(r,s)}\Big]^2 ds \\
&\le H(r,c_i)V(c_i) + \int_{c_i}^{r} g_1(s)h_2^2(r,s)ds
\end{aligned}
$$

Letting $r \to b_i^-$ and dividing both sides by $H(b_i, c_i)$ we obtain (2.10). □

Lemma 2.3. *Let* (C1)–(C4) *hold. Assume that there exist* $a_1 < c_1 < a_2 < c_2$ *such that* $q(x) \ge 0$ *for* $x \in G[a_1, c_1] \cup G[a_2, c_2]$ *and*

$$
e(x)\begin{cases} \le 0, & x \in G[a_1,c_1], \\ \ge 0, & x \in G[a_2,c_2], \end{cases}
$$

$y(x)$ *is a solution of* (1.1) *such that* $y(x) > 0$ *for* $x \in G[a_1, c_1]$ *and* $y(x) < 0$ *for* $x \in G[a_2, c_2]$. *Then for any* $H \in \Re$ *and* $i = 1, 2$,

$$
\frac{1}{H(c_i,a_i)}\int_{a_i}^{c_i} H(s,a_i)Q_1(s)ds \le -V(c_i) + \frac{1}{H(c_i,a_i)}\int_{a_i}^{c_i} g_1(s)h_1^2(s,a_i)ds. \tag{2.11}
$$

Proof. As in the proof of Lemma 2.2, we multiply (2.6) by $H(s, r)$ and integrate it with respect to s from r to c_i. We have

$$
\begin{aligned}
&\int_{r}^{c_i} H(s,r)Q_1(s)ds \\
&\le -\int_{r}^{c_i} H(s,r)V'(s)ds - \int_{r}^{c_i} H(r,s)\frac{1}{g_1(s)}V^2(s)ds \\
&= -H(c_i,r)V(c_i) + \int_{r}^{c_i} 2h_1(s,r)\sqrt{H(s,r)}V(s)ds - \int_{r}^{c_i} H(s,r)\frac{1}{g_1(s)}V^2(s)ds \\
&= -H(c_i,r)V(c_i) + \int_{r}^{c_i} g_1(s)h_1^2(s,r)ds \\
&\quad - \int_{c_i}^{r} \Big[\sqrt{\frac{H(s,r)}{g_1(s)}}V(s) - \sqrt{g_1(s)h_2^2(r,s)}\Big]^2 ds \\
&\le -H(c_i,r)V(c_i) + \int_{r}^{c_i} g_1(s)h_1^2(s,r)ds
\end{aligned}
$$

Letting $r \to a_i^+$ and dividing both sides by $H(c_i, a_i)$ we obtain (2.11). □

The following theorem is an immediate result from Lemmas 2.2 and 2.3.

Theorem 2.4. *Let* (C1)–(C4) *hold. Suppose that there exist* $a_1 < b_1 \le a_2 < b_2$ *such that* $q(x) \ge 0$ *for* $x \in G[a_1, b_1] \cup G[a_2, b_2]$ *and*

$$e(x) \begin{cases} \le 0, & x \in G[a_1, b_1], \\ \ge 0, & x \in G[a_2, b_2] \end{cases}$$

further, there exist some $c_i \in (a_i, b_i)$ *and some* $H \in \Re$ *such that*

$$\begin{aligned} &\frac{1}{H(c_i, a_i)} \int_{a_i}^{c_i} [H(s, a_i)Q_1(s) - g_1(s)h_1(s, a_i)]ds \\ &+ \frac{1}{H(b_i, c_i)} \int_{c_i}^{b_i} [H(b_i, s)Q_1(s) - g_1(s)h_2(b_i, s)]ds > 0 \end{aligned} \tag{2.12}$$

holds for $i = 1, 2$*, then every nontrivial solution of* (1.1) *has at least one zero either in* $G(a_1, b_1)$ *or in* $G(a_2, b_2)$*.*

Proof. Suppose to the contrary that there exists a solution $y(x)$ of (1.1) such that $y(x) > 0$ for $x \in G[T_0, +\infty)(T_0 \ge a_0)$, by the assumption, we can choose $a_1, b_1 \ge T_0(a_1 < b_1)$ such that $e(x) > 0, x \in G[a_1, b_1]$, then from Lemma 2.2 and Lemma 2.3 we see that (2.10) and (2.11) with $i = 1$ hold. Adding (2.10) and (2.11), we have that

$$\begin{aligned} &\frac{1}{H(c_1, a_1)} \int_{a_1}^{c_1} [H(s, a_1)Q_1(s) - g_1(s)h_1(s, a_1)]ds \\ &+ \frac{1}{H(b_1, c_1)} \int_{c_1}^{b_1} [H(b_1, s)Q_1(s) - g_1(s)h_2(b_1, s)]ds \le 0. \end{aligned} \tag{2.13}$$

which contradicts the assumption (2.12) with $i = 1$.

When $y(x)$ is eventually negative, we choose $a_2, b_2 \ge T_0$ such that $e(x) \le 0, x \in G[a_2, b_2]$ to reach a similar contradiction and hence completes the proof. ☐

Theorem 2.5. *Let* (C1)–(C4) *hold. Suppose that for any* $T \ge a_0$*, the following conditions hold:*

1. *there exist* $T \le a_1 < b_1 \le a_2 < b_2$ *such that*
$$e(x) \begin{cases} \le 0, & x \in G[a_1, b_1], \\ \ge 0, & x \in G[a_2, b_2] \end{cases}$$
and $q(x) \ge 0 (\not\equiv 0)$ *for* $x \in G[a_1, b_1] \cup G[a_2, b_2]$
2. *there exist some* $c_i \in (a_i, b_i), i = 1, 2,$ *and some* $H \in \Re$ *such that* $T \le a_1 < b_1 \le a_2 < b_2$ *and (2.12) holds.*

Then (1.1) *is oscillatory.*

Proof. Pick up a sequence $\{T_j\} \subset [a_0, +\infty)$, such that $j \to \infty$, $T_j \to \infty$. By the assumption, for each $j \in N$, there exist $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}$ such that $T_j \le a_1 < c_1 < b_1 \le a_2 < c_2 < b_2$ and (2.12) holds. From Theorem 2.4, every solution $y(x)$ has at least one zero on $G(a_1, b_1)$ or $G(a_2, b_2)$. Noting that $|x| > a_1 \ge T_j, j \in N$, we see that the zero set $\{x \in \Omega : y(x) = 0\}$ of $y(x)$ is is unbounded. Thus, every nontrivial solution of (1.1) is oscillatory. The proof is complete. ☐

Remark 2.6. With an appropriate choice of function H one can derive a number of oscillation criteria for (1.1).

As an immediate consequence of Theorem 2.5 we get the following oscillation criteria for (1.1).

Corollary 2.7. *Let* (C1)–(C4) *hold. Suppose that for any $T \geq a_0$, the following conditions hold:*

(1) *there exist $T \leq a_1 < b_1 \leq a_2 < b_2$ such that*
$$e(x) \begin{cases} \leq 0, & x \in G[a_1, b_1], \\ \geq 0, & x \in G[a_2, b_2], \end{cases}$$
and $q(x) \geq 0 (\not\equiv 0)$ for $x \in G[a_1, b_1] \cup G[a_2, b_2]$.

(2) *there exist some $c_i \in (a_i, b_i), i = 1, 2$, and some $H \in \Re$ such that $T \leq a_1 < b_1 \leq a_2 < b_2$ and the following two inequalities hold for $i = 1, 2$,*
$$\int_{a_i}^{c_i} [H(s, a_i)Q_1(s) - g_1(s)h_1^2(s, a_i)]ds > 0, \tag{2.14}$$
$$\int_{c_i}^{b_i} \big[H(b_i, s)Q_1(s) - g_1(s)h_2^2(b_i, s)\big]\, ds > 0. \tag{2.15}$$

Then (1.1) *is oscillatory.*

Moreover, let $H = H(r - s) \in \Re$, we have tha $\frac{\partial H(r-s)}{\partial r} = -\frac{\partial H(r-s)}{\partial s}$, and denote them by $h(r - s)$. The subclass of $\Re$ containing such $H(r - s)$ is denoted by $\Re_0$. Applying Theorem 2.5 to $\Re_0$, we obtain the following result.

Corollary 2.8. *Let* (C1)–(C4) *hold. Suppose that for any $T \geq a_0$, the following conditions hold:*

(1) *there exist $T \leq a_1 < 2c_1 - a_1 \leq a_2 < 2c_2 - a_2$ such that*
$$e(x) \begin{cases} \leq 0, & x \in G[a_1, 2c_1 - a_1], \\ \geq 0, & x \in G[a_2, 2c_2 - a_2], \end{cases}$$
and $q(x) \geq 0 (\not\equiv 0)$ for $x \in G[a_1, 2c_1 - a_1] \cup G[a_2, 2c_2 - a_2]$.

(2) *there exist some $H \in \Re_0$ such that $T \leq a_i < c_i$ for $i = 1, 2$ and the following inequality holds*
$$\int_{a_i}^{c_i} \big\{H(s - a_i)[Q_1(s) + Q_1(2c_i - s)] - [g_1(s) + g_1(2c_i - s)]h^2(s - a_i)\big\}\, ds > 0. \tag{2.16}$$

Then (1.1) *is oscillatory.*

Proof. Let $b_i = 2c_i - a_i$, then $H(b_i - c_i) = H(c_i - a_i) = H((b_i - a_i)/2)$, and for any $f \in L[a, b]$, we have
$$\int_{c_i}^{b_i} H(b_i - s)f(s)ds = \int_{a_i}^{c_i} H(s - a_i)f(2c_i - s)ds.$$
Thus that (2.16) holds implies that (2.12) holds for $H \in \Phi_0$ and therefor (1.1) is oscillatory by Theorem 2.4. □

Define
$$R(r) = \int_{a_0}^{r} \frac{1}{g_1(s)}ds, \quad r \geq a_0, \tag{2.17}$$
and let
$$H(r, s) = [R(r) - R(s)]^{\alpha}, \quad r \geq s \geq a_0, \tag{2.18}$$

where $\alpha > 1$ is a constant. Based on the above results, we obtain the following oscillation criteria of Kamenev's type.

Theorem 2.9. *Let* (C1)–(C4) *hold. Assume that* $\lim_{r\to\infty} R(r) = \infty$. *If for each* $T \ge a_0$, *the following conditions hold:*

(1) *there exist* $T \le a_1 < b_1 \le a_2 < b_2$ *such that*
$$e(x) \begin{cases} \le 0, & x \in G[a_1, b_1], \\ \ge 0, & x \in G[a_2, b_2], \end{cases}$$
and $q(x) \ge 0$ ($\not\equiv 0$) *for* $x \in G[a_1, b_1] \cup G[a_2, b_2]$

(2) *there exist* $c_i \in (a_i, b_i)$ *for* $i = 1, 2$, *such that* $T \le a_1 < b_1 \le a_2 < b_2$ *and the following inequalities hold for* $i = 1, 2$,
$$\frac{1}{[R(c_i) - R(a_i)]^{\alpha-1}} \int_{a_i}^{c_i} [R(s) - R(a_i)]^{\alpha} Q_1(s) ds \ge \frac{\alpha^2}{4(\alpha - 1)}, \tag{2.19}$$
$$\frac{1}{[R(b_i) - R(c_i)]^{\alpha-1}} \int_{c_i}^{b_i} [R(b_i) - R(s)]^{\alpha} Q_1(s) ds \ge \frac{\alpha^2}{4(\alpha - 1)}. \tag{2.20}$$

Then (1.1) *is oscillatory.*

Proof. It is easy to see that
$$h_1(r, s) = \alpha[R(r) - R(s)]^{\frac{\alpha-2}{2}} \frac{1}{2g_1(r)}, \quad h_2(r, s) = \alpha[R(r) - R(s)]^{\frac{\alpha-2}{2}} \frac{1}{2g_1(s)},$$
Hence we have
$$\begin{aligned} \int_{a_i}^{c_i} g_1(s) h_1^2(s, a_i) ds &= \int_{a_i}^{c_i} g_1(s) \alpha^2 [R(s) - R(a_i)]^{\alpha-2} \frac{1}{4g_1^2(s)} ds \\ &= \int_{a_i}^{c_i} [R(s) - R(a_i)]^{\alpha-2} \frac{\alpha^2}{4g_1(s)} ds \\ &= \frac{\alpha^2}{4(\alpha - 1)} [R(c_i) - R(a_i)]^{\alpha-1}. \end{aligned} \tag{2.21}$$
From (2.19) and (2.21) we have
$$\begin{aligned} &\frac{1}{[R(c_i) - R(a_i)]^{\alpha-1}} \int_{a_i}^{c_i} \left[H(s, a_i) Q_1(s) - g_1(s) h_1^2(s, a_i)\right] ds \\ &= \frac{1}{[R(c_i) - R(a_i)]^{\alpha-1}} \int_{a_i}^{c_i} [R(s) - R(a_i)]^{\alpha} Q_1(s) ds - \frac{\alpha^2}{4(\alpha - 1)} > 0; \end{aligned} \tag{2.22}$$
i.e., (2.14) holds. Similarly, (2.20) implies (2.15) holds. From Corollary 2.7, (1.1) is oscillatory. □

Example. Consider (1.1) with
$$A = \operatorname{diag}\Big(\frac{1}{\sqrt{r}}, \frac{1}{\sqrt{r}}\Big), \quad B^T = \Big(-\frac{2x_1}{r^2}, -\frac{2x_2}{r^2}\Big),$$
$$q(x) = \frac{\alpha}{r\sqrt{r}}, \quad f(y) = y + y^3, \quad e(x) = \frac{1}{r\sqrt{r}} \sin\sqrt{r},$$
where $r = \sqrt{x_1^2 + x_2^2}$, $r \ge 1$, $N = 2$. Let $k = 1$, hence
$$\lambda(r) = \frac{1}{\sqrt{r}}, \quad Q_1(r) = \frac{(2\alpha - 1)\pi}{\sqrt{r}}, \quad g_1(r) = 2\pi\sqrt{r}.$$

Choose $a_1 = n^2\pi^2$, $b_1 = (n+1)^2\pi^2$, $a_2 = (n+1)^2\pi^2$, $b_2 = (n+2)^2\pi^2$, and $H(r) = \sin^2\sqrt{r}$. It is easy to see that if $\alpha \geq 3/2$, then

$$\begin{aligned}
M_1(H) &= \int_{a_1}^{b_1} [g_1(s)h^2(s) - Q_1(s)H(s)]ds \\
&= \pi \int_{n^2\pi^2}^{(n+1)^2\pi^2} \frac{\cos^2\sqrt{s}}{2\sqrt{s}} ds - (2\alpha-1)\pi \int_{n^2\pi^2}^{(n+1)^2\pi^2} \frac{\sin^2\sqrt{s}}{\sqrt{s}} ds \\
&= \pi \int_{n\pi}^{(n+1)\pi} \cos^2 s ds - \frac{2\alpha-1}{2} \int_{n\pi}^{(n+1)\pi} \sin^2 s ds \\
&= \frac{\pi^2}{2} - \frac{(2\alpha-1)\pi^2}{4} \leq 0.
\end{aligned}$$

Similarly, for a_2, b_2 we can show that $M_2(H) \leq 0$. It follows from Theorem 2.1 that (1.1) is oscillatory when $\alpha \geq 3/2$.

3. Oscillation results when $\frac{\partial b_i}{\partial x_i}$ does not exist for some i

In this section, we establish oscillation criteria for (1.1) in case when $\frac{\partial b_i}{\partial x_i}$ does not exist for some i. For convenience, we let

$$Q_2(r) = \int_{S_r} \big[q(x) - \frac{1}{2k}\lambda(x)|B^T A^{-1}|^2\big] d\sigma, \quad g_2(r) = \frac{2\lambda(r)}{k}\omega r^{N-1},$$

We begin with the following lemma, the proof of this lemma is easy and thus omitted.

Lemma 3.1. *For two n-dimensional vectors $u, v \in \mathbb{R}^N$, and a positive constant c, then*

$$cuu^T + uv^T \geq \frac{c}{2}uu^T - \frac{1}{2c}vv^T. \tag{3.1}$$

Theorem 3.2. *Assume* (C1),(C3),(C4) *and*

(C2)' $b_i \in C^\mu_{\rm loc}(\Omega, \mathbb{R}), \mu \in (0,1)$, $i = 1, \dots, N$.

Suppose that for any $T \geq a_0$, there exist $T \leq a_1 < b_1 \leq a_2 < b_2$ such that

$$e(x) \begin{cases} \leq 0, & x \in G[a_1, b_1], \\ \geq 0, & x \in G[a_2, b_2] \end{cases}$$

and $q(x) \geq 0 (\not\equiv 0)$, $x \in G[a_1, b_1] \cup G[a_2, b_2]$ If there exist $H \in \Psi(a_i, b_i)$ such that

$$M_i(H) = \int_{a_i}^{b_i} \big\{g_2(s)h^2(s) - Q_2(s)H(s)\big\} ds < 0, \quad \textit{for } i = 1, 2,$$

where $\Psi(a_i, b_i)$ is defined in Theorem 2.1. Then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that there exists a solution $y(x)$ of (1.1) such that $y(x) > 0$ for $|x| \geq a_1 \geq a_0$. Define

$$W(x) = \frac{1}{f(y)}(A\nabla y)(x), \quad x \in G[a_1, +\infty), \tag{3.2}$$

$$V(r) = \int_{S_r} W(x) \cdot \gamma(x) d\sigma, \quad x \in G[a_1, +\infty), \tag{3.3}$$

where ∇y denotes the gradient of $y(x)$, $\gamma(x) = \frac{x}{|x|}, |x| \neq 0$ is the outward unit normal to S_r. From (1.1) and (3.2), it follows that

$$\begin{aligned} \nabla \cdot W(x) &= -\frac{f'(y)}{f^2(y)}(\nabla y)^T A \nabla y - \frac{1}{f(y)}[q(x)f(y) + B^T \nabla y - e(x)] \\ &\leq -kW^T A^{-1} W - q(x) - B^T A^{-1} W + \frac{e(x)}{f(y)} \\ &\leq -\frac{k}{\lambda(x)} W^T W - q(x) - B^T A^{-1} W + \frac{e(x)}{f(y)} \quad \text{(By Lemma 3.1)} \\ &\leq -\frac{k}{2\lambda(x)}|W|^2 + \frac{1}{2k}\lambda(x)|B^T A^{-1}|^2 - q(x) + \frac{e(x)}{f(y)}. \end{aligned} \tag{3.4}$$

where W^T denotes the transpose of W. Using Green's formula in (3.3), we get

$$\begin{aligned} V'(r) &= \int_{S_r} \nabla \cdot W(x) d\sigma \\ &\leq -\int_{S_r} q(x) d\sigma + \frac{1}{2k}\int_{S_r} \lambda(x)|B^T A^{-1}|^2 d\sigma \\ &\quad - \frac{k}{2\lambda(r)}\int_{S_r} |W|^2 d\sigma + \int_{S_r} \frac{e(x)}{y(x)} d\sigma. \end{aligned} \tag{3.5}$$

By Cauchy-Schwartz inequality,

$$\int_{S_r} |W(x)|^2 d\sigma \geq \frac{r^{1-N}}{\omega}\Big[\int_{S_r} W(x)\cdot\gamma(x) d\sigma\Big]^2.$$

Moreover, by (3.5) and (3.3),

$$V'(r) \leq -\int_{S_r}\Big[q(x) - \frac{1}{2k}\lambda(x)|B^T A^{-1}|^2\Big] d\sigma - \frac{1}{g_2(r)}V^2(r) + \int_{S_r}\frac{e(x)}{y(x)} d\sigma \tag{3.6}$$

The rest of proof is similar to that of Theorem 2.1 and hence omitted. □

Similar to the discussions in Section 2, we have the following results.

Lemma 3.3. *Let* (C1), (C2)', (C3), (C4) *hold. Assume that there exist* $c_1 < b_1 < c_2 < b_2$ *such that* $q(x) \geq 0$ *for* $x \in G[c_1, b_1] \cup G[c_2, b_2]$ *and*

$$e(x) \begin{cases} \leq 0, & x \in G[c_1, b_1], \\ \geq 0, & x \in G[c_2, b_2], \end{cases}$$

$y(x)$ *is a solution of* (1.1) *such that* $y(x) > 0$ *for* $x \in G[c_1, b_1]$ *and* $y(x) < 0$ *for* $x \in G[c_2, b_2]$. *Then for any* $H \in \Re$, *and* $i = 1, 2$,

$$\frac{1}{H(b_i, c_i)}\int_{c_i}^{b_i} H(b_i, s)Q_2(s) ds \leq V(c_i) + \frac{1}{H(b_i, c_i)}\int_{c_i}^{b_i} g_2(s)h_2^2(b_i, s) ds. \tag{3.7}$$

Lemma 3.4. *Let* (C1), (C2)', (C3), (C4) *hold. Assume that there exist* $a_1 < c_1 < a_2 < c_2$ *such that* $q(x) \geq 0$ *for* $x \in G[a_1, c_1] \cup G[a_2, c_2]$ *and*

$$e(x) \begin{cases} \leq 0, & x \in G[a_1, c_1], \\ \geq 0, & x \in G[a_2, c_2], \end{cases}$$

$y(x)$ is a solution of (1.1) such that $y(x) > 0$ for $x \in G[a_1, c_1]$ and $y(x) < 0$ for $x \in G[a_2, c_2]$. Then for any $H \in \Re$ and $i = 1, 2$,

$$\frac{1}{H(c_i, a_i)} \int_{a_i}^{c_i} H(s, a_i) Q_2(s) ds \le -V(c_i) + \frac{1}{H(c_i, a_i)} \int_{a_i}^{c_i} g_2(s) h_1^2(s, a_i) ds. \tag{3.8}$$

The following theorem is an immediate result from Lemmas 3.3 and 3.4.

Theorem 3.5. *Let (C1), (C2)', (C3), (C4) hold. Suppose that there exist $a_1 < b_1 \le a_2 < b_2$ such that $q(x) \ge 0$ for $x \in G[a_1, b_1] \cup G[a_2, b_2]$ and*

$$e(x) \begin{cases} \le 0, & x \in G[a_1, b_1], \\ \ge 0, & x \in G[a_2, b_2] \end{cases}$$

further, there exist some $c_i \in (a_i, b_i)$ and some $H \in \Re$ such that

$$\begin{aligned} &\frac{1}{H(c_i, a_i)} \int_{a_i}^{c_i} [H(s, a_i) Q_2(s) - g_2(s) h_1(s, a_i)] ds \\ &+ \frac{1}{H(b_i, c_i)} \int_{c_i}^{b_i} [H(b_i, s) Q_2(s) - g_2(s) h_2(b_i, s)] ds > 0, \quad i = 1, 2. \end{aligned} \tag{3.9}$$

Then every nontrivial solution of (1.1) has at least one zero either in $G(a_1, b_1)$ or in $G(a_2, b_2)$.

Theorem 3.6. *Let (C1), (C2)', (C3), (C4) hold. Suppose that for any $T \ge a_0$, the following conditions hold:*

1. *there exist $T \le a_1 < b_1 \le a_2 < b_2$ such that*
$$e(x) \begin{cases} \le 0, & x \in G[a_1, b_1], \\ \ge 0, & x \in G[a_2, b_2] \end{cases}$$
and $q(x) \ge 0 (\not\equiv 0)$ for $x \in G[a_1, b_1] \cup G[a_2, b_2]$
2. *there exist some $c_i \in (a_i, b_i), i = 1, 2$, and some $H \in \Re$ such that $T \le a_1 < b_1 \le a_2 < b_2$ and (3.9) holds.*

Then (1.1) is oscillatory.

Corollary 3.7. *Let (C1), (C2)', (C3), (C4) hold. Suppose that for any $T \ge a_0$, the following conditions hold:*

1. *there exist $T \le a_1 < b_1 \le a_2 < b_2$ such that*
$$e(x) \begin{cases} \le 0, & x \in G[a_1, b_1], \\ \ge 0, & x \in G[a_2, b_2], \end{cases}$$
and $q(x) \ge 0 (\not\equiv 0)$ for $x \in G[a_1, b_1] \cup G[a_2, b_2]$.
2. *there exist some $c_i \in (a_i, b_i), i = 1, 2$, and some $H \in \Re$ such that $T \le a_1 < b_1 \le a_2 < b_2$ and the following two inequalities hold for $i = 1, 2$,*
$$\int_{a_i}^{c_i} \left[H(s, a_i) Q_2(s) - g_2(s) h_1^2(s, a_i)\right] ds > 0, \tag{3.10}$$
$$\int_{c_i}^{b_i} \left[H(b_i, s) Q_2(s) - g_2(s) h_2^2(b_i, s)\right] ds > 0. \tag{3.11}$$

Then (1.1) is oscillatory.

Corollary 3.8. *Let* (C1), (C2)', (C3), (C4) *hold. Suppose that for any* $T \geq a_0$, *the following conditions hold:*

(1) *there exist* $T \leq a_1 < 2c_1 - a_1 \leq a_2 < 2c_2 - a_2$ *such that*
$$e(x)\begin{cases} \leq 0, & x \in G[a_1, 2c_1 - a_1], \\ \geq 0, & x \in G[a_2, 2c_2 - a_2], \end{cases}$$
and $q(x) \geq 0 (\not\equiv 0)$ *for* $x \in G[a_1, 2c_1 - a_1] \cup G[a_2, 2c_2 - a_2]$.

(2) *there exist some* $H \in \Re_0$ *such that* $T \leq a_i < c_i$ *for* $i = 1, 2$ *and the following inequality holds*
$$\int_{a_i}^{c_i} \left\{H(s - a_i)[Q_2(s) + Q_2(2c_i - s)] - [g_2(s) + g_2(2c_i - s)]h^2(s - a_i)\right\} ds > 0. \tag{3.12}$$

Then (1.1) *is oscillatory.*

Theorem 3.9. *Let* (C1), (C2)', (C3), (C4) *hold. Assume that* $\lim_{r\to\infty} R(r) = \infty$. *If for each* $T \geq a_0$, *the following conditions hold:*

(1) *there exist* $T \leq a_1 < b_1 \leq a_2 < b_2$ *such that*
$$e(x)\begin{cases} \leq 0, & x \in G[a_1, b_1], \\ \geq 0, & x \in G[a_2, b_2], \end{cases}$$
and $q(x) \geq 0 (\not\equiv 0)$ *for* $x \in G[a_1, b_1] \cup G[a_2, b_2]$

(2) *there exist* $c_i \in (a_i, b_i)$ *for* $i = 1, 2$, *such that* $T \leq a_1 < b_1 \leq a_2 < b_2$ *and the following inequalities hold for* $i = 1, 2$,
$$\frac{1}{[R(c_i) - R(a_i)]^{\alpha-1}} \int_{a_i}^{c_i} [R(s) - R(a_i)]^\alpha Q_2(s) ds \geq \frac{\alpha^2}{4(\alpha - 1)}, \tag{3.13}$$
$$\frac{1}{[R(b_i) - R(c_i)]^{\alpha-1}} \int_{c_i}^{b_i} [R(b_i) - R(s)]^\alpha Q_2(s) ds \geq \frac{\alpha^2}{4(\alpha - 1)}. \tag{3.14}$$
Where $R(r) = \int_{a_0}^{r} \frac{1}{g_2(s)} ds$.

Then (1.1) *is oscillatory.*

Remark 3.10. The results of the paper are presented in the form of a high degree of generality and thus they give wide possibilities of deriving the different oscillation criteria with an appropriate choice of the functions H. For instance, if we choose $H(r, s) = (r - s)^\alpha$, $[R(r) - R(s)]^\alpha$, $[\log(G(r)/G(s))]^\alpha$, or $[\int_s^r dz/\rho(z)]^\alpha$, etc., for $r \geq s \geq a_0$, where $\alpha > 1$ is a constant, $R(r) = \int_{a_0}^r ds/g_1(s)$, or $R(r) = \int_{a_0}^r ds/g_2(s)$, $G(r) = \int_r^\infty ds/g_1(s) < \infty$, or $G(r) = \int_r^\infty ds/g_2(s) < \infty$, for $r \geq a_0$, $\rho \in C([a_0, \infty), \mathbb{R}^+)$ satisfying $\int_{a_0}^\infty dz/\rho(z) = \infty$, then we can derive various explicit oscillation criteria.

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