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POSITIVE SOLUTIONS OF A THIRD-ORDER THREE-POINT BOUNDARY-VALUE PROBLEM

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Abstract. We obtain upper and lower estimates for positive solutions of a third-order three-point boundary-value problem. Sufficient conditions for the existence and nonexistence of positive solutions for the problem are also obtained. Then to illustrate our results, we include an example.

1. Introduction

Recently third-order multi-point boundary-value problems have attracted a lot of attention. In 2003, Anderson [2] considered the third-order boundary-value problem

$$u'''(t) = f(t, u(t)), \quad 0 \le t \le 1, \tag{1.1}$$

$$u(t_1) = u'(t_2) = \gamma u(t_3) + \delta u''(t_3) = 0. \tag{1.2}$$

In 2008, Graef and Yang [5] studied the third-order nonlocal boundary-value problem

$$u'''(t) = g(t)f(u(t)), \quad 0 \le t \le 1, \tag{1.3}$$

$$u(0) = u'(p) = \int_q^1 w(t)u''(t)dt = 0. \tag{1.4}$$

For more results on third-order boundary-value problems we refer the reader to [1, 4, 7, 9, 11, 12, 13].

In this paper, we consider the third-order three-point nonlinear boundary-value problem

$$u'''(t) = g(t)f(u(t)), \quad 0 \le t \le 1, \tag{1.5}$$

$$u(0) - \alpha u'(0) = u'(p) = \beta u'(1) + \gamma u''(1) = 0. \tag{1.6}$$

To our knowledge, the problem (1.5)-(1.6) has not been considered before. Note that the set of boundary conditions (1.6) is a very general one. For example, if we let $\alpha = \beta = 0$ and $p = \gamma = 1$, then (1.6) reduces to

$$u(0) = u'(1) = u''(1) = 0, \tag{1.7}$$

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which are often referred to as the $(1,2)$ focal boundary conditions. If we let $\alpha = 0$, $\beta = 0$, and $\gamma = 1$, then (1.6) reduces to

$$u(0) = u'(p) = u''(1) = 0. \tag{1.8}$$

The boundary-value problem that consists of the equation (1.5) and the boundary conditions (1.8) has been considered by Anderson and Davis in [3] and Graef and Yang in [6]. Our goal in this paper is to generalize some of the results from [3, 6] to the problem (1.5)-(1.6).

In this paper, we are interested in the existence and nonexistence of positive solutions of the problem (1.5)-(1.6). By a positive solution, we mean a solution $u(t)$ to the boundary-value problem such that $u(t) > 0$ for $0 < t < 1$.

In this paper, we assume that

- (H1) The functions $f : [0,\infty) \to [0,\infty)$ and $g : [0,1] \to [0,\infty)$ are continuous, and $g(t) \not\equiv 0$ on $[0,1]$.
- (H2) The parameters α, β, γ, and p are non-negative constants such that $\beta+\gamma > 0$, $0 < p \le 1$, and $2p(1+\alpha) \ge 1$.
- (H3) If $p = 1$, then $\gamma > 0$.

To prove some of our results, we will use the following fixed point theorem, which is due to Krasnosel'skii [10].

Theorem 1.1. *Let $(X, \|\cdot\|)$ be a Banach space over the reals, and let $P \subset X$ be a cone in X. Assume that Ω_1, Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let*

$$L : P \cap (\overline{\Omega_2} - \Omega_1) \to P$$

be a completely continuous operator such that, either

- (K1) *$\|Lu\| \le \|u\|$ if $u \in P \cap \partial\Omega_1$, and $\|Lu\| \ge \|u\|$ if $u \in P \cap \partial\Omega_2$; or*
- (K2) *$\|Lu\| \ge \|u\|$ if $u \in P \cap \partial\Omega_1$, and $\|Lu\| \le \|u\|$ if $u \in P \cap \partial\Omega_2$.*

Then L has a fixed point in $P \cap (\overline{\Omega_2} - \Omega_1)$.

Before the Krasnosel'skii fixed point theorem can be used to obtain any existence result, we have to find some nice estimates to positive solutions to the problem (1.5)–(1.6) first. These a priori estimates are essential to a successful application of the Krasnosel'skii fixed point theorem. It is based on these estimates that we can define an appropriate cone, on which Theorem 1.1 can be applied. Better estimates will result in sharper existence and nonexistence conditions.

We now fix some notation. Throughout we let $X = C[0,1]$ with the supremum norm

$$\|v\| = \max_{t\in[0,1]} |v(t)|, \quad \forall v \in X.$$

Obviously X is a Banach space. Also we define the constants

$$F_0 = \limsup_{x\to 0^+} \frac{f(x)}{x}, \quad f_0 = \liminf_{x\to 0^+} \frac{f(x)}{x},$$
$$F_\infty = \limsup_{x\to +\infty} \frac{f(x)}{x}, \quad f_\infty = \liminf_{x\to +\infty} \frac{f(x)}{x}.$$

These constants will be used later in our statements of the existence theorems.

This paper is organized as follows. In Section 2, we obtain some a priori estimates to positive solutions to the problem (1.5)-(1.6). In Section 3, we define a positive cone of the Banach space X using the estimates obtained in Section 2, and apply

Theorem 1.1 to establish some existence results for positive solutions of the problem (1.5)-(1.6). In Section 4, we present some nonexistence results. An example is given at the end of the paper to illustrate the existence and nonexistence results.

2. Green's Function and Estimates of Positive Solutions

In this section, we shall study Green's function for the problem (1.5)-(1.6), and prove some estimates for positive solutions of the problem. Throughout the section, we define the constant $M = \beta + \gamma - \beta p$. By conditions (H2) and (H3), we know that M is a positive constant.

Lemma 2.1. *If $u \in C^3[0,1]$ satisfies the boundary conditions* (1.6) *and $u'''(t) \equiv 0$ on $[0,1]$, then $u(t) \equiv 0$ on $[0,1]$.*

Proof. Since $u'''(t) \equiv 0$ on $[0,1]$, there exist constants a_1, a_2, and a_3 such that

$$u(t) = a_1 + a_2 t + a_3 t^2, \quad 0 \le t \le 1.$$

Because $u(t)$ satisfies the boundary conditions (1.6), we have

$$\begin{pmatrix} 1 & -\alpha & 0 \\ 0 & 1 & 2p \\ 0 & \beta & 2(\beta+\gamma) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.1}$$

The determinant of the coefficient matrix for the above linear system is $2M > 0$. Therefore, the system (2.1) has only the trivial solution $a_1 = a_2 = a_3 = 0$. Hence $u(t) \equiv 0$ on $[0,1]$. The proof is complete. □

We need the indicator function χ to write the expression of Green's function for the problem (1.5)-(1.6). Recall that if $[a,b] \subset R := (-\infty, +\infty)$ is a closed interval, then the indicator function χ of $[a,b]$ is given by

$$\chi_{[a,b]}(t) = \begin{cases} 1, & \text{if } t \in [a,b], \\ 0, & \text{if } t \notin [a,b]. \end{cases}$$

Now we define the function $G : [0,1] \times [0,1] \to [0,\infty)$ by

$$\begin{aligned} G(t,s) = {}& \frac{\beta+\gamma-\beta s}{2(\beta+\gamma-p\beta)}(2\alpha p + 2pt - t^2) + \frac{(t-s)^2}{2}\chi_{[0,t]}(s) \\ & - \frac{p-s}{2(\beta+\gamma-p\beta)}(2(\alpha+t)(\beta+\gamma) - \beta t^2)\chi_{[0,p]}(s). \end{aligned}$$

We are going to show that $G(t,s)$ is Green's function for the problem (1.5)-(1.6).

Lemma 2.2. *Let $h \in C[0,1]$. If*

$$y(t) = \int_0^1 G(t,s)h(s)ds, \quad 0 \le t \le 1,$$

then $y(t)$ satisfies the boundary conditions (1.6) *and $y'''(t) = h(t)$ for $0 \le t \le 1$.*

Proof. If

$$y(t) = \int_0^1 G(t,s)h(s)ds,$$

then

$$
\begin{aligned}
y(t) = &-\int_0^p \frac{p-s}{2M}(2(\alpha+t)(\beta+\gamma)-\beta t^2)h(s)ds + \int_0^t \frac{(t-s)^2}{2}h(s)ds \\
&+\int_0^1 \frac{\beta+\gamma-\beta s}{2M}(2\alpha p+2pt-t^2)h(s)ds.
\end{aligned} \tag{2.2}
$$

Differentiating the above expression, we have

$$
\begin{aligned}
y'(t) = &-\int_0^p \frac{p-s}{M}(\beta+\gamma-\beta t)h(s)ds + \int_0^t (t-s)h(s)ds \\
&+\int_0^1 \frac{\beta+\gamma-\beta s}{M}(p-t)h(s)ds,
\end{aligned} \tag{2.3}
$$

$$
y''(t) = \beta\int_0^p \frac{p-s}{M}h(s)ds + \int_0^t h(s)ds - \int_0^1 \frac{\beta+\gamma-\beta s}{M}h(s)ds, \tag{2.4}
$$

and

$$
y'''(t) = h(t), \quad 0 \le t \le 1.
$$

It is easy to see from (2.3) that $y'(p) = 0$.

By making the substitution $t = 0$ in (2.2) and (2.3), we get

$$
y(0) = -\int_0^p \frac{p-s}{M}\alpha(\beta+\gamma)h(s)ds + \int_0^1 \frac{\beta+\gamma-\beta s}{M}\alpha p h(s)ds \tag{2.5}
$$

and

$$
y'(0) = -\int_0^p \frac{p-s}{M}(\beta+\gamma)h(s)ds + \int_0^1 \frac{\beta+\gamma-\beta s}{M}p h(s)ds. \tag{2.6}
$$

It is clear from (2.5) and (2.6) that $y(0) - \alpha y'(0) = 0$.

By making the substitution $t = 1$ in (2.3) and (2.4), we get

$$
y'(1) = -\gamma\int_0^p \frac{p-s}{M}h(s)ds + \int_0^1 (1-s)h(s)ds + \int_0^1 \frac{\beta+\gamma-\beta s}{M}\cdot(p-1)h(s)ds
$$

and

$$
y''(1) = \beta\int_0^p \frac{p-s}{M}h(s)ds + \int_0^1 h(s)ds - \int_0^1 \frac{\beta+\gamma-\beta s}{M}h(s)ds.
$$

Simplifying the last two equations, we get

$$
y'(1) = -\gamma\int_0^p \frac{p-s}{M}h(s)ds + \gamma\int_0^1 \frac{p-s}{M}h(s)ds, \tag{2.7}
$$

$$
y''(1) = \beta\int_0^p \frac{p-s}{M}h(s)ds - \beta\int_0^1 \frac{p-s}{M}h(s)ds. \tag{2.8}
$$

It is easily seen from (2.7) and (2.8) that $\beta y'(1) + \gamma y''(1) = 0$. The proof is now complete. $\square$

Lemma 2.3. *Let $h \in C[0,1]$ and $y \in C^3[0,1]$. If $y(t)$ satisfies the boundary conditions* (1.6) *and $y'''(t) = h(t)$ for $0 \le t \le 1$, then*

$$
y(t) = \int_0^1 G(t,s)h(s)ds, \quad 0 \le t \le 1.
$$

Proof. Suppose that $y(t)$ satisfies the boundary conditions (1.6) and $y'''(t) = h(t)$ for $0 \le t \le 1$. Let

$$k(t) = \int_0^1 G(t,s)h(s)ds, \quad 0 \le t \le 1.$$

By Lemma 2.2 we have $k'''(t) = h(t)$ for $0 \le t \le 1$, and $k(t)$ satisfies the boundary conditions (1.6). If we let $m(t) = y(t) - k(t)$, $0 \le t \le 1$, then $m'''(t) = 0$ for $0 \le t \le 1$ and $m(t)$ satisfies the boundary conditions (1.6). By Lemma 2.1, we have $m(t) \equiv 0$ on $[0,1]$, which implies that

$$y(t) = \int_0^1 G(t,s)h(s)ds, \quad 0 \le t \le 1.$$

The proof is complete. □

We see from the last two lemmas that

$$y(t) = \int_0^1 G(t,s)h(s)ds \quad \text{for } 0 \le t \le 1$$

if and only if $y(t)$ satisfies the boundary conditions (1.6) and $y'''(t) = h(t)$ for $0 \le t \le 1$. Hence the problem (1.5)-(1.6) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s)g(s)f(u(s))ds, \quad 0 \le t \le 1, \tag{2.9}$$

and $G(t,s)$ is Green's function for the problem (1.5)-(1.6).

Now we investigate the sign property of $G(t,s)$. We start with a technical lemma.

Lemma 2.4. *If* (H2) *holds, then*

$$2p\alpha + 2pt - t^2 \ge 0, \quad 0 \le t \le 1.$$

Proof. Assuming that (H2) holds; if $0 \le t \le 1$, then

$$2p\alpha + 2pt - t^2 = t(1-t) + 2p\alpha(1-t) + (2p(1+\alpha) - 1)t \ge 0.$$

The proof is complete. □

Lemma 2.5. *If* (H2)–(H3) *hold, then we have*

(1) *If* $0 \le t \le 1$ *and* $0 \le s \le 1$, *then* $G(t,s) \ge 0$.
(2) *If* $0 < t < 1$ *and* $0 < s < 1$, *then* $G(t,s) > 0$.

Proof. We shall prove (1) only. We take four cases to discuss the sign property of $G(t,s)$.

(i) If $s \ge p$ and $s \ge t$, then

$$G(t,s) = \frac{\beta + \gamma - \beta s}{2M}(2p\alpha + 2pt - t^2) \ge 0.$$

(ii) If $s \ge p$ and $s \le t$, then

$$G(t,s) = \frac{\beta + \gamma - \beta s}{2M}(2p\alpha + 2pt - t^2) + \frac{(t-s)^2}{2} \ge 0.$$

(iii) If $s \le p$ and $s \ge t$, then

$$G(t,s) = \frac{1}{2}(2\alpha s + 2st - t^2) \ge 0.$$

(iv) If $s \le p$ and $s \le t$, then
$$G(t,s) = \alpha s + \frac{s^2}{2} \ge 0.$$
Therefore $G(t,s)$ is nonnegative in all four cases. The proof of (1) is complete.

If we take a closer look at the expressions of $G(t,s)$ in the four cases, then we will see easily that (2) is also true. We leave the details to the reader. □

Lemma 2.6. *If $u \in C^3[0,1]$ satisfies* (1.6), *and*
$$u'''(t) \ge 0 \quad \text{for } 0 \le t \le 1, \tag{2.10}$$
then

1. $u(t) \ge 0$ *for* $0 \le t \le 1$.
2. $u'(t) \ge 0$ *on* $[0,p]$ *and* $u'(t) \le 0$ *on* $[p,1]$.
3. $u(p) = \|u\|$.

Proof. Note that $G(t,s) \ge 0$ if $t,s \in [0,1]$. If $u'''(t) \ge 0$ on $[0,1]$, then for each $t \in [0,1]$ we have
$$u(t) = \int_0^1 G(t,s)u'''(s)\,ds \ge 0.$$
The proof of (1) is complete.

It follows from (2.3) that
$$u'(t) = \int_t^p (s-t)u'''(s)\,ds + \int_p^1 \frac{\beta+\gamma-\beta s}{M} \cdot (p-t)u'''(s)ds.$$
If $0 \le t \le p$, then it is obvious that $u'(t) \ge 0$. If $t \ge p$, then we put (2.3) into an equivalent form
$$u'(t) = -\int_p^t \frac{(s-p)(\gamma+\beta(1-t))}{M}u'''(s)\,ds - \int_t^1 \frac{\beta+\gamma-\beta s}{M}(t-p)u'''(s)\,ds,$$
from which we can see easily that $u'(t) \le 0$.

Part (3) of the lemma follows immediately from parts (1) and (2). The proof is now complete. □

Throughout the remainder of the paper, we define the continuous function $a : [0,1] \to [0,+\infty)$ by
$$a(t) = \frac{2p\alpha + 2pt - t^2}{2p\alpha + p^2}, \quad 0 \le t \le 1.$$
It can be shown that
$$a(t) \ge \min\{t, 1-t\}, \quad 0 \le t \le 1.$$
The proof of the last inequality is omitted.

Lemma 2.7. *If $u \in C^3[0,1]$ satisfies* (2.10) *and the boundary conditions* (1.6), *then $u(t) \ge a(t)u(p)$ on* $[0,1]$.

Proof. If we define
$$h(t) = u(t) - a(t)u(p), \quad 0 \le t \le 1,$$
then
$$h'''(t) = u'''(t) \ge 0, \quad 0 \le t \le 1.$$
Obviously we have $h(p) = h'(p) = 0$. To prove the lemma, it suffices to show that $h(t) \ge 0$ for $0 \le t \le 1$. We take two cases to continue the proof.

Case I: $h'(0) \le 0$. We note that $h'(p) = 0$ and h' is concave upward on $[0,1]$. Since $h'(0) \le 0$, we have $h'(t) \le 0$ on $[0,p]$ and $h'(t) \ge 0$ on $[p,1]$. Since $h(p) = 0$, we have $h(t) \ge 0$ on $[0,1]$.
Case II: $h'(0) > 0$. It is easy to see from the definition of $h(t)$ that

$$h(0) = \alpha h'(0).$$

Since $\alpha \ge 0$, we have $h(0) \ge 0$.

Because $h'(0) > 0$ and $h(0) \ge 0$, there exists $\delta \in (0,p)$ such that $h(\delta) > 0$.

By the mean value theorem, since $h(\delta) > h(p) = 0$, there exists $r_1 \in (\delta, p)$ such that $h'(r_1) < 0$. Now we have $h'(0) > 0$, $h'(r_1) < 0$, and $h'(p) = 0$. Because $h'(t)$ is concave upward on $[0,1]$, there exists $r_2 \in (0, r_1)$ such that

$$h'(t) > 0 \quad \text{on } [0,r_2), \quad h'(t) \le 0 \quad \text{on } [r_2,p], \quad h'(t) \ge 0 \quad \text{on } (p,1].$$

Since $h(0) \ge 0$ and $h(p) = 0$, we have $h(t) \ge 0$ on $[0,1]$.

We have shown that $h(t) \ge 0$ on $[0,1]$ in both cases. The proof is complete. □

In summary, we have

Theorem 2.8. *Suppose that* (H1)–(H3) *hold. If $u \in C^3[0,1]$ satisfies* (2.10) *and the boundary conditions* (1.6)*, then $u(p) = \|u\|$ and $u(t) \ge a(t)u(p)$ on $[0,1]$. In particular, if $u \in C^3[0,1]$ is a nonnegative solution to the boundary-value problem* (1.5)-(1.6)*, then $u(p) = \|u\|$ and $u(t) \ge a(t)u(p)$ on $[0,1]$.*

3. Existence of Positive Solutions

Now we give some notation. Define the constants

$$A = \int_0^1 G(p,s)g(s)a(s)\,ds, \quad B = \int_0^1 G(p,s)g(s)\,ds$$

and let

$$P = \{v \in X : v(p) \ge 0,\ a(t)v(p) \le v(t) \le v(p) \text{ on } [0,1]\}.$$

Obviously X is a Banach space and P is a positive cone of X. Define an operator $T : P \to X$ by

$$Tu(t) = \int_0^1 G(t,s)g(s)f(u(s))ds, \quad 0 \le t \le 1,\ u \in X.$$

It is well known that $T : P \to X$ is a completely continuous operator. And by the same argument as in Theorem 2.8 we can prove that $T(P) \subset P$.

Now the integral equation (2.9) is equivalent to the equality

$$Tu = u, \quad u \in P.$$

To solve problem (1.5)-(1.6) we need only to find a fixed point of T in P.

Theorem 3.1. *If $BF_0 < 1 < Af_\infty$, then the problem* (1.5)-(1.6) *has at least one positive solution.*

Proof. Choose $\epsilon > 0$ such that $(F_0 + \epsilon)B \le 1$. There exists $H_1 > 0$ such that

$$f(x) \le (F_0 + \epsilon)x \quad \text{for } 0 < x \le H_1.$$

For each $u \in P$ with $\|u\| = H_1$, we have

$$\begin{aligned}
(Tu)(p) &= \int_0^1 G(p,s)g(s)f(u(s))\,ds \\
&\le \int_0^1 G(p,s)g(s)(F_0+\epsilon)u(s)\,ds \\
&\le (F_0+\epsilon)\|u\| \int_0^1 G(p,s)g(s)ds \\
&= (F_0+\epsilon)\|u\|B \le \|u\|,
\end{aligned}$$

which means $\|Tu\| \le \|u\|$. So, if we let $\Omega_1 = \{u \in X : \|u\| < H_1\}$, then

$$\|Tu\| \le \|u\|, \quad \text{for } u \in P \cap \partial\Omega_1.$$

To construct Ω_2, we first choose $c \in (0, 1/4)$ and $\delta > 0$ such that

$$(f_\infty - \delta)\int_c^{1-c} G(p,s)g(s)a(s)\,ds > 1.$$

There exists $H_3 > 0$ such that $f(x) \ge (f_\infty - \delta)x$ for $x \ge H_3$. Let $H_2 = H_1 + H_3/c$. If $u \in P$ with $\|u\| = H_2$, then for $c \le t \le 1-c$, we have

$$u(t) \ge \min\{t, 1-t\}\|u\| \ge cH_2 \ge H_3.$$

So, if $u \in P$ with $\|u\| = H_2$, then

$$\begin{aligned}
(Tu)(p) &\ge \int_c^{1-c} G(p,s)g(s)f(u(s))\,ds \\
&\ge \int_c^{1-c} G(p,s)g(s)(f_\infty-\delta)u(s)ds \\
&\ge \int_c^{1-c} G(p,s)g(s)a(s)\,ds \cdot (f_\infty-\delta)\|u\| \ge \|u\|,
\end{aligned}$$

which implies $\|Tu\| \ge \|u\|$. So, if we let $\Omega_2 = \{u \in X \mid \|u\| < H_2\}$, then $\overline{\Omega_1} \subset \Omega_2$, and

$$\|Tu\| \ge \|u\|, \quad \text{for } u \in P \cap \partial\Omega_2.$$

Then the condition (K1) of Theorem 1.1 is satisfied, and so there exists a fixed point of T in P. The proof is complete. $\square$

Theorem 3.2. *If $BF_\infty < 1 < Af_0$, then* (1.5)-(1.6) *has at least one positive solution.*

The proof of the above theorem is similar to that of Theorem 3.1 and is therefore omitted.

4. Nonexistence Results and Example

In this section, we give some sufficient conditions for the nonexistence of positive solutions.

Theorem 4.1. *Suppose that* (H1)–(H3) *hold. If $Bf(x) < x$ for all $x \in (0, +\infty)$, then* (1.5)-(1.6) *has no positive solution.*

Proof. Assume the contrary that $u(t)$ is a positive solution of (1.5)-(1.6). Then $u \in P$, $u(t) > 0$ for $0 < t < 1$, and

$$\begin{aligned} u(p) &= \int_0^1 G(p,s)g(s)f(u(s))\,ds \\ &< B^{-1}\int_0^1 G(p,s)g(s)u(s)\,ds \\ &\le B^{-1}\int_0^1 G(p,s)g(s)ds \cdot u(p) \le u(p), \end{aligned}$$

which is a contradiction. □

In a similar fashion, we can prove the following theorem.

Theorem 4.2. *Suppose that* (H1)–(H3) *hold. If* $Af(x) > x$ *for all* $x \in (0, +\infty)$*, then* (1.5)-(1.6) *has no positive solution.*

We conclude this paper with an example.

Example 4.3. Consider the third-order boundary-value problem

$$u'''(t) = g(t)f(u(t)), \quad 0 < t < 1, \tag{4.1}$$

$$u(0) - u'(0) = u'(3/4) = u'(1) + u''(1) = 0, \tag{4.2}$$

where

$$\begin{gathered} g(t) = (1+t)/10, \quad 0 \le t \le 1, \\ f(u) = \lambda u \frac{1+3u}{1+u}, \quad u \ge 0. \end{gathered}$$

Here λ is a positive parameter. Obviously we have $F_0 = f_0 = \lambda$, $F_\infty = f_\infty = 3\lambda$. Calculations indicate that

$$A = \frac{198989}{2112000}, \quad B = \frac{9889}{102400}.$$

From Theorem 3.1 we see that if

$$3.538 \approx \frac{1}{3A} < \lambda < \frac{1}{B} \approx 10.355,$$

then problem (4.1)-(4.2) has at least one positive solution. From Theorems 4.1 and 4.2, we see that if

$$\lambda < \frac{1}{3B} \approx 3.45 \quad \text{or} \quad \lambda > \frac{1}{A} \approx 10.613,$$

then problem (4.1)-(4.2) has no positive solution.

This example shows that our existence and nonexistence conditions are quite sharp.

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