LINEAR MATRIX DIFFERENTIAL EQUATIONS OF HIGHER-ORDER AND APPLICATIONS

RAJAE BEN TAHER, MUSTAPHA RACHIDI

ABSTRACT. In this article, we study linear differential equations of higher-order whose coefficients are square matrices. The combinatorial method for computing the matrix powers and exponential is adopted. New formulas representing auxiliary results are obtained. This allows us to prove properties of a large class of linear matrix differential equations of higher-order, in particular results of Apostol and Kolodner are recovered. Also illustrative examples and applications are presented.

1. INTRODUCTION

Linear matrix differential equations are important in many fields of mathematics and their applications. Among the most simple and fundamental are the first order linear systems with constant coefficients,

$$X'(t) = AX(t), \quad \text{such that } X(0) = 1_d, \tag{1.1}$$

where 1_d is the identity matrix of $\mathcal{M}(d; \mathbb{C})$, $A \in \mathcal{M}(d; \mathbb{C})$ and $X \in \mathcal{C}^\infty(\mathbb{R}; \mathcal{M}(d; \mathbb{C}))$. For the sake of simplicity we set in the sequel $\mathcal{A}_d = \mathcal{M}(d; \mathbb{C})$. The system (1.1) has been extensively studied; its solutions depend closely on the computation of e^{tA} ($t \in \mathbb{R}$). To perform this computation, many theoretical and numerical methods have been developed (see [2, 3, 4, 6, 8, 11, 12, 14, 16, 17] for example). The combinatorial method is among those considered recently for obtaining some practical expressions of A^n ($n \geq d$) and e^{tA} ($t \in \mathbb{R}$) (see [2, 3, 14]). Techniques of this method are based on properties of some linear recursive sequences (see [10, 13] for example), known in the literature as r-generalized Fibonacci sequences (to abbreviate we write r-GFS).

Let $A_0, \ldots, A_{s-1}$ be in $\mathcal{A}_d$ and consider the linear matrix differential equation of higher-order

$$X^{(s)}(t) = A_0 X^{(s-1)}(t) + \cdots + A_{s-1} X(t), \tag{1.2}$$

subject to the initial conditions $X(0)$, $X'(0)$, …, $X^{(s-1)}(0)$. To study (1.2) it is customary to write it under its equivalent form as the linear first order system (1.1). More precisely, (1.2) takes the form $Y'(t) = BY(t)$, where $B \in \mathcal{A}_{ds}$ and

---

$Y \in \mathcal{C}^\infty(\mathbb{R}; \mathcal{A}_{ds})$ (see Section 4 for more details). For $s = 2$ properties of the linear matrix differential equation (1.2),

$$X''(t) = A_0 X'(t) + A_1 X(t), \tag{1.3}$$

have been studied (directly) by Apostol and Kolodner (see [1, 9]) without appealing to its equivalent form as equation (1.1).

The main purpose of this paper is to study solutions of a large class of linear matrix differential equations (1.2), whose coefficients are in the algebra $\mathcal{A}_d$. First we consider (1.2) when $A_0 = \cdots = A_{r-2} = \Theta_d$ (the zero matrix of $\mathcal{A}_d$) and $A_{r-1} = A \neq \Theta_d$. Solutions to these matrix differential equations are expressed in terms of the coefficients of the polynomial $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$ satisfying $P(A) = \Theta_d$ and matrices $A^0 = 1_d$, A, ..., A^{r-1}. Moreover, these solutions are also described with the aid of the zeros of the polynomial $P(z)$. Furthermore, the case $r = 2$ is improved and the fundamental results of Apostol-Kolodner are recovered and their extension is established. Secondly, in light of a survey of the general equation (1.2), we manage to supply their solutions under a combinatorial form.

In Section 2 we recall auxiliary results on the powers and exponential of an element $A \in \mathcal{A}_d$. In Section 3, we study the class of linear matrix differential equations of higher-order (1.2) associated to $A_0 = \cdots = A_{s-2} = \Theta_d$ and $A_{s-1} = A$ (we call this class the *Higher order linear matrix differential of Apostol type*) and recover the results of Apostol-Kolodner. In Section 4 we consider the combinatorial aspect of the matrix powers and exponential in order to explore the general setting of linear matrix differential equations (1.2). Finally, concluding remarks are stated and a future problem is formulated in Section 5.

2. Auxiliary results on the powers and exponential of a matrix

We review here some basic results on the matrix powers and exponential needed in the next sections. That is, we recall some results of [2, 3, 4] and set forth a new result in Proposition 2.2. To begin, let $A_0, \ldots, A_{r-1}$ and $S_0, \ldots, S_{r-1}$ be two finite sequences of $\mathcal{A}_d$ with $A_{r-1} \neq \Theta_d$. Consider the recursive sequence $\{Y_n\}_{n \geq 0}$ such that $Y_n = S_n$ for $0 \leq n \leq r - 1$ and

$$Y_{n+1} = A_0 Y_n + \cdots + A_{r-1} Y_{n-r+1}, \quad \text{for every } n \geq r - 1. \tag{2.1}$$

When $A_j A_k = A_k A_j$, for every $0 \leq j, k \leq r - 1$ and following the same straightforward computation as in [2, 14] we obtain,

$$Y_n = \rho(n, r) W_0 + \cdots + \rho(n - r + 1, r) W_{r-1}, \quad \text{for every } n \geq r, \tag{2.2}$$

where $W_p = A_{r-1} S_p + \cdots + A_p S_{r-1}$ $(p = 0, 1, \ldots, r - 1)$ and

$$\rho(n, r) = \sum_{k_0 + 2k_1 + \cdots + rk_{r-1} = n-r} \frac{(k_0 + \cdots + k_{r-1})!}{k_0! k_1! \ldots k_{r-1}!} A_0^{k_0} A_1^{k_1} \ldots A_{r-1}^{k_{r-1}}, \tag{2.3}$$

for every $n \geq r$, with $\rho(r, r) = 1$ and $\rho(n, r) = 0$ for $n \leq r - 1$ (see [2, 3, 11, 13, 14]). The computation of the powers A^n $(n \geq r)$ follows by a direct application of (2.2)-(2.3). Indeed, the polynomial equation $P(A) = \Theta_d$ shows that $A^{n+1} = a_0 A^n + \cdots + a_{r-1} A^{n-r+1}$, for every $n \geq r - 1$, and by the way the sequence $\{A^n\}_{n \geq 0}$ is an r-GFS in $\mathcal{A}_d$, whose coefficients and initial values are $A_0 = a_0, \ldots, A_{r-1} = a_{r-1}$,

$S_0 = A^0 = 1_d, \ldots, S_{r-1} = A^{r-1}$ (respectively). In light of [2, 14] we have the following characterization of the powers of A,

$$A^n = \sum_{p=0}^{r-1} \Big(\sum_{j=0}^{p} a_{r-p+j-1} \rho(n-j,r) \Big) A^p, \text{ for every } n \geq r, \qquad (2.4)$$

where

$$\rho(n,r) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-r} \frac{(k_0+\cdots+k_{r-1})!}{k_0!k_1!\ldots k_{r-1}!} a_0^{k_0} \ldots a_{r-1}^{k_{r-1}}, \quad \text{for } n \geq r, \quad (2.5)$$

with $\rho(r,r) = 1$ and $\rho(n,r) = 0$ for $n \leq r-1$ (see [2, 14]). The matrix exponential e^{tA} ($A \in \mathcal{A}_d$) is defined as usual by the series $e^{tA} = \sum_{n=0}^{+\infty}(t^n/n!)A^n$. It turns out following [2] that direct computation using (2.4)-(2.5) allows us to derive that the unique solution of the matrix differential equation (1.1), satisfying the initial condition $X(0) = 1_d$, is $X(t) = e^{tA} = \sum_{k=0}^{r-1} \Omega_k(t)A^k$ such that

$$\Omega_k(t) = \frac{t^k}{k!} + \sum_{j=0}^{k} a_{r-k+j-1}\omega_j(t), \quad (0 \leq k \leq r-1), \qquad (2.6)$$

with

$$\omega_j(t) = \sum_{n=0}^{+\infty} \rho(n-j,r)\frac{t^n}{n!} \quad (0 \leq j \leq r-1)$$

and $\rho(n,r)$ given by (2.5).

As shown here the preceding expressions of A^n ($n \geq r$) and e^{tA} are formulated in terms of the coefficients of the polynomial $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$ and the first r powers $A^0 = 1_d, A, \ldots, A^{r-1}$. The goal now is to express A^n ($n \geq r$) and e^{tA} using the roots of the polynomial $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$ satisfying $P(A) = \Theta_d$. To this aim, let $\lambda_1, \ldots, \lambda_s$ be the distinct roots of $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$ of multiplicities $m_1, \ldots, m_s$ (respectively). For every j ($1 \leq j \leq s$) we consider the sequence $\{b_{i,j}\}_{0 \leq i \leq m_j-1}$ such that $b_{i,j} = 0$ for $i > m_1 + \cdots + m_{j-1} + m_{j+1} + \cdots + m_s$ and otherwise

$$b_{i,j} = \sum_{h_1+\cdots+h_{j-1}+h_{j+1}+\cdots+h_s=i,h_d \leq m_d} \prod_{d=1,d\neq j}^{s} \binom{h_d}{m_d}(\lambda_j - \lambda_d)^{m_d-h_d}. \qquad (2.7)$$

For every j ($1 \leq j \leq s$), let $\{\alpha_{i,j}\}_{0 \leq i \leq m_j-1}$ be the sequence defined by $\alpha_{0,j} = 1$ and

$$\alpha_{i,j} = \frac{-1}{b_{0,j}} \left(b_{1,j}\alpha_{i-1,j} + b_{2,j}\alpha_{i-2,j} + \cdots + b_{i-1,j}\alpha_{1,j} + b_{i,j}\alpha_{0,j} \right), \qquad (2.8)$$

where the $b_{i,j}$ are given by (2.7). Recall that $\{b_{i,j}\}_{0 \leq i \leq m_j-1}$ and $\{\alpha_{i,j}\}_{0 \leq i \leq m_j-1}$ have been introduced in [3] for computing the powers of a matrix $A \in \mathcal{A}_d$. Besides the exploration of Expression (4.15) in [18] and Proposition 4.3 in [4] allow us to obtain an explicit formula for the $\alpha_{i,j}$ as follows,

$$\alpha_{i,j} = (-1)^i \sum_{h_1+h_2+\cdots+h_{j-1}+h_{j+1}\cdots+h_s=i,h_d \leq m_d} \prod_{d=1,d\neq j}^{s} \binom{h_d}{m_d+h_d-1}(\lambda_j - \lambda_d)^{-h_d}.$$

$$(2.9)$$

With the aid of the basis of the Lagrange-Sylvester interpolation polynomials as in [8], we obtain the following result.

Proposition 2.1. *Let A be in $\mathcal{A}_d$ such that $p(A) = \prod_{j=1}^{s}(A - \lambda_j 1_d)^{m_j} = \Theta_d$ ($\lambda_i \neq \lambda_j$, for $i \neq j$). Then*

$$e^{tA} = \sum_{j=1}^{s} e^{\lambda_j t} \sum_{k=0}^{m_j-1} f_{j,k}(t)(A - \lambda_j 1_d)^k q_j(A),$$

$$= \sum_{j=1}^{s} e^{\lambda_j t} \Big[\sum_{k=0}^{m_j-1} \frac{t^k}{k!}(A - \lambda_j 1_d)^k \Big] p_j(A), \tag{2.10}$$

$$A^n = \sum_{j=1}^{s} \sum_{k=0}^{\min(n,m_j-1)} \sum_{i=0}^{m_j-k-1} \lambda_j^{n-k} \alpha_{i,j} \binom{n}{k} (A - \lambda_j 1_d)^{k+i} q_j(A) \tag{2.11}$$

such that

$$p_j(z) = \prod_{d=1,d\neq j} ((z - \lambda_d)^{m_d}/(\lambda_j - \lambda_d)^{m_d}) \sum_{i=0}^{m_j-1} \alpha_{i,j}(z - \lambda_j)^i,$$

$q_j(z) = p(z)/(z - \lambda_j)^{m_j}$ *and*

$$f_{j,k}(t) = \sum_{i=0}^{k} \alpha_{i,j}(t^{k-i}/(k-i)!)(1/\prod_{d=1,d\neq j}(\lambda_j - \lambda_d)^{m_d}),$$

where the $\alpha_{i,j}$ are given by (2.9).

When $p(A) = (A - \lambda_j 1)^r = \Theta_d$ we can take $\alpha_{01} = 1$ and $\alpha_{i1} = 0$ for every $i \neq 0$.

Let $\Omega_0(t)$ be the coefficient of $A^0 = 1_d$ in the well known polynomial decomposition $e^{tA} = \sum_{k=0}^{r-1} \Omega_k(t) A^k$. A direct computation, using (2.10), yields a new formula for $\Omega_0(t)$ in terms of $\{\lambda_j\}_{\{1 \leq j \leq s\}}$ as follows.

Proposition 2.2. *Under the hypothesis of Proposition 2.1, we have*

$$\Omega_0(t) = \sum_{k=1}^{s} e^{\lambda_k t} Q_k(t) \prod_{j=1,j\neq k}^{s} \frac{(-\lambda_j)^{m_j}}{(\lambda_k - \lambda_j)^{m_j}}, \tag{2.12}$$

where

$$Q_k(t) = \sum_{p=0}^{m_k-1} \Big(\sum_{i=0}^{p} \alpha_{i,k} t^{p-i}/(p-i)! \Big)(-\lambda_k)^p,$$

with the $\alpha_{i,k}$ given by (2.9). *Moreover, we have $\frac{d\Omega_{k+1}}{dt}(t) = a_{r-k-2}\frac{d\omega_0}{dt}(t) + \Omega_k(t)$, with $\Omega_0(t) = 1 + a_{r-1}\omega_0(t)$.*

From Proposition 2.2 we derive that

$$\omega_0(t) = \frac{1 - \Omega_0(t)}{(-\lambda_1)^{m_1} \dots (-\lambda_s)^{m_s}},$$

where $\Omega_0(t)$ is given by (2.12). It seems for us that (2.12), which gives $\Omega_0(t)$ in terms of the eigenvalues $\{\lambda_j\}_{\{1 \leq j \leq s\}}$ of the matrix A, is not known in the literature under this form.

The preceding results will play a central role in the next sections devoted to some properties of the linear matrix differential equations of higher-order (1.2)-(1.3).

3. Higher-order matrix differential equations of Apostol type

We are concerned here with the higher-order matrix differential equations (1.2) when $A_0 = \cdots = A_{r-2} = \Theta_d$ and $A_{r-1} = A \neq \Theta_d$, where $A \in \mathcal{A}_d$. These linear matrix differential equations are called *of Apostol type*. For reason of clarity we proceed as follows. We start by reconsidering the case $r = 2$ and afterwards we focus on the case $r \geq 2$, particularly results of Apostol and Kolonder are recovered.

3.1. **Simple second-order differential equations.** Let A be in $\mathcal{A}_d$ satisfying the polynomial equation $P(A) = \Theta_d$, where $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$. It is well known that the second-order linear matrix differential equation

$$X''(t) = AX(t), \tag{3.1}$$

has a unique solution $X(t) = C(t)X(0) + S(t)X'(0)$, where $X(0)$ and $X'(0)$ are the prescribed initial values and $C(t)$ and $S(t)$ are the following series

$$C(t) = \sum_{k=0}^{+\infty} \frac{t^{2k}}{(2k)!} A^k, \quad S(t) = \sum_{k=0}^{+\infty} \frac{t^{2k+1}}{(2k+1)!} A^k$$

(see [1, 9]). If we substitute for A^k its expression given in (2.4) we manage to have the following improvement of the Apostol-Kolodner result.

Proposition 3.1. *Let A be in $\mathcal{A}_d$ satisfying the polynomial equation $P(A) = \Theta_d$, where $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$. Then, the unique solution of the matrix differential equation (3.1), with the prescribed initial values $X(0)$ and $X'(0)$ is $X(t) = \sum_{k=0}^{r-1} (C_k(t)X(0) + S_k(t)X'(0))A^k$, such that*

$$C_k(t) = \frac{t^{2k}}{(2k)!} + \sum_{n=r}^{+\infty} \frac{t^{2n}}{(2n)!} \rho_k(n), \quad S_k(t) = \frac{t^{2k+1}}{(2k+1)!} + \sum_{n=r}^{+\infty} \frac{t^{2n+1}}{(2n+1)!} \rho_k(n)$$

for $0 \leq k \leq r-1$, where $\rho_k(n) = \sum_{j=0}^{k} a_{r-k+j-1}\rho(n-j,r)$ with the $\rho(n,r)$ given by (2.5).

Consider the first-order differential equation (1.1) with $A \in \mathcal{A}_d$ satisfying $P(A) = \Theta_d$, where $P(X) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$. Kolodner's method (see [9]) shows that $P(D)X(t) = P(A)X(t) = 0$, where $D = d/dt$. Therefore, we have $X(t) = \sum_{j=0}^{r-1} \phi_j(t)A^j$, where the functions $\phi_j(t)$ $(0 \leq j \leq r-1)$ verify the scalar differential equation $P(D)x(t) = 0$, whose initial conditions are $D^k\phi_j(0) = \delta_{j,k}$ $(= 1$ for $j = k$ and 0 if not), (see [9, 11]). For the linear matrix differential equation (3.1) we have $D^2 X(t) = AX(t)$, and the preceding method shows that $P(D^2)X(t) = 0$. If we set $Q(z) = P(z^2)$, it is easy to show that each function $\phi_j(t)$ $(0 \leq j \leq 2r - 1)$ is also a solution of the scalar ordinary differential equation (of order $2r$) $Q(D)x(t) = 0$, satisfying the initial conditions $D^k\phi_j(0) = \delta_{j,k}$ $(0 \leq j \leq 2r - 1)$. Therefore, Equation (2.6) implies that

$$\phi_k(t) = \frac{t^k}{k!} + \sum_{j=0}^{k} b_{2r-k+j-1}\omega_j(t), \tag{3.2}$$

where $b_{2i} = 0$, $b_{2i+1} = a_i$ and $\omega_j(t) = \sum_{n=2r}^{+\infty} \rho(n-j,2r)t^n/n!$ $(0 \leq j \leq 2r-1)$, with $\rho(n,2r)$ given by (2.5) are such that the a_i and r are replaced by the b_i and $2r$ (respectively). In conclusion we have the following proposition.

Proposition 3.2. *Let $A \in \mathcal{A}_d$ such that $P(A) = \Theta_r$, where $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$. Then, the unique solution of the matrix differential equation (3.1), with the prescribed initial values $X(0)$ and $X'(0)$, is given by $X(t) = C(t)X(0) + S(t)X'(0)$ with*

$$C(t) = \sum_{k=0}^{r-1} \phi_{2k+1}(t)A^k, \quad S(t) = \sum_{k=0}^{r-1} \phi_{2k}(t)A^k, \tag{3.3}$$

where the $\phi_k(t)$ $(0 \le k \le r-1)$ are described by (3.2).

Proposition 3.2 represents a new formulation of the result of Kolodner (see [9]). As an application of the last propositions, let us consider the following example.

Example 3.3. Let $A \in \mathcal{A}_d$ satisfying $P(A) = A^2 - 1_d = \Theta_d$ (or equivalently $A^2 = 1_d$), then $\rho_0(2n) = 1, \rho_0(2n+1) = 0$ and $\rho_1(2n) = 0, \rho_1(2n+1) = 0$ for $n \ge 2$. And a straightforward verification, using Propositions 3.1 or 3.2, shows that we have $X(t) = (C_0(t)X(0) + S_0(t)X'(0)) 1_d + (C_1(t)X(0) + S_1(t)X'(0)) A$, where

$$C_0(t) = \frac{1}{2}(ch(t) + \cos(t)), \quad S_0(t) = t + \frac{1}{2}(sh(t) + \sin(t)),$$
$$C_1(t) = \frac{1}{2}(ch(t) - \cos(t)), \quad S_1(t) = t + \frac{1}{2}(sh(t) - \sin(t)).$$

3.2. **Higher order linear differential equations of Apostol type.** In this Subsection we extend results of Subsection 3.1 to the class of *Apostol* linear matrix differential equations of order $p \ge 2$,

$$X^{(p)}(t) = AX(t), \tag{3.4}$$

whose prescribed initial conditions are $X(0), X'(0), \ldots, X^{(p-1)}(0)$, where $A \in \mathcal{A}_d$ satisfying the polynomial equation $P(A) = \Theta_d$, where $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}1_d$. Consider the vector column $Z(t) =^t (X^{(p-1)}(t), \ldots, X(t))$ and the square matrix $dp \times dp$,

$$B = \begin{pmatrix} \Theta_d & \Theta_d & \ldots & \Theta_d & A \\ 1_d & \Theta_d & \ldots & & \Theta_d \\ \Theta_d & 1_d & \Theta_d & \ldots & \Theta_d \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \Theta_d & \ldots & \Theta_d & 1_d & \Theta_d \end{pmatrix}. \tag{3.5}$$

It is well known that the solution of the linear matrix differential equation (3.4) can be derived from the usual solution,

$$Z(t) = e^{tB}Z(0) \quad \text{where } Z(0) =^t (X^{(p-1)}(0), \ldots, X'(0), X(0)), \tag{3.6}$$

of the first order linear system (1.1) defined by the matrix (3.5). That is, we focus our task on the computation of e^{tB}, which is essentially based on the explicit formula of the powers $B^n (n \ge 0)$. Indeed, by induction we verify that, for every

$n \geq 0$, we have

$$B^{pn} = \begin{pmatrix} A^n & \Theta_d & \dots & \Theta_d \\ \Theta_d & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \Theta_d \\ \Theta_d & \dots & \Theta_d & A^n \end{pmatrix}, \quad B^{pn+i} = \begin{pmatrix} \Theta_d & \dots & A^{n+1} & \dots & \Theta_d \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \Theta_d & \Theta_d & \dots & \Theta_d & A^{n+1} \\ A^n & \Theta_d & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \Theta_d & \dots & A^n & \dots & \Theta_d \end{pmatrix},$$

$$(3.7)$$

for $1 \leq i \leq p-1$, where the power A^{n+1} of the first line is located at the $(p-i+1)^{th}$ column and the power A^n of the first column is located at the $(i+1)^{th}$ line. That is, the matrix $B^{np+i} = (E_{i,j})_{1 \leq i,j \leq p}$ such that $1 \leq i \leq p-1$ and $E_{i,j} \in \mathcal{A}_d$, is described as follows : $E_{j,p-i+j} = A^{n+1}$ for $j = 1, \dots, i+1$, $E_{i+j,j} = A^{n+1}$ for $j = 1, \dots, p-i-1$ and $E_{i,j} = \Theta_d$ otherwise. The solution $X(t)$ of the matrix differential equation (3.4), is derived by a direct computation. More precisely, the natural generalization of Proposition 3.1 is formulated as follows.

Theorem 3.4. *Let A be in $\mathcal{A}_d$. Then, the linear matrix differential equation (3.4) of order $p \geq 2$, with the prescribed initial conditions $X(0)$, $X'(0)$, $\dots$, $X^{(p-1)}(0)$ has a unique solution $X(t)$ satisfying $X(t) = C_0(t, A)X(0) + C_1(t, A)X'(0) + \cdots + C_{p-1}(t, A)X^{(p-1)}(0)$, where*

$$C_j(t, A) = \sum_{n=0}^{+\infty} \frac{t^{pn+j}}{(np+j)!} A^n, \quad \text{for every } j \ (0 \leq j \leq p-1). \qquad (3.8)$$

Expression (3.7) of the powers of the matrix (3.5), can be easily derived using the general method of Section 4, where this later is based on the extension of the technique of [5, 7]. On the other hand, properties of the family of the functions $C_j(t, A)$ $(0 \leq j \leq p-1)$, may be obtained from (3.8). Indeed, a simple verification gives the corollary.

Corollary 3.5. *Under the hypothesis of Theorem 3.4, the functions $C_j(t)$ $(0 \leq j \leq p-1)$ defined by (3.8) satisfy the following differential relations,*

$$D^k C_j(t, A) = C_{j-k}(t, A), \text{ for } 0 \leq k \leq j \leq p-1, \text{ and } D^k C_j(0) = \delta_{j,k},$$

where $D = d/dt$. Moreover, for every j $(0 \leq j \leq p-1)$ we have $C_j(t) = D^{p-j-1}C_{p-1}(t)$.

A direct application of (2.4)-(2.5) and Theorem 3.4, permits us to establish that solutions of the linear matrix differential equation (3.4) are expressed in terms of the coefficients a_j $(0 \leq j \leq r-1)$ of the polynomial $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$ satisfying $P(A) = \Theta_d$. More precisely, replacing A^n in (3.8) by its expression given in (2.4) yields the following proposition.

Proposition 3.6. *Let A be in $\mathcal{A}_d$ such that $P(A) = \theta_d$, where $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$. Then, the solution of the linear differential equation (3.4), with the prescribed initial conditions $X(0)$, $X'(0)$, $\dots$, $X^{(p-1)}(0)$, is*

$$X(t) = \sum_{k=0}^{r-1} \Big[\sum_{j=0}^{p-1} C_{j,k}(t) X^{(j)}(0) \Big] A^k$$

such that

$$C_{j,k}(t) = \frac{t^{pk+j}}{(pk+j)!} + \sum_{n=r}^{+\infty} \frac{t^{pn+j}}{(pn+j)!}\rho_k(n),$$

with $\rho_k(n) = \sum_{j=0}^{k} a_{r-k+j-1}\rho(n-j,r)$, *where* $\rho(n,r)$ *is given by* (2.5).

Now we obtain an explicit formula for the solution of the linear matrix differential equation (3.4), in terms of the roots of the polynomial $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$ satisfying $P(A) = \Theta_d$. To this aim, in Expression (3.8) we substitute for A^n ($n \geq r$) its expression set forth in Proposition 2.1. Then, a hard straightforward computation leads us to the following result.

Theorem 3.7. *Let A be in $\mathcal{A}_d$ such that $P(A) = \prod_{j=1}^{s}(A - \lambda_j 1_d)^{m_j} = \Theta_d$ ($\lambda_i \neq \lambda_j$ for $i \neq j$). Then, the unique solution of the linear matrix differential equation (3.4), with the prescribed initial conditions $X(0), X'(0), \ldots, X^{(p-1)}(0)$, is $X(t) = \sum_{i=0}^{p-1} C_i(t,A)X^{(i)}(0)$ such that*

$$C_i(t,A) = \sum_{j=1}^{s} \varphi_j(t;A), \text{ with } \varphi_j(t;A) = \sum_{k=0}^{m_j-1} \frac{d^k V_i}{dz^k}(t,\lambda_j)\rho_{jk}(A),$$

where $V_i(t,z) = \sum_{n=0}^{+\infty} \frac{t^{pn+i}}{(pn+i)!}z^n$ *and*

$$\rho_{j,k}(z) = \frac{1}{k!}\Big(\sum_{d=0}^{m_j-k-1} \alpha_{d,j}(z-\lambda_j)^{k+d}\Big)\Big(\prod_{t=1,t\neq j}^{s} \frac{(z-\lambda_t)^{m_t}}{(\lambda_j-\lambda_t)^{m_t}}\Big),$$

with $\alpha_{d,j}$ *given by* (2.9).

The expression of the $C_j(t,A)$ ($0 \leq j \leq s$) given in Theorem 3.7 seems quite complicated, yet its application is a powerful tool in many important practical situations as shown is the following corollaries and Example 3.10, it also leads to have explicit formulas. Indeed, when $p(A) = \prod_{j=1}^{r}(A - \lambda_j 1_d) = \Theta_d$ ($\lambda_i \neq \lambda_j$, then we show that $\rho_{j0} = \alpha_{0j} = 1$, and the following corollary is derived.

Corollary 3.8. *Let A be in $\mathcal{A}_d$ such that $p(A) = \prod_{j=1}^{r}(A - \lambda_j 1_d) = \Theta_d$ ($\lambda_i \neq \lambda_j$ for $i \neq j$). Then, the unique solution of Equation (3.4), with the prescribed initial conditions $X(0), X'(0), \ldots, X^{(p-1)}(0)$ is $X(t) = \sum_{i=0}^{p-1} C_i(t,A)X^{(i)}(0)$, where*

$$C_i(t,z) = \sum_{j=1}^{r}\Big(\sum_{n=0}^{+\infty} \frac{t^{pn+i}}{(pn+i)!}\lambda_j^n\Big)\prod_{t=1,t\neq j}^{r} \frac{(z-\lambda_t)}{(\lambda_j-\lambda_t)}.$$

In the particular case where $p = 2$, the result of Apostol (see [1]) is derived as a simple consequence of the above corollary. Another important result of Theorem 3.7 can be established when $p(A) = (A - \lambda_1 1_d)^r = \Theta_d$. Indeed, as it was noticed above, in this case we have $\alpha_{01} = 1$ and $\alpha_{ij} = 0$ for each $i \neq 0$, whence $\rho_{1,k}(z) = \frac{1}{k!}(z-\lambda_1)^k$.

Corollary 3.9. *Suppose that $A \in \mathcal{A}_d$ satisfies $p(A) = (A - \lambda_1 1_d)^r = \Theta_d$. Then, the unique solution of Equation (3.4), with the prescribed initial conditions $X(0), X'(0), \ldots, X^{(p-1)}(0)$, is $X(t) = \sum_{i=0}^{p-1} C_i(t,A)X^{(i)}(0)$, where*

$$C_i(t,A) = \sum_{j=1}^{r} \frac{1}{(j-1)!}\frac{d^{j-1}V_i}{d\lambda^{j-1}}(t,\lambda_1)(A-\lambda_1 1_d)^{j-1}.$$

Example 3.10. Let $A \in \mathcal{A}_d$ such that $P(A) = \Theta_d$, where $P(z) = (z-\lambda)^2(z-\mu)$ (with $\lambda \neq \mu$). Then, the solution of the third-order matrix differential equation $X'''(t) = AX(t)$, with the prescribed initial conditions $X(0)$, $X'(0)$ and $X''(0)$, is given by $X(t) = C_0(t,A)X(0) + C_1(t,A)X'(0) + C_2(t,A)X''(0)$. And by Corollary 3.8 we have $C_1(t,A) = DC_2(t,A)$, $C_0(t,A) = D^2C_2(t,A)$ $(D = d/dt)$, where

$$C_2(t,A) = V_2(t,\lambda)\big[1_d + \frac{A - \lambda 1_d}{\mu - \lambda}\big]\phi_1(A) + \frac{dV_2}{dz}(t,\lambda)(A-\lambda 1_d)\phi_1(t,\lambda) + V_2(t,\mu)\phi_2(A),$$

with

$$V_2(t,z) = \sum_{n=0}^{+\infty} \frac{t^{3n+2}}{(3n+2)!}z^n, \quad \phi_1(z) = \frac{z - \mu}{\lambda - \mu}$$

and $\phi_2(z) = \frac{(z-\lambda)^2}{(\mu-\lambda)^2}$.

Remark 3.11. *As shown before, our method for studying these kinds of matrix differential equations is more direct and does not appeal to other technics. Meanwhile, it seems for us that the method of Verde Star based on the technics of divided differences can be also applied for studying such equations (see [17]).*

4. Solutions of Equation (1.2): combinatorial setting

We are interested here in the combinatorial solutions of (1.2), where the exponential generating function of the family of sequences $\{\rho(n-j,r)\}_{n\geq 0}$ $(0 \leq j \leq r-1)$ is defined by (2.3).

Let $A_0, \ldots, A_{r-1}$ and $S_0, \ldots, S_{r-1}$ be two finite sequences of $\mathcal{A}_d$ such that $A_{r-1} \neq 0$. Let $\mathcal{C}^\infty(\mathbb{R}\,;\mathcal{A}_d)$ be the $\mathbb{C}$-vector space of functions of class $\mathcal{C}^\infty$. Consider the class of linear matrix differential equations of higher-order (1.2), with solutions in $X \in \mathcal{C}^\infty(\mathbb{R}\,;\mathcal{A}_d)$ and subject to the initial conditions, $X(0)$, $X'(0)$, ..., $X^{(r-1)}(0)$. Set $Z(t) =^t (X^{(r-1)}(t), \ldots, X(t))$ and $Z(0) =^t (X^{(r-1)}(0), \ldots, X(0))$. A standard computation shows that (1.2) is equivalent to the following first-order matrix differential equation,

$$Z'(t) = BZ(t), \tag{4.1}$$

where $B \in \mathcal{M}(r,\mathcal{A}_d)$ is the companion matrix

$$B = \begin{pmatrix} A_0 & A_1 & \ldots & & A_{r-1} \\ 1_d & \Theta_d & \ldots & & \Theta_d \\ \Theta_d & 1_d & \Theta_d & \ldots & \Theta_d \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \Theta_d & \ldots & \Theta_d & 1_d & \Theta_d \end{pmatrix}. \tag{4.2}$$

It is well known that the solution of the linear matrix differential equation (1.2) is $Z(t) = e^{tB}Z(0)$, where the expression of e^{tB} depends on the computation of the powers B^n with the aid of $A_0, \ldots, A_{r-1}$. To this aim, we apply the recent technique for computing the powers of the usual companion matrix (4.2) (see [5, 7]). Indeed, let $\{Y_{n,s}\}_{n\geq 0}$ $(0 \leq s \leq r-1)$ be the class of sequences (2.1) in $\mathcal{A}_d$ such that $Y_{n,s} = \delta_{n,s}1_d$ $(\delta_{n,s} = 1$ if $n = s$ and 0 if not$)$ for $0 \leq n \leq r-1$ and

$$Y_{n+1,s} = A_0Y_{n,s} + A_1Y_{n-1,s} + \cdots + A_{r-1}Y_{n-r+1,s}, \quad \text{for } n \geq r-1. \tag{4.3}$$

Lemma 4.1. *Let $A_0, \ldots, A_{r-1}$ be in $\mathcal{A}_d$ with $A_{r-1} \neq 0$. Then*

$$B^n = \begin{pmatrix} Y_{n+r-1,r-1} & Y_{n+r-1,r-2} & \cdots & Y_{n+r-1,0} \\ Y_{n+r-2,r-1} & Y_{n+r-2,r-2} & \cdots & Y_{n+r-2,0} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n,r-1} & Y_{n,r-2} & \cdots & Y_{n,0} \end{pmatrix}, \quad \text{for every } n \geq 0. \quad (4.4)$$

Expression (4.4) implies that $e^{tB} = (C_{i,j}(t))_{0 \leq i,\, j \leq r-1}$, where $C_{i,j}(t)$ $(0 \leq i,\, j \leq r-1)$ is the exponential generating functions of the sequence $\{Y_{n+r-1-i,r-1-j}\}_{n \geq 0}$ $(0 \leq i,\, j \leq r-1)$. Thus, the characterization of solutions of the linear matrix differential equation (1.2), can be formulated as follows.

Theorem 4.2. *Let $A_0, \ldots, A_{r-1}$ be in $\mathcal{A}_d$ such that $A_{r-1} \neq 0$. Then, the solution of the linear matrix differential equation (1.2) is $X(t) = \sum_{j=0}^{r-1} C_j(t) X^{(j)}(0)$, where $C_j(t) = \sum_{n=0}^{+\infty} Y_{n,j} t^n / n!$ $(0 \leq j \leq r-1)$ is the exponential generating function of the sequences $\{Y_{n,j}\}_{n \geq 0}$ $(0 \leq j \leq r-1)$ defined by (4.3).*

A result similar to Theorem 4.2 has been given by Verde Star under another form, by using the divided difference method for solving linear differential equations (1.2) (see Section 5 of [19]). Indeed, Equation (1.2) is analogous to the Equation (5.6) given by Verde Star whose solutions (5.13), submitted to the initial conditions $C_0, C_1, \ldots, C_s$, are expressed in terms of a sequence $\{P_{k+j}\}_{0 \leq j \leq s, k \geq 0}$ (see [19]), satisfying a linear recursive relation analogous to (4.3), moreover $\{P_{k+j}\}_{0 \leq j \leq s, k \geq 0}$ is nothing else but the sequence $\{Y_{k,s-j}\}_j$ defined previously by (4.3). Comparison of our procedure with the Verde Star's method leads to infer that the functions $C_0(t), C_1(t), \ldots, C_{r-1}(t)$ of Theorem 4.2 are identical to the functions $G_{0,1}(t), G_{0,2}(t), \ldots, G_{0,s}(t)$ considered in [19].

Furthermore when the condition $A_j A_k = A_k A_j$ $(0 \leq j, k \leq r-1)$ is satisfied, expressions (2.2)–(2.3) show that the combinatoric formula of sequences (4.3) can be written explicitly as follows,

$$Y_{n,j} = \sum_{k=0}^{j} A_{r-j+k-1} \rho(n-k,r), \quad \text{for } n \geq r, \quad (4.5)$$

where the $\rho(n,r)$ are defined in (2.3) for $n \geq r$, with $\rho(r,r) = 1_d$ and $\rho(n,r) = \Theta_d$ for $n \leq r-1$. Therefore, Expression (4.5) implies that $C_j(t) = \frac{t^j}{j!} 1_d + \sum_{k=0}^{j} A_{r-j+k-1} g_k(t)$, where $g_k(t) = \sum_{n=0}^{+\infty} \rho(n-k,r) t^n / n!$ is the exponential generating functions of the sequence $\{\rho(n-k,r)\}_{n \geq 0}$. Moreover, we verify easily that the functions $g_j(t)$ $(0 \leq k \leq j)$ satisfy $g_j^{(j-k)}(t) = \frac{d^{j-k} g_j}{dt^{j-k}}(t) = g_k(t)$. Hence, an extension of Proposition 3.1 and Theorem 3.4 can be formulated as follows.

Proposition 4.3. *Let $A_0, \ldots, A_{r-1}$ be in $\mathcal{A}_d$ such that $A_i A_k = A_k A_i$ for $0 \leq i$, $k \leq r-1$. Then the solution of the linear matrix differential equation (1.2), subject to the prescribed initial values $X(0), X'(0), \ldots, X^{(r-1)}(0)$, is the following*

$$X(t) = \sum_{j=0}^{r-1} \Delta_j(t) X^{(j)}(0) = \sum_{j=0}^{r-1} \left(\frac{t^j}{j!} + \Pi_j(D) g_j(t) \right) X^{(j)}(0), \quad (4.6)$$

such that

$$\Delta_j(t) = \frac{t^j}{j!} + \sum_{k=0}^{j} A_{r-j+k-1} g_k(t), \quad \Pi_j(D) = \frac{t^j}{j!} + \sum_{k=0}^{j} A_{r-j+k-1} D^{j-k},$$

where $D = d/dt$ and $g_j(t)$ is the exponential generating function of the sequence $\{\rho(n-j,r)\}_{n \geq 0}$.

Proposition 4.3 shows that solutions of the linear matrix differential equation (1.2) may be given in terms of the exponential generating function of the class of sequences $\{Y_{n,j}\}_{n \geq 0}$ $(0 \leq j \leq r-1)$ defined by (4.3). In the same way, expressions (2.2)-(2.3) may be used to obtain the combinatoric formulas of the sequences in (4.3). Moreover, solutions of the linear matrix differential equation (1.2), subject to the prescribed initial values $X(0), X'(0), \ldots, X^{(r-1)}(0)$, can be expressed in terms the exponential generating functions of the class of sequences $\{\rho(n-k,r)\}_{n \geq 0}$ $(0 \leq k \leq r-1)$.

Remark 4.4. Consider a unitary topological $\mathbb{C}$-algebra $\mathcal{A}$ instead of the $\mathbb{C}$-algebra of the square matrices $\mathcal{A}_d$. Suppose that $A \in \mathcal{A}$ is an algebraic element satisfying $P(A) = 0$, where $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$ $(a_j \in \mathbb{C}$ for $0 \leq j \leq r-1)$. Then, all results of the preceding Sections are still valid. On the other hand, it's easy to show that Theorem 3.4 and Corollary 3.5, on the solutions of the linear matrix differential equation (3.4), do not depend on the condition that A is algebraic.

Remark 4.5 (Future problem). Solutions of the linear matrix differential equation (1.3), can be also described using some recursive relations and the exponential generating functions of the combinatorial sequences (2.1). One of the main problems is to study the spectral aspect of solutions of these classes of differential equations. More precisely, suppose that the two (non trivial) matrices A_0, A_1 appearing in (1.3) satisfy the following polynomial equations $P_1(A_0) = P_2(A_1) = \Theta_d$, where $P_1(z) = \prod_{j=0}^{s_1}(z - \lambda_j)^{p_j}$ and $P_2(z) = \prod_{j=0}^{s_2}(z - \mu_j)^{q_j}$. The problem can be formulated as follows : *study the solutions of the linear matrix differential equation* (1.3), *in terms of the* λ_j $(0 \leq j \leq s_1)$ *and* μ_j $(0 \leq j \leq s_2)$. Some investigations are currently done for this purpose.

References

[1] T.M. Apostol; *Explicit formulas for solutions of the second order matrix differential equation $Y'' = AY$,* Amer. Math. Monthly 82 (1975), pp. 159-162.

[2] R. Ben Taher and M. Rachidi; *Linear recurrence relations in the algebra of matrices and applications,* Linear Algebra and Its Applications, Vol. 330 (1-3) (2001), pp. 15-24.

[3] R. Ben Taher and M. Rachidi; *Some explicit formulas for the polynomial decomposition of the matrix exponential and applications,* Linear Algebra and Its Applications, 350, Issue 1-3 (2002), pp. 171-184.

[4] R. Ben Taher, I. Bensaoud and M. Rachidi; *Dynamic solutions and computation of the powers and exponential of matrix,* International Journal of Mathematics, Game Theory, and Algebra 13 (2003), No. 6, p. 455-463.

[5] R. Ben Taher and M. Rachidi; *On the matrix powers and exponential by the r-generalized Fibonacci sequences methods. The companion matrix case,* Linear Algebra and Applications, 370 (2003), pp. 341-353.

[6] H.W. Cheng and S.S.-T. Yau; *On more explicit formulas for matrix exponential,* Linear Algebra Appl. 262 (1997), pp. 131-163.

[7] B. El Wahbi, M. Mouline and M. Rachidi; *Solving nonhomogeneous recurrence relations by matrix methods,* The Fibonacci Quarterly, 40-2 (2002), pp. 109-117.

[8] F. R. Gantmacher; *Theory of matrices*, Chelsea Publishing Company, New York (1959).

[9] I. I. Kolodner; *On exp(tA) with A satisfying a polynomial*, J. Math.Anal. and Appl. 52 (1975), pp. 514-524.

[10] C. Levesque; *On the m − th order linear recurrences*, Fibonacci Quarterly 23 (4), (1985) : 290-295.

[11] I. E. Leonardo; *The matrix exponential*, SIAM Review Vol. 38, No. 3 (1996), pp. 507-512.

[12] C. Moler and C. Van Loan; *Nineteen dubious ways to compute the exponential of matrix, twenty-Five years later,* SIAM Review Vol.45 (1), (2003), pp. 3-49.

[13] M. Mouline and M. Rachidi; *Application of Markov Chains properties to r-Generalized Fibonacci Sequences,* Fibonacci Quarterly 37 (1999), pp. 34-38.

[14] M. Mouline and M. Rachidi; *Suites de Fibonacci généralisées, Théorème de Cayley-Hamilton et chaines de Markov,* Rendiconti del Seminario Matematico di Messina Serie II, Tomo XIX, No. 4 (1996/97), pp. 107-115.

[15] E. J. Putzer; *Avoiding the Jordan canonical form in the discussion of linear systems with constant coefficients,* Amer. Math. Monthly 73 (1966), pp. 2-7.

[16] Hans-J. Runckel and U. Pitelkow; *Practical Computation of matrix functions,* Linear Algebra Appl. 49 (1983), pp. 161-178.

[17] L. Verde-Star; *Operator identities and the solution of linear matrix difference and differential equations,* Studies in Applied Mathematics 91 (1994), pp. 153-177.

[18] L. Verde-Star; *Divided differences and linearly recurrent sequences,* Stud. Appl. Math. 95 (1995), pp. 433-456.

[19] L. Verde-Star; *Solutions of linear differential equations by the method of divided differences,* Adv. in Applied Math. 16 (1995), pp. 484-508.

Rajae Ben Taher

Département de Mathématiques et Informatique, Faculté des Sciences,
Université Moulay Ismail, B.P. 4010, Beni M'hamed, Méknés, Morocco
    E-mail address: `bentaher@fs-umi.ac.ma`

Mustapha Rachidi

Mathematics Section LEGT - F. Arago, Académie de Reims, 1, Rue F. Arago, 51100 Reims, France
    E-mail address: `mu.rachidi@hotmail.fr`