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POSITIVE SOLUTIONS FOR SYSTEMS OF NONLINEAR SINGULAR DIFFERENTIAL EQUATIONS

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Abstract. By constructing a special cone and using the fixed point theorem of cone expansion and compression, this paper shows the existence of positive solutions for two-point boundary-value problems of nonlinear singular differential systems. To illustrate the applications of our main results, some examples are given.

1. Introduction

Recently, singular boundary value problems (SBVP for short) have been studied extensively (see [1, 3, 4, 5, 6, 7, 8, 9, 10] and references therein). Under the superlinear effect, Wei and Zhang [8] obtained necessary and sufficient conditions for the existence of $C^2[0,1]$ and $C^3[0,1]$ positive solutions for fourth-order singular boundary value problems by using the fixed point theorem of cone expansion and compression. Under the sublinear effect, Wei [7] obtained necessary and sufficient conditions for the existence of positive solutions for fourth-order singular boundary value problems by using the upper and lower solution method and the maximal principal. However, in this paper, we will investigate the existence of positive solutions of second and fourth order singular boundary value problems of nonlinear singular differential systems. We obtain necessary and sufficient conditions for the existence of $C^2[0,1]\times C[0,1]$ and $C^3[0,1]\times C^1[0,1]$ positive solutions for the coupled systems. Two examples are given to show the applications of our results.

In this article, we investigate the boundary-value problem

$$\begin{gathered}
u^{(4)} = f(t,u,v);\\
-v'' = g(t,u,v);\\
u(0) = u(1) = u''(0) = u''(1) = 0;\\
v(0) = v(1) = 0,
\end{gathered}\tag{1.1}$$

where $t \in (0,1)$, $f, g \in C[(0,1)\times[0,\infty)\times[0,\infty),[0,\infty)]$; that is, f, g may be singular at $t = 0$ and $t = 1$.

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Let

$$E = \{(u,v) \in C^2[0,1] \times C[0,1] : u(0) = u(1) = u''(0) = u''(1) = 0,\\ v(0) = v(1) = 0\},$$

with the norm $\|(u,v)\| = \|u\|_2 + \|v\|_0$, where $\|u\|_2 = \max_{t\in J} |u''(t)|$, $\|v\|_0 = \max_{t\in J} |v(t)|$, $J = [0,1]$. Then $(E, \|\cdot\|)$ is a Banach space. In this paper, E will be the basic space to study (1.1). Define

$$P = \Big\{(u,v) \in E : v(t) \geq t(1-t)v(s),\ u(t) \geq t(1-t)u(s),\\ u(t) \geq -t(1-t)u''(s)/30,\ u''(t) \leq t(1-t)u''(s) \leq 0 \text{ and}\\ u(t), v(t) \text{ are nonnegative concave functions, for all } t, s \in J\Big\}.$$

It is easy to see that P is a cone of E.

A pair (u,v) is said to be a $C^2[0,1] \times C[0,1]$ positive solution of (1.1) if $u \in C^2[0,1] \cap C^{(4)}(0,1)$, $v \in C[0,1] \cap C^2(0,1)$ satisfy (1.1) and $u(t) > 0$, $u''(t) \leq 0$, $v(t) > 0$ for $t \in (0,1)$. In addition, if (u,v) is a $C^2[0,1] \times C[0,1]$ positive solution of (1.1) and both $u'''(0^+)$, $u'''(1^-)$, $v'(0^+)$ and $v'(1^-)$ exist, then (u,v) is said to be a $C^3[0,1] \times C^1[0,1]$ positive solution of (1.1).

Now, we state a lemma which will be used in Section 2.

Lemma 1.1 ([1]). *Let P be a cone of real Banach space E, Ω_1, Ω_2 be bounded open sets of E, and θ be in $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is completely continuous such that one of the following two conditions is satisfied:*

(i) $\|Ax\| \leq \|x\|$ *for* $x \in P \cap \partial\Omega_1$; $\|Ax\| \geq \|x\|$ *for* $x \in P \cap \partial\Omega_2$.
(ii) $\|Ax\| \leq \|x\|$ *for* $x \in P \cap \partial\Omega_2$; $\|Ax\| \geq \|x\|$ *for* $x \in P \cap \partial\Omega_1$.

Then, A has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Main Results

Let us list some conditions to be used later.

(H1) $g \in C[(0,1) \times [0,\infty) \times [0,\infty), [0,\infty)]$ and satisfy

$$\int_0^1 t(1-t)f(t, t(1-t), 1)dt < \infty, \quad \int_0^1 t(1-t)g(t, t(1-t), 1)dt < \infty.$$

(H2)

$$\int_0^1 f(t, t(1-t), t(1-t))dt < \infty, \quad \int_0^1 g(t, t(1-t), t(1-t))dt < \infty.$$

(H3) f is quasi-homogeneous with respect to the last two variables, that is, there are constants $\lambda_1, \mu_1, \alpha_1, \beta_1, N_1, M_1, N_2, M_2$ with $0 \leq \lambda_1 \leq \mu_1 < +\infty$, $0 \leq \alpha_1 \leq \beta_1 \leq 1$, $\mu_1 + \beta_1 < 1$, $0 < N_1 \leq 1 \leq M_1$, $0 < N_2 \leq 1 \leq M_2$ such that for all $0 < t < 1$, $u \geq 0$, $v \geq 0$ satisfying
 (a) $c^{\mu_1} f(t,u,v) \leq f(t,cu,v) \leq c^{\lambda_1} f(t,u,v)$, $0 < c \leq N_1$;
 $c^{\lambda_1} f(t,u,v) \leq f(t,cu,v) \leq c^{\mu_1} f(t,u,v)$, $c \geq M_1$;
 (b) $c^{\beta_1} f(t,u,v) \leq f(t,u,cv) \leq c^{\alpha_1} f(t,u,v)$, $0 < c \leq N_2$;
 $c^{\alpha_1} f(t,u,v) \leq f(t,u,cv) \leq c^{\beta_1} f(t,u,v)$, $c \geq M_2$.

(H4) g is quasi-homogeneous with respect to the last two variables; that is, there are constants $\lambda_2, \mu_2, \alpha_2, \beta_2, N_3, M_3, N_4, M_4$ with $0 \le \lambda_2 \le \mu_2 < +\infty$, $0 \le \alpha_2 \le \beta_2 \le 1$, $\mu_2 + \beta_2 < 1$, $0 < N_3 \le 1 \le M_3$, $0 < N_4 \le 1 \le M_4$ such that for all $0 < t < 1$, $u \ge 0$, $v \ge 0$ satisfying
 (a) $c^{\mu_2} g(t,u,v) \le g(t,cu,v) \le c^{\lambda_2} g(t,u,v)$, $0 < c \le N_3$;
 $c^{\lambda_2} g(t,u,v) \le g(t,cu,v) \le c^{\mu_2} g(t,u,v)$, $c \ge M_3$;
 (b) $c^{\beta_2} g(t,u,v) \le g(t,u,cv) \le c^{\alpha_2} g(t,u,v)$, $0 < c \le N_4$;
 $c^{\alpha_2} g(t,u,v) \le g(t,u,cv) \le c^{\beta_2} g(t,u,v)$, $c \ge M_4$.

(H5) There exist $0 < \gamma_i < 1, k_i \ge 0 (i = 1,2)$ such that
$$f(t,u,v) \ge k_1(u+v)^{\gamma_1}, \quad g(t,u,v) \ge k_2(u+v)^{\gamma_2}$$
for any $t \in J$, $(u,v) \in P$.

The main results of this paper are as follows.

Theorem 2.1. *Suppose* (H3)-(H5) *hold. Then* (1.1) *has a* $C^2[0,1] \times C[0,1]$ *positive solution* (u,v)*, if and only if* (H1) *holds.*

Theorem 2.2. *Suppose* (H3)-(H5) *hold. Then* (1.1) *has a* $C^3[0,1] \times C^1[0,1]$ *positive solution* (u,v)*, if and only if* (H2) *holds.*

To prove Theorems 2.1 and 2.2, we need some preliminary lemmas.

Lemma 2.3. *The functions* $u \in C^2[0,1] \cap C^{(4)}(0,1), v \in C[0,1] \cap C^2(0,1)$ *form a solution to* (1.1) *if and only if* (u,v) *is a fixed point of the integral operator* $A(u,v) = (A_1(u,v), A_2(u,v))$ *in* $C^2[0,1] \times C[0,1]$*, where*
$$\begin{gathered}
A_1(u,v)(t) = \int_0^1 G(t,s) \int_0^1 G(s,\tau) f(\tau, u(\tau), v(\tau)) d\tau ds, \\
A_2(u,v)(t) = \int_0^1 G(t,s) g(s, u(s), v(s)) ds, \\
G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1; \\ s(1-t), & 0 \le s \le t \le 1, \end{cases}
\end{gathered} \tag{2.1}$$

The proof of the above lemma is obvious; we omit it.

Lemma 2.4. *Assume that* (H1), (H3), (H4) *hold. Then* $A : P \to P$ *is a completely continuous operator.*

Proof. First of all, we show that $A(P) \subset P$. Note that $G(t,s) \ge t(1-t)G(s,\tau)$ for all $\tau, s \in J$. Then
$$\begin{aligned}
A_1(u,v)(t) &= \int_0^1 G(t,\tau) \int_0^1 G(\tau,\xi) f(\xi, u(\xi), v(\xi)) d\xi d\tau \\
&\ge t(1-t) \int_0^1 G(s,\tau) \int_0^1 G(\tau,\xi) f(\xi, u(\xi), v(\xi)) d\xi d\tau \\
&= t(1-t) A_1(u,v)(s), \quad \forall t, s \in J, \ (u,v) \in P.
\end{aligned}$$
$$\begin{aligned}
(A_1(u,v))''(t) &= -\int_0^1 G(t,\tau) f(\tau, u(\tau), v(\tau)) d\tau \\
&\le -t(1-t) \int_0^1 G(s,\tau) f(\tau, u(\tau), v(\tau)) d\tau
\end{aligned}$$

$$= t(1-t)(A_1(u,v))''(s) \le 0, \quad \forall t, s \in J,\ (u,v) \in P.$$

$$\begin{aligned}
A_2(u,v)(t) &= \int_0^1 G(t,s)g(s,u(s),v(s))ds \\
&\ge t(1-t)\int_0^1 G(s,\tau)g(\tau,u(\tau),v(\tau))d\tau \\
&= t(1-t)A_2(u,v)(s), \quad \forall t, s \in J,\ (u,v) \in P.
\end{aligned}$$

$$(A_2(u,v))''(t) = -g(t,u(t),v(t)) \le 0, \quad \forall t \in J,\ (u,v) \in P.$$

It is easy to see that $G(t,s) \ge s(1-s)t(1-t)$. Then

$$\begin{aligned}
A_1(u,v)(t) &= -\int_0^1 G(t,\tau)(A_1(u,v))''(\tau)d\tau \\
&\ge -\int_0^1 G(t,\tau)\tau(1-\tau)(A_1(u,v))''(s)d\tau \\
&\ge -t(1-t)\int_0^1 \tau^2(1-\tau)^2(A_1(u,v))''(s)d\tau \\
&= -\frac{1}{30}t(1-t)(A_1(u,v))''(s), \quad \forall t, s \in J,\ (u,v) \in P.
\end{aligned}$$

Therefore $A(P) \subset P$.

Next we show that A is bounded. Suppose V is an any bounded set of P, then there exists a $M > 0$ such that $\|(u,v)\| \le M$ for any $(u,v) \in V$. It follows from $u(t) = \int_0^1 G(t,s)(-u''(s))ds$ that $u(t) \le \frac{1}{2}t(1-t)\|u\|_2$. On the other hand, it follows from $u(t) \ge t(1-t)u(s)$, for all $t, s \in J$ that $u(t) \ge t(1-t)\|u\|_0$. Hence

$$t(1-t)\|u\|_0 \le u(t) \le \frac{1}{2}t(1-t)\|u\|_2, \quad t \in J. \tag{2.2}$$

Choose positive numbers $c_1 \ge \max\{M_1, \frac{M}{2N_1}\}$ and $c_2 \ge \max\{M_2, \frac{M}{N_2}\}$. For any $(u,v) \in V$, $t \in J$, we can get

$$\begin{aligned}
|(A_1(u,v))''(t)| &= \int_0^1 G(t,s)f(s,u(s),v(s))ds \\
&\le \int_0^1 s(1-s)f(s, c_1\frac{u(s)}{c_1 s(1-s)}s(1-s), c_2\frac{v(s)}{c_2})ds \\
&\le c_1^{\mu_1}c_2^{\beta_1}\int_0^1 (\frac{u(s)}{c_1 s(1-s)})^{\lambda_1}(\frac{v(s)}{c_2})^{\alpha_1}s(1-s)f(s,s(1-s),1)ds \\
&\le c_1^{\mu_1}(\frac{1}{2c_1})^{\lambda_1}\|u\|_2{}^{\lambda_1}c_2^{\beta_1}\|v\|_0^{\alpha_1}c_2^{-\alpha_1}\int_0^1 s(1-s)f(s,s(1-s),1)ds \\
&\le 2^{-\lambda_1}c_1^{\mu_1-\lambda_1}c_2^{\beta_1-\alpha_1}M^{\lambda_1+\alpha_1}\int_0^1 s(1-s)f(s,s(1-s),1)ds < +\infty.
\end{aligned}$$

Let $c_3 \ge \max\{M_3, \frac{M}{2N_3}\}$ and $c_4 \ge \max\{M_4, \frac{M}{N_4}\}$. For any $(u,v) \in V$, $t \in J$, we have

$$\begin{aligned}
|A_2(u,v)(t)| &= \int_0^1 G(t,s)g(s,u(s),v(s))ds \\
&\le \int_0^1 s(1-s)g(s, c_3\frac{u(s)}{c_3 s(1-s)}s(1-s), c_4\frac{v(s)}{c_4})ds
\end{aligned}$$

$$\le 2^{-\lambda_2} c_3^{\mu_2-\lambda_2} c_4^{\beta_2-\alpha_2} M^{\lambda_2+\alpha_2} \int_0^1 s(1-s)g(s,s(1-s),1)ds < +\infty.$$

Consequently, A is bounded on P.

Thirdly, we show that AV is equicontinuous for arbitrary bounded set $V \subset P$. Choose positive numbers $c_1 \ge \max\{M_1, \frac{M}{2N_1}\}$, $c_2 \ge \max\{M_2, \frac{M}{N_2}\}$, it follows from

$$A_1(u,v)(t) = \int_0^1 G(t,s) \int_0^1 G(s,\tau) f(\tau, u(\tau), v(\tau)) d\tau ds$$

that

$$\begin{aligned}
(A_1(u,v))''(t) &= -\int_0^1 G(t,s) f(s,u(s),v(s))ds \\
&= -\int_0^t s(1-t) f(s,u(s),v(s))ds - \int_t^1 t(1-s) f(s,u(s),v(s))ds,
\end{aligned}$$

$$\begin{aligned}
(A_1(u,v))'''(t) &= \int_0^t s f(s,u(s),v(s))ds - \int_t^1 (1-s) f(s,u(s),v(s))ds \\
&\le \int_0^t s f(s,u(s),v(s))ds + \int_t^1 (1-s) f(s,u(s),v(s))ds \\
&\le c_0 \Big(\int_0^t s f(s,s(1-s),1)ds + \int_t^1 (1-s) f(s,s(1-s),1)ds \Big),
\end{aligned}$$

where $c_0 = 2^{-\lambda_1} c_1^{\mu_1-\lambda_1} c_2^{\beta_1-\alpha_1} M^{\lambda_1+\alpha_1}$. Assume

$$H(t) = c_0 \Big(\int_0^t s f(s,s(1-s),1)ds + \int_t^1 (1-s) f(s,s(1-s),1)ds \Big).$$

So we can obtain

$$\begin{aligned}
\int_0^1 H(t)dt &= c_0 \Big(\int_0^1 dt \int_0^t s f(s,s(1-s),1)ds \\
&\quad + \int_0^1 dt \int_t^1 (1-s) f(s,s(1-s),1)ds \Big) \\
&= 2c_0 \int_0^1 s(1-s) f(s,s(1-s),1)ds < +\infty.
\end{aligned} \tag{2.3}$$

Thus for any given $t_1, t_2 \in J$ with $t_1 \le t_2$ and all $(u,v) \in V$, we obtain

$$\|(A_1(u,v))''(t_2) - (A_1(u,v))''(t_1)\| = |\int_{t_1}^{t_2} A_1(u,v))'''(t)dt| \le \int_{t_1}^{t_2} H(t)dt.$$

From this inequality, (2.3) and the absolute continuity of integral, it follows that A_1V is equicontinuous on J.

On the other hand

$$\begin{aligned}
|(A_2(u,v))'(t)| &= -\int_0^t s g(s,u(s),v(s))ds + \int_t^1 (1-s) g(s,u(s),v(s))ds \\
&\le \int_0^t s g(s,u(s),v(s))ds + \int_t^1 (1-s) g(s,u(s),v(s))ds
\end{aligned}$$

Let $G(t)=\int_0^t sg(s,u(s),v(s))ds+\int_t^1(1-s)g(s,u(s),v(s))ds$, $c_3\geq\max\{M_3,\frac{M}{2N_3}\}$ and $c_4\geq\max\{M_4,\frac{M}{N_4}\}$, then

$$\begin{aligned}\int_0^1 G(t)dt&=2\int_0^1 s(1-s)g(s,u(s),v(s))ds\\&\leq 2^{1-\lambda_2}c_3^{\mu_2-\lambda_2}c_4^{\beta_2-\alpha_2}M^{\lambda_2+\alpha_2}\int_0^1 s(1-s)g(s,s(1-s),1)ds<+\infty.\end{aligned}\tag{2.4}$$

Thus for any given $t_1,t_2\in J$ with $t_1\leq t_2$ and all $(u,v)\in V$, we obtain

$$\|A_2(u,v)(t_2)-A_2(u,v)(t_1)\|=\Big|\int_{t_1}^{t_2}(A_2(u,v))'(t)dt\Big|\leq\int_{t_1}^{t_2}G(t)dt.$$

From this inequality, (2.4), and the absolute continuity of integral, it follows that A_2V is equicontinuous on J. Therefore AV are equicontinuous on J. It follows from the Ascoli-Arzela theorem that A_1V and A_2V is relatively compact.

Finally, we show that $A:P\to P$ is a continuous operator. Notice that A is continuous on $C^2[0,1]\times C[0,1]$ if and only if A_1 is continuous on $C^2[0,1]$ and A_2 is continuous on $C[0,1]$.

Suppose $\{(u_n,v_n)\}\subset P,(u,v)\in P$ and $\|u_n-u\|_2\to 0$, $\|v_n-v\|_0\to 0$ as $n\to\infty$. It follows from (2.2) that $\|u\|_0\leq\frac{1}{2}\|u\|_2$. So we can get $\|u_n-u\|_0\to 0(n\to\infty)$ from $\|u_n-u\|_2\to 0(n\to\infty)$. Then $u_n(t)\to u(t)$ and $v_n(t)\to v(t)$ as $n\to\infty$ uniformly with respect to $t\in J$. Therefore

$$\begin{aligned}&|(A_1(u_n,v_n))''(t)-(A_1(u,v))''(t)|\\&=\Big|\int_0^1 G(t,s)f(s,u_n(s),v_n(s))ds-\int_0^1 G(t,s)f(s,u(s),v(s))ds\Big|\\&\leq\int_0^1 G(t,s)\big|f(s,u_n(s),v_n(s))-f(s,u(s),v(s))\big|ds.\end{aligned}$$

From (H1), (H3) and the Lebesgue dominated convergence theorem, it follows that

$$|(A_1(u_n,v_n))''(t)-(A_1(u,v))''(t)|\to 0\quad\text{as } n\to\infty.$$

Hence one can conclude that

$$\|A_1(u_n,v_n)-A_1(u,v)\|_2\to 0\quad\text{as } n\to\infty.$$

In fact, if this is not true, then there exist ϵ_0 and $\{u_{n_i}\}\subset\{u_n\},\{v_{n_i}\}\subset\{v_n\}$ such that $\|A_1(u_{n_i},v_{n_i})-A_1(u,v)\|_2\geq\epsilon_0$ $(i=1,2\dots)$. Since $\{A_1(u_n,v_n)\}$ is relatively compact, there exists a sequence of $\{A_1(u_n,v_n)\}$ which convergence in $C^2[0,1]$ to some y. Not loss of generality, we may assume that $\{A_1(u_{n_i},v_{n_i})\}$ itself converge to y, then $y=A_1(u,v)$. This is a contradiction. Consequently A_1 is continuous. In the same way, we can get A_2 is continuous, too. This completes the proof. □

Lemma 2.5 ([2])**.** *Suppose $(u,v)\in P$ and $\mu\in(0,\frac{1}{2})$. Then $u(t)+v(t)\geq\mu(1-\mu)(\|u\|_0+\|v\|_0),t\in[\mu,1-\mu]$.*

The proof of the above lemma is obvious; we omit it.

In the following we prove Theorem 2.1.

Proof. **Sufficiency.** From Lemma 2.5, we can choose $\mu = \frac{1}{4}$. Then $u(t) + v(t) \geq \frac{3}{16}(\|u\|_0 + \|v\|_0), t \in [\frac{1}{4}, \frac{3}{4}]$. First of all, we prove

$$\|A(u,v)\| \geq \|(u,v)\|, \quad \forall (u,v) \in \partial P_r, \tag{2.5}$$

where $P_r = \big\{\|(u,v)\| < r\big\}$,

$$r \leq \min\Big\{2N_1, N_2, 2N_3, N_4, \big(\frac{k_1}{2(160)^{\gamma_1}} \int_{1/4}^{3/4} [s(1-s)]^{1+\gamma_1} ds\big)^{\frac{1}{1-\gamma_1}},$$
$$\big(\frac{k_2}{2(160)^{\gamma_2}} \int_{1/4}^{3/4} [s(1-s)]^{1+\gamma_2} ds\big)^{\frac{1}{1-\gamma_2}}\Big\}.$$

From the definition of P, we know that $\|u\|_0 \geq \frac{1}{30} t(1-t)\|u\|_2$ for any $(u,v) \in P, t \in J$. By condition (H5) and Lemma 2.5, we obtain

$$\begin{aligned}
-(A_1(u,v))''(t) &= \int_0^1 G(t,s) f(s,u(s),v(s)) ds \\
&\geq \frac{1}{4} \int_{1/4}^{3/4} s(1-s) f(s,u(s),v(s)) ds \\
&\geq \frac{1}{4} \int_{1/4}^{3/4} k_1 s(1-s)(u(s)+v(s))^{\gamma_1} ds \\
&\geq \frac{k_1}{4} \int_{1/4}^{3/4} s(1-s)(\frac{3}{16}(\|u\|_0 + \|v\|_0))^{\gamma_1} ds \\
&\geq \frac{k_1}{4(160)^{\gamma_1}} r^{\gamma_1} \int_{1/4}^{3/4} [s(1-s)]^{1+\gamma_1} ds \\
&\geq \frac{r}{2} = \frac{\|(u,v)\|}{2}, \quad \forall t \in J, (u,v) \in \partial P_r.
\end{aligned}$$

Consequently

$$\|A_1(u,v)\|_2 \geq \frac{\|(u,v)\|}{2}, \quad \forall (u,v) \in \partial P_r.$$

For any $t \in J$, $(u,v) \in \partial P_r$, by virtue of (H5) and Lemma 2.5, one can see

$$\begin{aligned}
A_2(u,v)(t) &= \int_0^1 G(t,s) g(s,u(s),v(s)) ds \\
&\geq \frac{1}{4} \int_{1/4}^{3/4} s(1-s)) g(s,u(s),v(s)) ds \\
&\geq \frac{k_2}{4} \int_{1/4}^{3/4} s(1-s)(\frac{3}{16}(\|u\|_0 + \|v\|_0))^{\gamma_2} ds \\
&\geq \frac{k_2}{4(160)^{\gamma_2}} r^{\gamma_2} \int_{1/4}^{3/4} [s(1-s)]^{1+\gamma_2} ds \\
&\geq \frac{r}{2} = \frac{\|(u,v)\|}{2}, \quad \forall t \in J, (u,v) \in \partial P_r.
\end{aligned}$$

Therefore, $\|A_2(u,v)\|_0 \geq \frac{\|(u,v)\|}{2}$, for all $(u,v) \in \partial P_r$. Consequently, (2.5) holds.

Next we claim that

$$\|A(u,v)\| \leq \|(u,v)\|, \forall (u,v) \in \partial P_R, \tag{2.6}$$

where

$$
\begin{aligned}
R \geq \max\Big\{&2N_1M_1,\ M_2N_2,\ 2N_3M_3,\ M_4N_4,\\
&\big(2^{1-\mu_1}N_1^{\lambda_1-\mu_1}N_2^{\alpha_1-\beta_1}\int_0^1 s(1-s)f(s,s(1-s),1)ds\big)^{\frac{1}{1-(\mu_1+\beta_1)}},\\
&\big(2^{1-\mu_2}N_3^{\lambda_2-\mu_2}N_4^{\alpha_2-\beta_2}\int_0^1 s(1-s)g(s,s(1-s),1)ds\big)^{\frac{1}{1-(\mu_2+\beta_2)}}\Big\},
\end{aligned}
$$

$P_R = \{\|(u,v)\| < R\}$.

Let $c_1 = \frac{R}{2N_1}$ and $c_2 = \frac{N_2}{R}$. Then for any $(u,v) \in \partial P_R$, by virtue of (H3), we have

$$
\begin{aligned}
-(A_1(u,v))''(t) &= \int_0^1 G(t,s)f(s,u(s),v(s))ds\\
&\leq \int_0^1 s(1-s)f(s,u(s),v(s))ds\\
&= \int_0^1 s(1-s)f(s,c_1\frac{u(s)}{c_1s(1-s)}s(1-s),\frac{c_2v(s)}{c_2})ds\\
&\leq \int_0^1 s(1-s)c_1^{\mu_1}(\frac{u(s)}{c_1s(1-s)})^{\lambda_1}(\frac{1}{c_2})^{\beta_1}(c_2v(s))^{\alpha_1}f(s,s(1-s),1)ds\\
&\leq 2^{-\lambda_1}c_1^{\mu_1-\lambda_1}c_2^{\alpha_1-\beta_1}R^{\lambda_1+\alpha_1}\int_0^1 s(1-s)f(s,s(1-s),1)ds\\
&= 2^{-\mu_1}N_1^{\lambda_1-\mu_1}N_2^{\alpha_1-\beta_1}R^{\mu_1+\beta_1}\int_0^1 s(1-s)f(s,s(1-s),1)ds\\
&\leq \frac{R}{2} = \frac{\|(u,v)\|}{2}.
\end{aligned}
$$

Therefore, for any $(u,v) \in \partial P_R$, we have

$$
\|A_1(u,v)\|_2 \leq \frac{\|(u,v)\|}{2}.
$$

For any $(u,v) \in \partial P_R, t \in J$, by virtue of (H4), one can also see

$$
\begin{aligned}
A_2(u,v)(t) &= \int_0^1 G(t,s)g(s,u(s),v(s))ds\\
&\leq \int_0^1 s(1-s)g(s,u(s),v(s))ds\\
&\leq 2^{-\mu_2}N_3^{\lambda_2-\mu_2}N_4^{\alpha_2-\beta_2}R^{\mu_2+\beta_2}\int_0^1 s(1-s)g(s,s(1-s),1)ds\\
&\leq \frac{R}{2} = \frac{\|(u,v)\|}{2}.
\end{aligned}
$$

Then for any $(u,v) \in \partial P_R$, we have

$$
\|A_2(u,v)\|_0 \leq \frac{\|(u,v)\|}{2}.
$$

Consequently, (2.6) holds. By Lemma 1.1 and Lemma 2.4, we obtain that A has a fixed point (u,v) in $\overline{P_R} \setminus P_r$ and satisfies $u''(t) < 0$, $u(t) > 0$, $v(t) > 0$, for all $t \in (0,1)$.

Necessity. Let $u \in C^2[0,1] \cap C^{(4)}(0,1), v \in C[0,1] \cap C^2(0,1)$ be a positive solution of (1.1). It follows from $u(0) = u(1) = 0$ and $u''(t) \le 0$ for $t \in J$ that there exist $0 < m_1 < 1 < m_2$ such that $m_1 t(1-t) \le u(t) \le m_2 t(1-t)$. In the same way, there also exist $0 < n_1 < 1 < n_2$ such that $n_1 t(1-t) \le v(t) \le n_2 t(1-t)$. There exists $t_0 \in (0,1)$ such that $v'(t_0) = 0$. This together with $v''(t) \le 0$ for $t \in (0,1)$ yields that $v'(t) \ge 0$ as $t \in (0, t_0)$ and $v'(t) \le 0$ as $t \in (t_0, 1)$. Choose positive numbers $c_3 \le \min\{N_3, \frac{1}{m_2 M_3}\}, c_4 \le \min\{N_4, \frac{1}{M_4 n_2}\}$. Then

$$
\begin{aligned}
g(t, t(1-t), 1) &= g(t, c_3 \frac{t(1-t)}{c_3 u(t)} u(t), c_4 \frac{1}{c_4 v(t)} v(t)) \\
&\le c_3^{\lambda_2} (\frac{t(1-t)}{c_3 u(t)})^{\mu_2} c_4^{\alpha_2} (\frac{1}{c_4 v(t)})^{\beta_2} g(t, u(t), v(t)) \\
&\le c_3^{\lambda_2} (\frac{1}{c_3 m_1})^{\mu_2} c_4^{\alpha_2 - \beta_2} (\frac{1}{v(t)})^{\beta_2} g(t, u(t), v(t)) \\
&= c_3^{\lambda_2 - \mu_2} c_4^{\alpha_2 - \beta_2} m_1^{-\mu_2} (v(t))^{-\beta_2} g(t, u(t), v(t)).
\end{aligned}
$$

Namely,

$$
(v(t))^{\beta_2} g(t, t(1-t), 1) \le c_3^{\lambda_2 - \mu_2} c_4^{\alpha_2 - \beta_2} m_1^{-\mu_2} g(t, u(t), v(t)). \tag{2.7}
$$

Hence, integrate (2.7) from t_0 to t to obtain

$$
\begin{aligned}
\int_{t_0}^{t} (v(s))^{\beta_2} g(s, s(1-s), 1) ds &\le c_3^{\lambda_2 - \mu_2} c_4^{\alpha_2 - \beta_2} m_1^{-\mu_2} \int_{t_0}^{t} g(s, u(s), v(s)) ds \\
&= c_3^{\lambda_2 - \mu_2} c_4^{\alpha_2 - \beta_2} m_1^{-\mu_2} v'(t).
\end{aligned}
$$

Since $v(t)$ is decreasing on $[t_0, 1]$, we get

$$
(v(t))^{\beta_2} \int_{t_0}^{t} g(s, s(1-s), 1) ds \le -c_3^{\lambda_2 - \mu_2} c_4^{\alpha_2 - \beta_2} m_1^{-\mu_2} v'(t);
$$

namely,

$$
\int_{t_0}^{t} g(s, s(1-s), 1) ds \le -c_3^{\lambda_2 - \mu_2} c_4^{\alpha_2 - \beta_2} m_1^{-\mu_2} \frac{v'(t)}{(v(t))^{\beta_2}}. \tag{2.8}
$$

Note that $\beta_2 < 1$, then integrate (2.8) from t_0 to 1 to have

$$
\int_{t_0}^{1} dt \int_{t_0}^{t} g(s, s(1-s), 1) ds \le -c_3^{\lambda_2 - \mu_2} c_4^{\alpha_2 - \beta_2} m_1^{-\mu_2} \int_{t_0}^{1} \frac{v'(t)}{(v(t))^{\beta_2}} dt.
$$

Therefore,

$$
\int_{t_0}^{1} (1-s) g(s, s(1-s), 1) ds \le c_3^{\lambda_2 - \mu_2} c_4^{\alpha_2 - \beta_2} m_1^{-\mu_2} (1 - \beta_2)^{-1} (v(t_0))^{1-\beta_2} < \infty.
$$

On the other hand, we can also prove

$$
\int_{0}^{t_0} s g(s, s(1-s), 1) ds < \infty.
$$

Thus

$$
\int_{0}^{1} s(1-s) g(s, s(1-s), 1) ds < \infty.
$$

Next, we prove that

$$\int_0^1 s(1-s)f(s,s(1-s),1)ds<\infty.$$

Let $c_1\le\min\{N_1,\frac{1}{m_2M_1}\}, c_2\le\min\{N_2,\frac{1}{M_2n_2}\}$, then

$$\begin{aligned}
f(t,t(1-t),1)&=f(t,c_1\frac{t(1-t)}{c_1u(t)}u(t),c_2\frac{1}{c_2v(t)}v(t))\\
&\le c_1^{\lambda_1}(\frac{t(1-t)}{c_1u(t)})^{\mu_1}c_2^{\alpha_1}(\frac{1}{c_2v(t)})^{\beta_1}f(t,u(t),v(t))\\
&\le c_1^{\lambda_1}(\frac{1}{c_1m_1})^{\mu_1}c_2^{\alpha_1-\beta_1}(\frac{1}{v(t)})^{\beta_1}f(t,u(t),v(t))\\
&=c_1^{\lambda_1-\mu_1}c_2^{\alpha_1-\beta_1}m_1^{-\mu_1}(v(t))^{-\beta_1}f(t,u(t),v(t)).
\end{aligned}$$

There exists $t_0\in(0,1)$ such that $u'''(t_0)=0$ from $u''(0)=u''(1)=0$. This together with $u^{(4)}\ge0$ for $t_0\in(0,1)$ yields that $u'''(t)\ge0$ as $t\in(0,t_0)$ and $u'''(t)\le0$ as $t\in(t_0,1)$. Integrate $u^{(4)}(t)=f(t,u(t),v(t))$ from t to t_0, we can get

$$-u^{(3)}(t)=\int_t^{t_0}f(s,u(s),v(s))ds, t\in(0,t_0).$$

However,

$$\begin{aligned}
\int_0^{t_0}tf(t,t(1-t),1)dt&=\int_0^{t_0}dt\int_t^{t_0}f(s,s(1-s),1)ds\\
&\le\int_0^{t_0}\int_t^{t_0}c_1^{\lambda_1-\mu_1}c_2^{\alpha_1-\beta_1}m_1^{-\mu_1}(v(s))^{-\beta_1}f(s,u(s),v(s))ds.
\end{aligned}$$

On the other hand

$$\begin{aligned}
v(t)&=\int_0^1G(t,s)g(s,u(s),v(s))ds\\
&\ge\frac14\int_{1/4}^{3/4}s(1-s)g(s,s(1-s)c_5\frac{u(s)}{c_5s(1-s)},c_6\frac{v(s)}{c_6})ds.
\end{aligned}$$

Let $c_5\ge\max\{M_1,\frac{m_2}{N_1}\}$, $c_6\ge\max\{M_2,\frac{\|v\|_0}{N_2}\}$. Then

$$v(t)\ge\frac14c_5^{\lambda_2-\mu_2}c_6^{\alpha_2-\beta_2}{m_1}^{\mu_2}\int_{1/4}^{3/4}s(1-s)((v(s))^{\beta_2}g(s,s(1-s),1)ds.$$

However, one can see that $v(t)\ge\frac{1}{16}\|v\|_0$ as $t\in[\frac14,\frac34]$. Hence, for $t\in J$,

$$v(t)\ge2^{-(2+4\beta_2)}c_5^{\lambda_2-\mu_2}c_6^{\alpha_2-\beta_2}{m_1}^{\mu_2}\|v\|_0^{\beta_2}\int_{1/4}^{3/4}s(1-s)g(s,s(1-s),1)ds.$$

Let

$$\begin{aligned}
k_0=c_1^{\lambda_1-\mu_1}c_2^{\alpha_1-\beta_1}{m_1}^{-\mu_1}\Big(&2^{-(2+4\beta_2)}c_5^{\alpha_2-\mu_2}c_6^{\alpha_2-\beta_2}{m_1}^{\mu_2}\|v\|_0^{\beta_2}\\
&\times\int_{1/4}^{3/4}s(1-s)g(s,s(1-s),1)ds\Big)^{-\beta_1}.
\end{aligned}$$

Thus

$$\int_0^{t_0} tf(t,t(1-t),1)dt \le k_0 \int_0^{t_0} dt \int_t^{t_0} f(s,u(s),v(s))ds = k_0(-u''(t_0)) < \infty.$$

In the same way, we can also prove

$$\int_{t_0}^1 (1-t)f(t,t(1-t),1)dt < +\infty.$$

Hence

$$\int_0^1 t(1-t)f(t,t(1-t),1)dt < +\infty.$$

□

Proof of Theorem 2.2. **Sufficiency.** First of all, we prove that

$$\int_0^1 f(t,t(1-t),t(1-t))dt < +\infty$$

implies

$$\int_0^1 t(1-t)f(t,t(1-t),1)dt < +\infty.$$

Choose positive number $c \ge \max\{M_2, \frac{1}{4N_2}\}$. Then

$$\begin{aligned} f(t,t(1-t),t(1-t)) &= f(t,t(1-t),c\frac{t(1-t)}{c}) \\ &\ge c^{\alpha_1-\beta_1}(t(1-t))^{\beta_1} f(t,t(1-t),1) \\ &\ge c^{\alpha_1-\beta_1} t(1-t)f(t,t(1-t),1). \end{aligned}$$

Consequently, we can get

$$\int_0^1 f(t,t(1-t),t(1-t))dt \ge c^{\alpha_1-\beta_1}\int_0^1 t(1-t)f(t,t(1-t),1)dt,$$

namely,

$$\int_0^1 t(1-t)f(t,t(1-t),1)dt < +\infty.$$

On the other hand, we can also prove that

$$\int_0^1 g(t,t(1-t),t(1-t))dt < +\infty.$$

This implies

$$\int_0^1 t(1-t)g(t,t(1-t),1)dt < +\infty.$$

From above inequalities, we know that (1.1) exists a $C^2[0,1] \times C[0,1]$ positive solution (u,v). Therefore it suffices to show that $u'''(0^+), u'''(1^-), v'(0^+)$ and $v'(1^-)$ exist. The same reason as the proof of Theorem 2.1 of necessity asserts that there exist $0 < m_1 < 1 < m_2$ and $0 < n_1 < 1 < n_2$ satisfying $m_1 t(1-t) \le u(t) \le m_2 t(1-t)$ and $n_1 t(1-t) \le v(t) \le n_2 t(1-t), t \in J$.

Let $c_1 \ge \max\{M_1, \frac{m_2}{N_1}\}$ and $c_2 \ge \max\{M_2, \frac{n_2}{N_2}\}$, then we have

$$\int_0^1 |u^{(4)}(t)|dt = \int_0^1 f(t,u(t),v(t))dt$$

$$
\begin{aligned}
&= \int_0^1 f(t, c_1\frac{u(t)}{c_1 t(1-t)} t(1-t), c_2\frac{v(t)}{c_2 t(1-t)} t(1-t))dt \\
&\leq c_1^{\mu_1}(\frac{u(t)}{c_1 t(1-t)})^{\lambda_1} c_2^{\beta_1}(\frac{v(t)}{c_2 t(1-t)})^{\alpha_1} \int_0^1 f(t, t(1-t), t(1-t))dt \\
&\leq c_1^{\mu_1-\lambda_1} c_2^{\beta_1-\alpha_1} m_2{}^{\lambda_1} n_2{}^{\alpha_1} \int_0^1 f(t, t(1-t), t(1-t))dt < +\infty.
\end{aligned}
$$

This guarantees $u'''(0^+)$ and $u'''(1^-)$ exist.

On the other hand, let $c_3 \geq \max\{M_3, \frac{m_2}{N_3}\}$ and $c_4 \geq \max\{M_4, \frac{n_2}{N_4}\}$, then

$$
\begin{aligned}
\int_0^1 |-v''(t)|dt &= \int_0^1 g(t, u(t), v(t))dt \\
&= \int_0^1 g(t, c_3\frac{u(t)}{c_3 t(1-t)} t(1-t), c_4\frac{v(t)}{c_4 t(1-t)} t(1-t))dt \\
&\leq c_3^{\mu_2-\lambda_2} c_4^{\beta_2-\alpha_2} m_2{}^{\lambda_2} n_2{}^{\alpha_2} \int_0^1 g(t, t(1-t), t(1-t))dt < +\infty.
\end{aligned}
$$

This means that $v'(0^+)$ and $v'(1^-)$ exist.

Necessity. Let (u, v) be a $C^3[0,1] \times C^1[0,1]$ positive solution of (1.1). The same reason as the beginning of the proof of sufficiency asserts that there exist $0 < m_1 < 1 < m_2$ and $0 < n_1 < 1 < n_2$ satisfying $m_1 t(1-t) \leq u(t) \leq m_2 t(1-t)$ and $n_1 t(1-t) \leq v(t) \leq n_2 t(1-t), t \in J$. Suppose $c_1 \leq \min\{N_1, \frac{1}{M_1 m_2}\}$, $c_2 \leq \min\{N_2, \frac{1}{M_2 n_2}\}$, $c_3 \leq \min\{N_3, \frac{1}{M_3 m_2}\}$ and $c_4 \leq \min\{N_4, \frac{1}{M_4 n_2}\}$. Then we have

$$
\begin{aligned}
f(t, t(1-t), t(1-t)) &= f(t, c_1\frac{t(1-t)}{c_1 u(t)} u(t), c_2\frac{t(1-t)}{c_2 v(t)} v(t)) \\
&\leq c_1{}^{\lambda_1}(\frac{t(1-t}{c_1 u(t)})^{\mu_1} c_2{}^{\alpha_1}(\frac{t(1-t}{c_1 v(t)})^{\beta_1} f(t, u(t), v(t)) \\
&\leq c_1{}^{\lambda_1-\mu_1} c_2{}^{\alpha_1-\beta_1} m_1{}^{-\mu_1} n_1{}^{-\beta_1} f(t, u(t), v(t)).
\end{aligned}
$$

Consequently,

$$
\int_0^1 f(t, t(1-t), t(1-t))dt \leq c_1{}^{\lambda_1-\mu_1} c_2{}^{\alpha_1-\beta_1} m_1{}^{-\mu_1} n_1{}^{-\beta_1}[u'''(1^-) - u'''(0^+)] < +\infty.
$$

On the other hand, we can also prove

$$
\int_0^1 g(t, t(1-t), t(1-t))dt \leq c_3{}^{\lambda_2-\mu_2} c_4{}^{\alpha_2-\beta_2} m_1{}^{-\mu_2} n_1{}^{-\beta_2}[v'(1^-) - v'(0^+)] < +\infty.
$$

Therefore, our conclusion follows. □

In the following we give some examples to illustrate the theorems obtained in Section 2.

Example 2.6. Consider (1.1) with

$$
f(t, u, v) = p(t)u^{10}v^{1/3} + (u+v)^{1/3}, \quad g(t, u, v) = a(t)u^2 v^{2/5} + (u+v)^{2/5},
$$

where $p, a \in C[(0,1), R^+]$ and

$$
\int_0^1 [p(t)(t(1-t))^{11} + (t(1-t))(t(1-t)+1)^{1/3}]dt < +\infty,
$$

$$\int_0^1 [a(t)(t(1-t))^3 + (t(1-t))(t(1-t)+1)^{2/5}]dt < +\infty.$$

It is obvious that f, g satisfy (H3)-(H5). So it is easy to see, by Theorem 2.1, that (1.1) has a $C^2[0,1] \times C[0,1]$ positive solution.

Example 2.7. In (1.1), let

$$f(t,u,v) = q(t)u^3v^{1/4} + (u+v)^{1/4}, \quad g(t,u,v) = e(t)u^3v^{2/3} + (u+v)^{2/3},$$

where $q, e \in C[(0,1), R^+]$ and

$$\int_0^1 [q(t)(t(1-t))^{13/4} + (2t(1-t))^{1/4}]dt < +\infty,$$
$$\int_0^1 [e(t)(t(1-t))^{11/3} + (2t(1-t))^{2/3}]dt < +\infty.$$

It is obvious that f, g satisfy (H3)-(H5). So it is easy to see, by Theorem 2.2, that (1.1) has a $C^3[0,1] \times C^1[0,1]$ positive solution.

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