Electronic Journal of Differential Equations, Vol. 2008(2008), No. 72, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

SOLVABILITY OF CHARACTERISTIC BOUNDARY-VALUE PROBLEMS FOR NONLINEAR EQUATIONS WITH ITERATED WAVE OPERATOR IN THE PRINCIPAL PART

SERGO KHARIBEGASHVILI, BIDZINA MIDODASHVILI

Abstract. A characteristic boundary-value problem for a hyperbolic equation with power nonlinearity and iterated wave operator in the principal part is considered in a conical domain. Depending on the exponent of nonlinearity and spatial dimensionality of the equation, the existence and uniqueness of the solution of a boundary-value problem is established. The non-solvability of this problem is also considered here.

1. Introduction

In the Euclidean space $\mathbb{R}^{n+1}$ of independent variables $x_1, x_2, \dots, x_n, t$, consider the nonlinear equation
$$L_\lambda u := \square^2 u = \lambda f(u) + F, \tag{1.1}$$
where λ is a given real constant, $f : \mathbb{R} \to \mathbb{R}$ is a given continuous nonlinear function, $f(0) = 0$, F is a given, and u is an unknown real functions, and for $n \geq 2$,
$$\square = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\,.$$

Let $D_T : |x| < t < T - |x|$ be a domain, which is the intersection of the light cone of future $K_O^+ : t > |x|$ with the apex in the origin $O(0, 0, \dots, 0)$ and light cone of past $K_A^- : t < T - |x|$ with apex in point $A(0, \dots, 0, T)$, $T = const > 0$.

For equation (1.1) consider the boundary-value problem on determination of its solution $u(x_1, \dots, x_n, t)$ in domain D_T with the boundary condition
$$u\big|_{\partial D_T} = 0. \tag{1.2}$$

It should be noted that for nonlinear hyperbolic equations the local or global solvability of the Cauchy problem with initial conditions for $t = 0$ and mixed problems has been studied in numerous publications; see, [2, 4, 5, 8, 9, 12, 13, 14, 15, 16, 17, 18, 25, 27, 28, 29, 30, 32, 33, 34].

Regarding the nonlinear wave equation $\square u = \lambda f(u) + F$, we have the following results: The characteristic problem in the light cone of future $K_O^+ : t > |x|$, with

---

2000 *Mathematics Subject Classification.* 35L05, 35L35, 35L75.
Key words and phrases. Characteristic boundary-value problem; hyperbolic equations; wave operator; power nonlinearity; nonexistence.

Submitted February 18, 2008. Published May 15, 2008.

boundary condition $u|_{\partial K_O^+} = g$, in the linear case with $\lambda = 0$, is well-posed and has global solvability in some appropriate function spaces; see [1, 3, 6, 10, 26]. Meanwhile, the nonlinear case, when $f(u)$ has exponential nature and $\lambda \neq 0$, has been considered in [19, 20, 21].

Assume $\overset{\circ}{C}{}^k(\overline{D}_T, \partial D_T) = \{u \in C^k(\overline{D}_T) : u|_{\partial D_T} = 0\}$, $k \geq 1$. Let $u \in \overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T)$ be a classical solution of problem (1.1)-(1.2). Multiplying the both parts of (1.1) by an arbitrary function $\phi \in \overset{\circ}{C}{}^2(\overline{D}_T, \partial D_T)$ and integrating obtained equation by parts in domain D_T we obtain

$$\int_{D_T} \Box u \Box \phi \, dx \, dt = \lambda \int_{D_T} f(u)\phi \, dx \, dt + \int_{D_T} F\phi \, dx \, dt. \tag{1.3}$$

Here we used the equality

$$\int_{D_T} \Box u \Box \phi \, dx \, dt = \int_{\partial D_T} \frac{\partial \phi}{\partial N} \Box u ds - \int_{\partial D_T} \phi \frac{\partial}{\partial N} \Box u ds + \int_{D_T} \phi \Box^2 u \, dx \, dt$$

and the fact that since ∂D_T is characteristic manifold, then derivative on the conormal

$$\frac{\partial}{\partial N} = \gamma_{n+1} \frac{\partial}{\partial t} - \sum_{i=1}^{n} \gamma_i \frac{\partial}{\partial x_i},$$

where $\gamma = (\gamma_1, \dots, \gamma_n, \gamma_{n+1})$ is the unit vector of external normal relative to ∂D_T, is an inner differential operator on characteristic manifold ∂D_T and, thus, if $v \in \overset{\circ}{C}{}^1(\overline{D}_T, \partial D_T)$, then $\frac{\partial v}{\partial N}|_{\partial D_T} = 0$.

Let us introduce the Hilbert space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$ as a completion with respect to the norm

$$\|u\|^2_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)} = \int_{D_T} \big[u^2 + (\frac{\partial u}{\partial t})^2 + \sum_{i=1}^{n} (\frac{\partial u}{\partial x_i})^2 + (\Box u)^2\big] \, dx \, dt \tag{1.4}$$

of classical space $\overset{\circ}{C}{}^2(\overline{D}_T, \partial D_T)$. It follows from (1.4) that if $u \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$, then $u \in \overset{\circ}{W}{}^1_2(D_T)$ and $\Box u \in L_2(D_T)$. Here $W^1_2(D_T)$ is the known Sobolev space [24, p. 56], consisting of elements from $L_2(D_T)$, which have first order generalized derivatives in $L_2(D_T)$, and $\overset{\circ}{W}{}^1_2(D_T) = \{u \in W^1_2(D_T) : u|_{\partial D_T} = 0\}$, where equality $u|_{\partial D_T} = 0$ should be understood in the sense of the theory of trace [24, p. 70].

Let us assume (1.3) as the basis of determination of generalized solution of problem (1.1)-(1.2).

Definition 1.1. Let $F \in L_2(D_T)$. We call function $u \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$ a weak generalized solution of problem (1.1)-(1.2) if $f(u) \in L_2(D_T)$ and for any function $\phi \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$ it is valid integral equality (1.3); i.e.

$$\int_{D_T} \Box u \Box \phi \, dx \, dt = \lambda \int_{D_T} f(u)\phi \, dx \, dt + \int_{D_T} F\phi \, dx \, dt \quad \forall \phi \in \overset{\circ}{W}{}^1_{2,\Box}(D_T). \tag{1.5}$$

It is easy to verify that if the solution u of problem (1.1)-(1.2) in the sense of the above definition belongs to the class $C^4(\overline{D}_T)$, then it will be a classical solution of this problem.

2. Solvability of (1.1)-(1.2) with $f(u) = |u|^\alpha \operatorname{sgn} u$

Assume that for a positive constant $\alpha \neq 1$, the nonlinear function f in (1.1) has the form

$$f(u) = |u|^\alpha \operatorname{sgn} u\,. \tag{2.1}$$

Then in accordance to (2.1), equation (1.1) and (1.5) take the form

$$L_\lambda u := \Box^2 u = \lambda |u|^\alpha \operatorname{sgn} u + F \tag{2.2}$$

and

$$\int_{D_T} \Box u \Box \phi \, dx\, dt = \lambda \int_{D_T} \phi |u|^\alpha \operatorname{sgn} u \, dx\, dt + \int_{D_T} F\phi\, dx\, dt, \quad \forall \phi \in \mathring{W}^1_{2,\Box}(D_T). \tag{2.3}$$

Lemma 2.1. *With the norm of the space $\mathring{W}^1_{2,\Box}(D_T)$ given in* (1.4),

$$\|u\|_{\mathring{W}^1_{2,\Box}(D_T)} \leq c\|\Box u\|_{L_2(D_T)} \quad \forall u \in \mathring{W}^1_{2,\Box}(D_T) \tag{2.4}$$

where c is positive constant independent on u.

Proof. Since the space $\mathring{C}^2(\overline{D}_T, \partial D_T)$ is the dense subspace of space $\mathring{W}^1_{2,\Box}(D_T)$ it is sufficient to prove that for all $u \in mathaccent"7017C^2(\overline{D}_T, \partial D_T)$,

$$\|u\|^2_{W^1_{2,\Box}(D^+_{T/2})} \leq c^2 \|\Box u\|^2_{L_2(D^+_{T/2})}, \quad \|u\|^2_{W^1_{2,\Box}(D^-_{T/2})} \leq c^2 \|\Box u\|^2_{L_2(D^-_{T/2})}, \tag{2.5}$$

where $D^+_{T/2} = D_T \cap \{t < T/2\}$, $D^-_{T/2} = D_T \cap \{t > T/2\}$ and the norm $\|\cdot\|_{W^1_{2,\Box}(D^\pm_{T/2})}$ is given by (1.4) with $D^\pm_{T/2}$ instead of D_T.

Let us prove the first inequality of (2.5), the second inequality can be proved in the same way. Assume $\Omega_\tau := \overline{D}^+_{T/2} \cap \{t = \tau\}$, $D^+_\tau = D^+_{T/2} \cap \{t < \tau\}$, $S^+_\tau = \{(x,t) \in \partial D^+_\tau : t = |x|\}$, $0 < \tau \leq T/2$ and $\gamma = (\gamma_1, \dots, \gamma_n, \gamma_{n+1})$ be the unit vector of outer normal relative to ∂D^+_τ. For $u \in C^{2^0}(\overline{D}_T, \partial D_T)$, taking into account equalities $u|_{S^+_\tau} = 0$, $\Omega_\tau = \partial D^+_\tau \cap \{t = \tau\}$ and $\gamma|_{\Omega_\tau} = (0, \dots, 0, 1)$, integrating by parts it is easy to obtain

$$\begin{aligned} \int_{D^+_\tau} \frac{\partial^2 u}{\partial t^2} \frac{\partial u}{\partial t} \, dx\, dt &= \frac{1}{2} \int_{D^+_\tau} \frac{\partial}{\partial t} (\frac{\partial u}{\partial t})^2 \, dx\, dt = \frac{1}{2} \int_{\partial D^+_\tau} (\frac{\partial u}{\partial t})^2 \gamma_{n+1} ds \\ &= \frac{1}{2} \int_{\Omega_\tau} (\frac{\partial u}{\partial t})^2 dx + \frac{1}{2} \int_{S^+_\tau} (\frac{\partial u}{\partial t})^2 \gamma_{n+1} ds, \quad \tau \leq T/2, \end{aligned} \tag{2.6}$$

$$\begin{aligned} \int_{D^+_\tau} \frac{\partial^2 u}{\partial x_i^2} \frac{\partial u}{\partial t} \, dx\, dt &= \int_{\partial D^+_\tau} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial t} \gamma_i ds - \frac{1}{2} \int_{D^+_\tau} \frac{\partial}{\partial t} (\frac{\partial u}{\partial x_i})^2 \, dx\, dt \\ &= \int_{\partial D^+_\tau} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial t} \gamma_i ds - \frac{1}{2} \int_{\partial D^+_\tau} (\frac{\partial u}{\partial x_i})^2 \gamma_{n+1} ds \\ &= \int_{\partial D^+_\tau} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial t} \gamma_i ds - \frac{1}{2} \int_{S^+_\tau} (\frac{\partial u}{\partial x_i})^2 \gamma_{n+1} ds - \frac{1}{2} \int_{\Omega_\tau} (\frac{\partial u}{\partial x_i})^2 dx, \end{aligned} \tag{2.7}$$

with $\tau \leq T/2$. It follows from (2.6) and (2.7) that

$$\begin{aligned} &\int_{D_\tau^+} \Box u \frac{\partial u}{\partial t}\, dx\, dt \\ &= \int_{S_\tau^+} \frac{1}{2\gamma_{n+1}} \Big[\sum_{i=1}^n \big(\frac{\partial u}{\partial x_i}\gamma_{n+1} - \frac{\partial u}{\partial t}\gamma_i\big)^2 + (\frac{\partial u}{\partial t})^2 \big(\gamma_{n+1}^2 - \sum_{j=1}^n \gamma_j^2\big)\Big] ds \\ &\quad + \frac{1}{2}\int_{\Omega_\tau} \big[(\frac{\partial u}{\partial t})^2 + \sum_{i=1}^n (\frac{\partial u}{\partial x_i})^2\big] dx, \quad \tau \leq T. \end{aligned} \tag{2.8}$$

Since $u|_{S_\tau^+} = 0$ and operator $(\gamma_{n+1}\frac{\partial}{\partial x_i} - \gamma_i \frac{\partial}{\partial t})$, $1 \leq i \leq n$, is an inner differential operator on S_τ^+, then we have the equalities

$$\Big(\frac{\partial u}{\partial x_i}\gamma_{n+1} - \frac{\partial u}{\partial t}\gamma_i\Big)\big|_{S_\tau^+} = 0, \quad i = 1, \dots, n. \tag{2.9}$$

Therefore, taking into account that $\gamma_{n+1}^2 - \sum_{j=1}^n \gamma_j^2 = 0$ on the characteristic manifold S_τ^+, in view of (2.8) and (2.9), we have

$$\int_{\Omega_\tau} \big[(\frac{\partial u}{\partial t})^2 + \sum_{i=1}^n (\frac{\partial u}{\partial x_i})^2\big] dx = 2\int_{D_\tau^+} \Box u \frac{\partial u}{\partial t}\, dx\, dt, \quad \tau \leq T/2. \tag{2.10}$$

Assuming $w(\delta) = \int_{\Omega_\delta} [(\frac{\partial u}{\partial t})^2 + \sum_{i=1}^n (\frac{\partial u}{\partial x_i})^2] dx$, and using inequality $2\Box u \frac{\partial u}{\partial t} \leq \varepsilon(\frac{\partial u}{\partial t})^2 + \frac{1}{\varepsilon}|\Box u|^2$, which is valid for any positive ε, from (2.10) we obtain

$$w(\delta) \leq \varepsilon \int_0^\delta w(\sigma) d\sigma + \frac{1}{\varepsilon}\|\Box\|^2_{L_2(D_\delta^+)}, \quad 0 < \delta \leq T/2. \tag{2.11}$$

From (2.11), taking into account that value $\|\Box\|^2_{L_2(D_\delta^+)}$ as a function of δ is non-decreasing, in view of Gronwall's lemma [11, p. 13] it follows that

$$w(\delta) \leq \frac{1}{\varepsilon}\|\Box\|^2_{L_2(D_\delta^+)} \exp \delta\varepsilon.$$

Hence, taking into account the fact that $\inf_{\varepsilon > 0} \frac{1}{\varepsilon} \exp \delta\varepsilon = e\delta$ and it is reached at $\varepsilon = \frac{1}{\delta}$, we obtain

$$w(\delta) \leq e\delta\|\Box\|^2_{L_2(D_\delta^+)}, \ \ 0 < \delta \leq T/2.$$

From (2), in turn, it follows that

$$\int_{D_{T/2}^+} [(\frac{\partial u}{\partial t})^2 + \sum_{i=1}^n (\frac{\partial u}{\partial x_i})^2]\, dx\, dt = \int_0^{T/2} w(\delta) d\delta \leq \frac{e}{8}T^2\|\Box u\|^2_{L_2(D_{T/2}^+)}. \tag{2.12}$$

Using the equalities $u|_{S_{T/2}} = 0$ and $u(x,t) = \int_{|x|}^t \frac{\partial u(x,t)}{\partial t} d\tau$, $(x,t) \in \overline{D}_{T/2}^+$, which are valid for any function $u \in C^{2^0}(\overline{D}_T, \partial D_T)$, by standard reasoning [24, p. 63] we easily obtain

$$\int_{D_{T/2}^+} u^2(x,t)\, dx\, dt \leq \frac{1}{4}T^2 \int_{D_{T/2}^+} (\frac{\partial u}{\partial t})^2\, dx\, dt. \tag{2.13}$$

By virtue of (2.12) and (2.13), we have

$$\begin{aligned}\|u\|^2_{\mathring{W}^1_{2,\Box}(D^+_{T/2})} &= \int_{D^+_{T/2}} \big[u^2 + (\frac{\partial u}{\partial t})^2 + \sum_{i=1}^n (\frac{\partial u}{\partial x_i})^2 + (\Box u)^2\big]\,dx\,dt\\ &\le \big(1 + \frac{e}{8}T^2 + \frac{e}{32}T^4\big)\|\Box\|^2_{L_2(D^+_{T/2})},\end{aligned}$$

whence it follows the first inequality of (2.5) with constant $c^2 = 1 + \frac{e}{8}T^2 + \frac{e}{32}T^4$. The proof is complete. $\Box$

Lemma 2.2. *Assume $F \in L_2(D_T)$, $0 < \alpha < 1$, and in the case when $\alpha > 1$ additionally require that $\lambda < 0$. Then for a weak generalized solution $u \in \mathring{W}^1_{2,\Box}(D_T)$ of* (1.1)-(1.2) *in the case with nonlinearity of form* (2.1)*; i.e., problem* (2.2)-(1.2) *in the sense of integral equality* (2.3) *with $|u|^\alpha \in L_2(D_T)$, it is valid a priori estimate*

$$\|u\|_{\mathring{W}^1_{2,\Box}(D_T)} \le c_1\|F\|_{L_2(D_T)} + c_2 \tag{2.14}$$

with non-negative constants $c_i(T,\alpha,\lambda)$, $i = 1,2$, which do not depend on u, F and $c_1 > 0$.

Proof. First let $\alpha > 1$ and $\lambda < 0$. Assuming in (2.3) that $\phi = u \in \mathring{W}^1_{2,\Box}(D_T)$ and taking into account (1.4), for any $\varepsilon > 0$ we have

$$\begin{aligned}\|\Box u\|^2_{L_2(D_T)} &= \int_{D_T} (\Box u)^2\,dx\,dt\\ &= \lambda\int_{D_T} |u|^{\alpha+1}\,dx\,dt + \int_{D_T} Fu\,dx\,dt\\ &\le \int_{D_T} Fu\,dx\,dt\\ &\le \frac{1}{4\varepsilon}\int_{D_T} F^2\,dx\,dt + \varepsilon\|u\|^2_{L_2(D_T)}\\ &\le \frac{1}{4\varepsilon}\|F\|^2_{L_2(D_T)} + \varepsilon\|u\|^2_{\mathring{W}^1_{2,\Box}(D_T)}.\end{aligned} \tag{2.15}$$

Due to (2.4) and the above inequality we have

$$\|u\|^2_{\mathring{W}^1_{2,\Box}(D_T)} \le c^2\|\Box u\|^2_{L_2(D_T)} \le \frac{c^2}{4\varepsilon}\|F\|^2_{L_2(D_T)} + c^2\varepsilon\|u\|^2_{\mathring{W}^1_{2,\Box}(D_T)},$$

from which for $\varepsilon = \frac{1}{2c^2} < \frac{1}{c^2}$, we obtain

$$\|u\|^2_{\mathring{W}^1_{2,\Box}(D_T)} \le \frac{c^2}{4\varepsilon(1-\varepsilon c^2)}\|F\|^2_{L_2(D_T)} = c^4\|F\|^2_{L_2(D_T)}.$$

From this inequality in the case $\alpha > 1$ and $\lambda < 0$ follows inequality (2.14) with $c_1 = c^2$ and $c_2 = 0$.

Now let $0 < \alpha < 1$. Using the known inequality

$$ab \le \frac{\varepsilon a^p}{p} + \frac{b^q}{q\varepsilon^{q-1}}$$

with parameter $\varepsilon > 0$ for $a = |u|^{\alpha+1}$, $b = 1$, $p = \frac{2}{\alpha+1} > 1$, $q = \frac{2}{1-\alpha}$, $\frac{1}{p} + \frac{1}{q} = 1$, in the same way as for inequality (2.15), we have

$$\begin{aligned}
&\|\Box u\|^2_{L_2(D_T)} \\
&= \int_{D_T} (\Box u)^2\, dx\, dt \\
&= \lambda \int_{D_T} |u|^{\alpha+1}\, dx\, dt + \int_{D_T} Fu\, dx\, dt \\
&\le |\lambda| \int_{D_T} \big[\varepsilon \frac{1+\alpha}{2} |u|^2 + \frac{1-\alpha}{2\varepsilon^{q-1}}\big]\, dx\, dt + \frac{1}{4\varepsilon} \int_{D_T} F^2\, dx\, dt + \varepsilon \int_{D_T} u^2\, dx\, dt \\
&= \frac{1}{4\varepsilon} \|F\|^2_{L_2(D_T)} + \varepsilon(|\lambda| \frac{1+\alpha}{2} + 1) \|u\|^2_{L_2(D_T)} + |\lambda| \frac{1-\alpha}{2\varepsilon^{q-1}} \operatorname{meas} D_T.
\end{aligned} \tag{2.16}$$

In view of (1.4) and (2.4) it follows from (2.16) that

$$\begin{aligned}
&\|u\|^2_{\mathring{W}^1_{2,\Box}(D_T)} \\
&\le c^2 \|\Box u\|^2_{L_2(D_T)} \\
&\le \frac{c^2}{4\varepsilon} \|F\|^2_{L_2(D_T)} + \varepsilon c^2 (|\lambda| \frac{1+\alpha}{2} + 1) \|u\|^2_{\mathring{W}^1_{2,\Box}(D_T)} + c^2 |\lambda| \frac{1-\alpha}{2\varepsilon^{q-1}} \operatorname{meas} D_T,
\end{aligned}$$

where $q = \frac{2}{1-\alpha}$; whence for $\varepsilon = \frac{1}{2} c^{-2} (|\lambda| \frac{1+\alpha}{2} + 1)^{-1}$,

$$\begin{aligned}
&\|u\|^2_{\mathring{W}^1_{2,\Box}(D_T)} \\
&\le \big[1 - \varepsilon c^2 \big(|\lambda| \frac{1+\alpha}{2} + 1\big)\big]^{-1} \Big(\frac{c^2}{4\varepsilon} \|F\|^2_{L_2(D_T)} + c^2 |\lambda| \frac{1-\alpha}{2\varepsilon^{q-1}} \operatorname{meas} \operatorname{meas} D_T \Big) \\
&= c^4 \big(|\lambda| \frac{1+\alpha}{2} + 1\big) \|F\|^2_{L_2(D_T)} + 2c^2 |\lambda| \frac{1-\alpha}{2\varepsilon^{q-1}} \operatorname{meas} D_T.
\end{aligned} \tag{2.17}$$

From (2.17), in the case when $0 < \alpha < 1$, follows inequality (2.14) with $c_1 = c^2 (|\lambda| \frac{1+\alpha}{2} + 1)^{1/2}$ and $c_2 = c(2|\lambda| \frac{1-\alpha}{2\varepsilon^{q-1}} \operatorname{meas} D_T)^{1/2}$, where $q = \frac{1}{1-\alpha}$. The proof is complete. $\Box$

Remark 2.3. From the proof of Lemma 2.2 it follows that in estimate (2.14) the constants c_1 and c_2 are equal:

$$\alpha > 1, \quad \lambda < 0: \quad c_1 = c^2, \quad c_2 = 0; \tag{2.18}$$

$$0 < \alpha < 1, \quad -\infty < \lambda < +\infty:$$

$$c_1 = c^2 (|\lambda| \frac{1+\alpha}{2} + 1)^{1/2}, \quad c_2 = c(2|\lambda| \frac{1-\alpha}{2\varepsilon^{q-1}} \operatorname{meas} D_T)^{\frac{1}{2}}, \tag{2.19}$$

where constant $c = (1 + \frac{e}{2} T^2 + \frac{e}{2} T^4)^{1/2}$ is taken from estimate (2.4), and $q = \frac{2}{1-\alpha}$.

Remark 2.4. Below, we will consider a linear problem appropriate for (1.1)-(1.2); i.e., when $\lambda = 0$. In this case for $F \in L_2(D_T)$ it is analogously introduced a concept of the weak generalized solution $u \in \mathring{W}^1_{2,\Box}(D_T)$ of this problem, when

$$(u, \phi)_\Box := \int_{D_T} \Box u \Box \phi\, dx\, dt = \int_{D_T} F \phi\, dx\, dt \quad \forall \phi \in \mathring{W}^1_{2,\Box}(D_T). \tag{2.20}$$

Remark 2.5. In view of (1.4) and (2.4), taking into account that

$$\begin{aligned}|(\Box u,\Box\phi)_{L_2(D_T)}| &= \Big|\int_{D_T}\Box u\Box\phi\,dx\,dt\Big|\\ &\le \|\Box u\|_{L_2(D_T)}\|\Box\phi\|_{L_2(D_T)}\\ &\le \|\Box u\|_{\mathring{W}^1_{2,\Box}(D_T)}\|\Box\phi\|_{\mathring{W}^1_{2,\Box}(D_T)},\end{aligned}$$

the bilinear form

$$(u,\phi)_\Box := \int_{D_T}\Box u\Box\phi\,dx\,dt$$

in (2.20) can be considered as a scalar product in the Hilbert space $\mathring{W}^1_{2,\Box}(D_T)$. Therefore, since for $F\in L_2(D_T)$

$$\Big|\int_{D_T}F\phi\,dx\,dt\Big| \le \|F\|_{L_2(D_T)}\|\phi\|_{L_2(D_T)} \le \|F\|_{L_2(D_T)}\|\phi\|_{\mathring{W}^1_{2,\Box}(D_T)},$$

then due to the Riesz theorem [7, p. 83] there is unique function u in the space $\mathring{W}^1_{2,\Box}(D_T)$, which satisfies equality (2.20) for any $\phi\in\mathring{W}^1_{2,\Box}(D_T)$ and for the norm of which it is valid estimate

$$\|u\|_{\mathring{W}^1_{2,\Box}(D_T)} \le \|F\|_{L_2(D_T)}. \tag{2.21}$$

Thus, introducing notation $u = L_0^{-1}F$, we obtain that to the linear problem appropriate to (1.1)-(1.2); i.e., when $\lambda = 0$, corresponds the linear, bounded operator

$$L_0^{-1} : L_2(D_T) \to \mathring{W}^1_{2,\Box}(D_T),$$

for the norm of which, by (2.21), it is valid the estimate

$$\|L_0^{-1}\|_{L_2(D_T)\to\mathring{W}^1_{2,\Box}(D_T)} \le \|F\|_{L_2(D_T)}. \tag{2.22}$$

Taking into account Definition 1.1 and Remark 2.5, Equality (2.3) and Problem (2.2)-(1.2) can be rewritten in the equivalent form

$$u = L_0^{-1}[\lambda|u|^\alpha \operatorname{sgn} u + F] \tag{2.23}$$

in the Hilbert space $\mathring{W}^1_{2,\Box}(D_T)$.

Remark 2.6. The embedding operator $I : \mathring{W}^1_2(D_T) \to L_q(D_T)$ is a linear continuous compact operator for $1 < q < \frac{2(n+1)}{n-1}$, when $n \ge 2$ [24, p. 81]. At the same time the operator of Nemytskii $N : L_q(D_T) \to L_2(D_T)$, which acts according to the formula $Nu = \lambda|u|^\alpha \operatorname{sgn} u$, $\alpha > 1$, is continuous and bounded for $q \ge 2\alpha$ [22, p. 349], [23, pp. 66, 67]. Thus, if $1 < \alpha < \frac{n+1}{n-1}$, then there exists such number q, that $1 < 2\alpha \le q < \frac{2(n+1)}{n-1}$ and hence the operator

$$N_1 = NI : \mathring{W}^1_2(D_T) \to L_2(D_T) \tag{2.24}$$

is continuous and compact operator. In this case since $u \in \mathring{W}^1_2(D_T)$ then it is clear that $f(u) = |u|^\alpha \operatorname{sgn} u \in L_2(D_T)$. Further, since in view of (1.4) the space $\mathring{W}^1_{2,\Box}(D_T)$ is continuously embedded in the space $\mathring{W}^1_2(D_T)$, then taking into account (2.24) the operator

$$N_2 = NII_1 : \mathring{W}^1_{2,\Box}(D_T) \to L_2(D_T), \tag{2.25}$$

where $I_1 : \mathring{W}^1_{2,\Box}(D_T) \to \mathring{W}^1_2(D_T)$ is the embedding operator, continuous and compact for $1 < \alpha < \frac{n+1}{n-1}$. For $0 < \alpha < 1$ operator (2.25) is also continuous and compact, since according to the Rellich theorem [24, p. 64] the space $\overset{\circ}{W}{}_2^1(D_T)$ is continuously and compactly embedded into $L_2(D_T)$, and the space $L_2(D_T)$, in turn, is continuously embedded into $L_p(D_T)$ for $p < 2$.

Let us rewrite equation (2.23) in the form

$$u = Au := L_0^{-1}(N_2 u + F), \tag{2.26}$$

where the operator $N_2 : \overset{\circ}{W}{}_{2,\Box}^1(D_T) \to L_2(D_T)$, for $0 < \alpha < \frac{n+1}{n-1}$, $\alpha \neq 1$, is continuous and compact in view of the Remark 2.6. Then taking into account (2.22) operator $A : \overset{\circ}{W}{}_{2,\Box}^1(D_T) \to \overset{\circ}{W}{}_{2,\Box}^1(D_T)$ in (2.26) is also continuous and compact. At the same time according to a priori estimate (2.14) of the Lemma 2.2, in which the constants c_1 and c_2 are given by equalities (2.18) and (2.19), for any parameter $\tau \in [0,1]$ and for any solution $u \in \overset{\circ}{W}{}_{2,\Box}^1(D_T)$ of equation $u = \tau Au$ with this parameter it is valid a priori estimation (2.14) with constants $c_1 > 0$ and $c_2 \geq 0$, not depending on u, τ and F. Therefore, according to the Lere-Schauder theorem [31, p. 375] equation (2.26), and consequently problem (2.2)-(1.2) has at least one weak generalized solution u in the space $\overset{\circ}{W}{}_{2,\Box}^1(D_T)$. This is summarized in the following result.

Theorem 2.7. *Let $0 < \alpha < \frac{n+1}{n-1}$, $\alpha \neq 1$, $\lambda \neq 0$ and $\lambda < 0$ when $\alpha > 1$. Then for any $F \in L_2(D_T)$ problem (2.2)-(1.2) has at least one weak generalized solution $u \in \overset{\circ}{W}{}_{2,\Box}^1(D_T)$.*

3. Uniqueness of solution for (1.1)-(1.2) when $f(u) = |u|^\alpha \operatorname{sgn} u$

Let $F \in L_2(D_T)$, and u_1, u_2 be two weak generalized solutions of (2.2)-(1.2) in the space $\overset{\circ}{W}{}_{2,\Box}^1(D_T)$. According to (2.3),

$$\int_{D_T} \Box u_i \Box \phi \, dx\, dt = \lambda \int_{D_T} \phi |u_i|^\alpha \operatorname{sgn} u_i \, dx\, dt + \int_{D_T} F\phi \, dx\, dt \quad \forall \phi \in \overset{\circ}{W}{}_{2,\Box}^1(D_T) \tag{3.1}$$

and $|u_i|^\alpha \in L_2(D_T)$, $i = 1,2$. For the difference $v = u_2 - u_1$ from (3.1) it follows that

$$\int_{D_T} \Box v \Box \phi \, dx\, dt = \lambda \int_{D_T} \phi(|u_2|^\alpha \operatorname{sgn} u_2 - |u_1|^\alpha \operatorname{sgn} u_1) \, dx\, dt \quad \forall \phi \in \overset{\circ}{W}{}_{2,\Box}^1(D_T). \tag{3.2}$$

Assuming $\phi = v \in \overset{\circ}{W}{}_{2,\Box}^1(D_T)$ in the above equality, we obtain

$$\int_{D_T} (\Box v)^2 \, dx\, dt = \lambda \int_{D_T} (|u_2|^\alpha \operatorname{sgn} u_2 - |u_1|^\alpha \operatorname{sgn} u_1)(u_2 - u_1) \, dx\, dt. \tag{3.3}$$

Let us note that for the finite values of u_1 and u_2 with $\alpha > 0$ it is valid the inequality

$$(|u_2|^\alpha \operatorname{sgn} u_2 - |u_1|^\alpha \operatorname{sgn} u_1)(u_2 - u_1) \geq 0. \tag{3.4}$$

From (3.3) and inequality (3.4), which is true for almost all points $(x,t) \in D_T$ with $u_i \in \overset{\circ}{W}{}_{2,\Box}^1(D_T)$, $i = 1,2$, in the case when $\alpha > 0$ and $\lambda < 0$ it follows that

$$\int_{D_T} (\Box v)^2 \, dx\, dt \leq 0,$$

whence, due to (2.4), we obtain $v = 0$; i.e. $u_1 = u_2$. This result is summarized in the next theorem.

Theorem 3.1. *Let $\alpha > 0$, $\alpha \neq 1$ and $\lambda < 0$. Then for any $F \in L_2(D_T)$, Problem* (2.2)-(1.2) *cannot have more than one generalized solution in $\mathring{W}^1_{2,\Box}(D_T)$.*

The following result follows from Theorems 2.7 and 3.1.

Theorem 3.2. *Let $0 < \alpha < \frac{n+1}{n-1}$, $\alpha \neq 1$ and $\lambda < 0$. Then for any $F \in L_2(D_T)$, Problem* (2.2)-(1.2) *has an unique weak generalized solution $u \in \mathring{W}^1_{2,\Box}(D_T)$.*

4. Non-solvability of (1.1)-(1.2) when $f(u) = |u|^\alpha$

Now assume that in (1.1), and therefore in (1.3), that $f(u) = |u|^\alpha$, $\alpha > 1$.

Theorem 4.1. *Let $F^0 \in L_2(D_T), \|F^0\|_{L_2(D_T)} \neq 0$, $F^0 \geq 0$, and $F = \mu F^0$, μ is a positive constant. Then when $f(u) = |u|^\alpha$ with $\alpha > 1$ and $\lambda > 0$, there exists a number $\mu_0 = \mu_0(F^0, \lambda, \alpha) > 0$ suh that for $\mu > \mu_0$, problem* (1.1)-(1.2) *can not have a weak generalized solution in the space $\mathring{W}^1_{2,\Box}(D_T)$.*

Proof. Let us assume that there is a solution $u \in \mathring{W}^1_{2,\Box}(D_T)$ of problem (1.1)-(1.2) exists for any fixed $\mu > 0$. Then (1.5) takes the form

$$\int_{D_T} \Box u \Box \phi \, dx\, dt = \lambda \int_{D_T} |u|^\alpha \phi \, dx\, dt + \mu \int_{D_T} F^0 \phi \, dx\, dt \quad \forall \phi \in \mathring{W}^1_{2,\Box}(D_T). \tag{4.1}$$

It is easy to verify that

$$\int_{D_T} \Box u \Box \phi \, dx\, dt = \int_{D_T} u \Box^2 \phi \, dx\, dt \quad \forall \phi \in \mathring{C}^4(\overline{D}_T, \partial D_T), \tag{4.2}$$

where $\mathring{C}^4(\overline{D}_T, \partial D_T) = \{u \in C^4(\overline{D}_T) : u|_{\partial D_T} = 0\} \subset \mathring{W}^1_{2,\Box}(D_T)$. Indeed, since $u \in \mathring{W}^1_{2,\Box}(D_T)$, and the space $\mathring{C}^2(\overline{D}_T, \partial D_T)$ is dense in $\mathring{W}^1_{2,\Box}(D_T)$, there exists such sequence $u_k \in \mathring{C}^2(\overline{D}_k, \partial D_k)$ that

$$\lim_{k\to\infty} \|u_k - u\|_{\mathring{W}^1_{2,\Box}(D_T)} = 0. \tag{4.3}$$

Taking into account that

$$\int_{D_T} \Box u_k \Box \phi \, dx\, dt = \int_{\partial D_T} \frac{\partial u_k}{\partial N} \Box \phi ds - \int_{\partial D_T} u_k \frac{\partial}{\partial N} \Box \phi ds + \int_{D_T} u_k \Box^2 \phi \, dx\, dt, \tag{4.4}$$

where the derivative on the conormal $\frac{\partial}{\partial N} = \gamma_{n+1} \frac{\partial}{\partial t} - \sum_{i=1}^n \gamma_i \frac{\partial}{\partial x_i}$ is an inner differential operator on characteristic manifold ∂D_T, and, therefore $\frac{\partial u_k}{\partial N}|_{\partial D_T} = 0$, since $u_k|_{\partial D_T} = 0$, then from (4.4) we obtain

$$\int_{D_T} \Box u_k \Box \phi \, dx\, dt = \int_{D_T} u_k \Box^2 \phi \, dx\, dt, \tag{4.5}$$

where $\gamma = (\gamma_1, \dots, \gamma_n, \gamma_{n+1})$ is the unit vector of outer normal relative to ∂D_T. Passing in (4.5) to the limit with $k \to \infty$, in view of (1.4) and (4.3), we obtain (4.2).

Taking into account (4.2) let us rewrite equality (4.1) in the form

$$\lambda \int_{D_T} |u|^\alpha \phi \, dx\, dt = \int_{D_T} u \Box^2 \phi \, dx\, dt - \mu \int_{D_T} F^0 \phi \, dx\, dt \quad \forall \phi \in \mathring{C}^4(\overline{D}_T, \partial D_T). \tag{4.6}$$

Below we use the method of test functions [22, p. 10-12]. Let us select such a test function $\phi \in \overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T)$, that $\phi|_{D_T} > 0$. If in Young's inequality with parameter $\varepsilon > 0$

$$ab \le \frac{\varepsilon}{\alpha} a^\alpha + \frac{1}{\alpha' \varepsilon^{\alpha'-1}} b^{\alpha'}, \quad a, b \ge 0,\ \alpha' = \frac{\alpha}{\alpha - 1}$$

we take $a = |u|\phi^{1/\alpha}$, $b = \frac{|\Box^2 \phi|}{\phi^{\frac{1}{\alpha}}}$, then due to the fact that $\frac{\alpha'}{\alpha} = \alpha' - 1$, we have

$$|u \Box^2 \phi| = |u|\phi^{\frac{1}{\alpha}} \frac{|\Box^2 \phi|}{\phi^{\frac{1}{\alpha}}} \le \frac{\varepsilon}{\alpha} |u|^\alpha \phi + \frac{1}{\alpha' \varepsilon^{\alpha'-1}} \frac{|\Box^2 \phi|^{\alpha'}}{\phi^{\alpha'-1}}. \tag{4.7}$$

By (4.7) and (4.6) we have the inequality

$$(\lambda - \frac{\varepsilon}{\alpha}) \int_{D_T} |u|^\alpha \phi \, dx \, dt \le \frac{1}{\alpha' \varepsilon^{\alpha'-1}} \int_{D_T} \frac{|\Box^2 \phi|^{\alpha'}}{\phi^{\alpha'-1}} \, dx \, dt - \mu \int_{D_T} F^0 \phi \, dx \, dt;$$

whence for $\varepsilon < \lambda\alpha$ we obtain

$$\int_{D_T} |u|^\alpha \phi \, dx \, dt \le \frac{\alpha}{(\lambda\alpha - \varepsilon)\alpha' \varepsilon^{\alpha'-1}} \int_{D_T} \frac{|\Box^2 \phi|^{\alpha'}}{\phi^{\alpha'-1}} \, dx \, dt - \frac{\alpha\mu}{\lambda\alpha - \varepsilon} \int_{D_T} F^0 \phi \, dx \, dt. \tag{4.8}$$

Taking into account the equalities $\alpha' = \frac{\alpha}{\alpha - 1}$, $\alpha = \frac{\alpha'}{\alpha' - 1}$, and

$$\min_{0 < \varepsilon < \lambda\alpha} \frac{\alpha}{(\lambda\alpha - \varepsilon)\alpha' \varepsilon^{\alpha'-1}} = \frac{1}{\lambda^{\alpha'}},$$

which is reached at $\varepsilon = \lambda$, it follows from (4.8) that

$$\int_{D_T} |u|^\alpha \phi \, dx \, dt \le \frac{1}{\lambda^{\alpha'}} \int_{D_T} \frac{|\Box^2 \phi|^{\alpha'}}{\phi^{\alpha'-1}} \, dx \, dt - \frac{\alpha' \mu}{\lambda} \int_{D_T} F^0 \phi \, dx \, dt.$$

Let us note that is not difficult to the existence of test function ϕ, such that

$$\phi \in \overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T), \quad \phi|_{D_T} > 0, \quad \kappa = \int_{D_T} \frac{|\Box^2 \phi|^{\alpha'}}{\phi^{\alpha'-1}} \, dx \, dt < +\infty \,. \tag{4.9}$$

Indeed, it is easy to verify that the function

$$\phi(x,t) = \left[(t^2 - |x|^2)((T-t)^2 - |x|^2)\right]^m$$

for sufficiently large positive m satisfies conditions (4.9).

According to the conditions in this theorem, $F^0 \in L_2(D_T)$, $\|F^0\|_{L_2(D_T)} \ne 0$, $F^0 \ge 0$, and $\operatorname{meas} D_T < +\infty$. Then due to the fact that $\phi|_{D_T} > 0$ we have

$$0 < \kappa_1 = \int_{D_T} F^0 \phi \, dx \, dt < +\infty. \tag{4.10}$$

Let us denote by $g(\mu)$ the right side of inequality (4), which is a linear function with respect to μ, then in view of (4.9) and (4.10) we have

$$g(\mu) < 0 \text{ for } \mu > \mu_0 \quad \text{and} \quad g(\mu) > 0 \text{ for } \mu < \mu_0, \tag{4.11}$$

where

$$g(\mu) = \frac{\kappa_0}{\lambda^{\alpha'}} - \frac{\alpha' \mu}{\lambda} \kappa_1, \quad \mu_0 = \frac{\lambda}{\alpha' \lambda^{\alpha'}} \frac{\kappa_0}{\kappa_1} > 0.$$

According to (4.11) with $\mu > \mu_0$ the right side of inequality (4) is negative, while the left side is non-negative. This contradiction completes the proof. $\Box$

References

[1] A. V. Bitsadze; *Some Classes of Partial Differential Equations*. Izdat. "Nauka", Moscow, 1981.

[2] M. M. Cavalcanti, V. N. D. Cavalcanti, I. Lasiecka; *Well posedness and optimal decay rates for the wave equation with nonlinear boundary damping - source interaction*. J. Differential Equations **236** (2007), 407-459.

[3] F. Cagnac; *Problème de Cauchy sur un conoïde caractéristique*. Ann. Mat. Pura Appl. (4) **104** (1975), 355-393.

[4] R. C. Charão and R. Ikehata; *Decay of solutions for a semilinear system of elastic waves in an exterior domain with damping near infinity*. Nonlinear Analysis **67** (2007), 398-429.

[5] J. W. Cholewa and T. Dlotco; *Strongly damped wave equation in uniform spaces*. Nonlinear Analysis **64** (2006), 174-187.

[6] R. Courant; *Partial Differential Equations*. Izdat. "Mir", Moscow, 1964.

[7] D. Gilbarg and N. Trudinger; *Elliptic Partial Differential Equations of Second Order*. Izdat. "Nauka", Moscow, 1989.

[8] V. Georgiev, H. Lindblad, and C.D. Sogge; *Weighted Strichartz estimates and global existence for semilinear wave equations*. Amer. J. Math. **119**(1997), no.6, 1291-1319.

[9] J. Ginibre, A. Soffer, and G. Velo; *The global Cauchy problem for the critical nonlinear wave equation*. J Funct. Anal. **110**(1992), no.1, 96-130.

[10] J. Hadamard; *Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques*. Hermann, Paris, 1932.

[11] D. Henry, Geometric Theory of Semilinear Parabolic Equations. Izdat. "Mir", Moscow, 1985.

[12] L. Hörmander; *Lectures on Nonlinear Hyperbolic Differential Equations*. Mathématics & Applications (Berlin), Vol.26, Springer-Verlag, Berlin, 1997.

[13] R. Ikehata and K. Tanizawa; *Global existence of solutions for semilinear damped wave equations in $\mathbb{R}^N$ with noncompactly supported initial data*. Nonlinear Analysis **61** (2005), 1189-1208.

[14] F. John; *Blow-up of solutions of nonlinear wave equations in three space dimensions*. Manuscripta Math. **28**(1979), no. 1-3, 235-268.

[15] F. John; *Blow-up for quasilinear wave equations in three space dimensions*. Comm. Pure Appl. Math. **34**(1981), no. 1, 29-51.

[16] F. John And S. Klainerman; *Almost global existence to nonlinear wave equations in three space dimensions*. Comm. Pure Appl. Math. **37** (1984), no. 4, 443-455.

[17] K. Jörgens; *Das Anfangswertproblem im Grossen für eine Klasse nichtlinearer Wellengleichungen*. Math.Z. **77**(1961), 295-308 (German).

[18] T. Kato; *Blow-up of solutions of some nonlinear hyperbolic equations*. Comm. Pure Appl. Math. **33** (1980), no. 4, 501-505.

[19] S. Kharibegashvili; *On the existence or the absence of global solutions of the Cauchy characteristic problem for some nonlinear hyperbolic equations*. J. Boundary Value Problems **2005**(2005), no.3, 359-376.

[20] S. Kharibegashvili; *On the nonexistence of global solutions of the characteristic Cauchy problem for a nonlinear wave equation in a conical domain*. Differential Equations, **42**(2006), No.2, 279-290. Translated from Differential'nye Uravneniya, **42**(2006), No.2, 261-271.

[21] S. Kharibegashvili; *Some multidimensional problems for hyperbolic partial differential equations and systems*. Mem. Differential Equations Math. Phys. **37**(2006), 1-136.

[22] M. A. Krasnosel'skiĭ, P. P. Zabreĭko, E. I. Pustyl'nik, and P. E. Sobolevskiĭ; *Integral Operators in Spaces of Summable Functions*. Izdat. "Nauka", Moscow, 1966.

[23] A. Kufner and S. Fuchik, Nonlinear Differential Equations. Izdat. "Nauka", Moscow, 1988.

[24] O. A. Ladyzhenskaya; *Boundary Value Problems of Mathematical Physics*. Izdat. "Nauka", Moscow, 1973.

[25] H. A. Levine; *Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathfrak{F}(u)$*. Trans. Amer. Math. Soc. **192** (1974), 1-21.

[26] L. Lundberg; *The Klein-Gordon equation with light-cone data*. Comm. Math. Phys. **62**(1978), no.2, 107-118.

[27] E. Mitidieri and S. I. Pohozaev; *A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities*. (Russian) Trudy Mat. Inst. Steklova, **234**(2001), 1-384; English transl.: Proc. Steklov Inst. Math. 2001, no. 3(**234**), 1-362.

[28] T. G. Sideris; *Nonexistence of global solutions to semilinear wave equations in high dimensions*. J Differential Equations **52** (1984), no.3, 378-406.

[29] W. A. Strauss; *Nonlinear scattering theory at low energy*. J Funct. Anal. **41** (1981), no. 1, 110-133.

[30] G. Todorova and E. Vitillaro; *Blow-up for nonlinear dissipative wave equations in* $\mathbb{R}^n$. J. Math. Anal. Appl. **303** (2005), no.1, 242-257.

[31] V. A. Trenogin; *Functional Analysis*. Izdat. "Nauka", Moscow, 1993.

[32] P.-F. Yao; *Global smooth solutions for the quasilinear wave equation with boundary dissipation*. J. Differential Equations **241** (2007), 62-93.

[33] B. Yordanov and Qi. S. Zhang; *Finite time blow up for critical wave equations in high dimensions*. J. Funct. Anal. **231** (2006), 361-374.

[34] J. Zhu; *Blow-up of solutions of a semilinear hyperbolic equation and a parabolic equation with general forcing term and boundary condition*. Nonlinear Analysis **67** (2007), 33-38.

Sergo Kharibegashvili
A. Razmadze Mathematical Institute, 1, M. Aleksidze St., Tbilisi 0193, Georgia
E-mail address: kharibegashvili@yahoo.com

Bidzina Midodashvili
A. Razmadze Mathematical Institute, 1, M. Aleksidze St., Tbilisi 0193, Georgia
E-mail address: bidmid@hotmail.com