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POWER SERIES SOLUTION FOR THE MODIFIED KDV EQUATION

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ABSTRACT. We use the method developed by Christ [3] to prove local well-posedness of a modified Korteweg de Vries equation in $\mathcal{F}L^{s,p}$ spaces.

1. INTRODUCTION

The modified Korteweg de Vries (mKdV) equation on a torus $\mathbb{T}$ has the form

$$\begin{gathered}\partial_t u + \partial_x^3 u + u^2 \partial_x u = 0\\ u(\cdot,0) = u_0\end{gathered} \tag{1.1}$$

where $(x,t) \in \mathbb{T}\times\mathbb{R}$, u is a real-valued function. If u is a smooth solution of (1.1), then $\|u(\cdot,t)\|_{L^2(\mathbb{T})} = \|u_0\|_{L^2(\mathbb{T})}$ for all t; therefore, $\widetilde{u}(x,t) = u(x + \frac{1}{2\pi}\|u_0\|^2_{L^2(\mathbb{T})}t, t)$ is a solution of

$$\begin{gathered}\partial_t u + \partial_x^3 u + \Big(u^2 - \frac{1}{2\pi}\int_{\mathbb{T}} u^2(x,t)dx\Big)\partial_x u = 0\\ u(\cdot,0) = u_0\end{gathered} \tag{1.2}$$

Thus, (1.2) and (1.1) are essentially equivalent. Using Fourier restriction norm method, Bourgain [1] proved that (1.2) is locally well-posed for initial data $u_0 \in H^s(\mathbb{T})$ when $s \geq 1/2$, and the solution map is uniformly continuous. In [2], he also showed that the solution map is not C^3 in $H^s(\mathbb{T})$ when $s < 1/2$. Takaoka and Tsutsumi [10] proved local-wellposedness of (1.2) when $1/2 > s > 3/8$, and they showed that solution map is not uniformly continuous for this range of s. For (1.1), Kappeler and Topalov [8] used inverse scattering method to show wellposedness when $s \geq 0$ and Christ, Colliander and Tao [4] showed that uniformly continuous dependence on the initial data does not hold when $s < 1/2$. Thus, there is a gap between known local well-posedness results and the space $H^{-1/2}(\mathbb{T})$ suggested by the standard scaling argument.

Recently, Grünrock and Vega [7] showed local well-posedness of the mKdV equation on $\mathbb{R}$ with initial data in

$$\widehat{H}^r_s(\mathbb{R}) := \{f \in \mathcal{D}'(\mathbb{R}) : \|f\|_{\widehat{H}^r_s} := \|\langle\cdot\rangle^s \hat{f}(\cdot)\|_{L^{r'}} < \infty\},$$

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when $2 \geq r > 1$ and $s \geq \frac{1}{2} - \frac{1}{2r}$. (for $r > \frac{4}{3}$, this was obtained by Grünrock [5]). This is an extension of the result of Kenig, Ponce and Vega [9] that local-wellposedness holds in $H^s(\mathbb{R})$ when $s \geq 1/4$. Furthermore, as $\widehat{H}^r_s$ scales like H^σ with $\sigma = s + \frac{1}{2} - \frac{1}{r}$, this result covers spaces that have scaling exponent $-\frac{1}{2}+$.

There is also a related recent work of Grünrock and Herr [6] on the derivative nonlinear Schrödinger equation on $\mathbb{T}$. Both [7] and [6] used a version of Bourgain's method.

In this paper, we apply the new method of solution developed by Christ [3] to investigate the local well-posedness of (1.2) with initial data in

$$\mathcal{F}L^{s,p}(\mathbb{T}) := \{f \in \mathcal{D}'(\mathbb{T}) : \|f\|_{\mathcal{F}L^{s,p}} := \|\langle\cdot\rangle^s \hat{f}(\cdot)\|_{l^p} < \infty\}.$$

Let $B(0,R)$ be the ball of radius R centered at 0 in $\mathcal{F}L^{s,p}(\mathbb{T})$. Our main result is the following.

Theorem 1.1. *Suppose $s \geq 1/2$, $1 \leq p < \infty$ and $p'(s+1/4) > 1$. Let W be the solution map for smooth initial data of (1.2). Then for any $R > 0$ there is $T > 0$ such that the solution map W extends to a uniformly continuous map from $B(0,R)$ to $C([0,T], \mathcal{F}L^{s,p}(\mathbb{T}))$.*

We note that the $\mathcal{F}L^{s,p}(\mathbb{T})$ spaces that are covered by Theorem 1.1 have scaling index $\frac{1}{4}+$. The restriction $s \geq 1/2$ is due to the presence of the derivative in the nonlinear term, and is only used to bound the operator S_2 in section 3. The same restriction on s is also required in the work on the derivative nonlinear Schrödinger equation on $\mathbb{T}$ by Grünrock and Herr [6]. We believe that the range of p in Theorem 1.1 is not sharp.

Concerning (1.1), we have the following result.

Corollary 1.2. *Suppose $s \geq 1/2$, $1 \leq p < \infty$ and $p'(s+1/4) > 1$. Let $\widetilde{W}$ be the solution map for smooth initial data of (1.1). Then for any $R > 0$ there is $T > 0$ such that for any $c > 0$, the solution map $\widetilde{W}$ extends to a uniformly continuous map from $B(0,R) \cap \{\varphi : \|\varphi\|_{L^2} = c\} \subset \mathcal{F}L^{s,p}(\mathbb{T})$ to $C([0,T], \mathcal{F}L^{s,p}(\mathbb{T}))$.*

As in [3], the solution map W obtained in Theorem 1.1 gives a weak solution of (1.2) in the following sense. Let T_N be defined by $T_N u = (\chi_{[-N,N]}\widehat{u})^\vee$. Let $\mathcal{N}u := \big(u^2 - \frac{1}{2\pi}\int_{\mathbb{T}} u^2(x,t)dx\big)\,\partial_x u$ be the limit in $C([0,T], \mathcal{D}'(\mathbb{T}))$ of $\mathcal{N}(T_N u)$ as $N \to \infty$, provided it exists.

Proposition 1.3. *Let s and p be as in Theorem 1.1. Let $\varphi \in \mathcal{F}L^{s,p}$ and $u := W\varphi \in C([0,T], \mathcal{F}L^{s,p})$. Then $\mathcal{N}u$ exists and u satisfies (1.2) in the sense of distribution in $(0,T) \times \mathbb{T}$.*

To prove these results, we formally expand the solution map into a sum of multilinear operators. These multilinear operators are described in the section 2. Then we will show that if $u(\cdot,0) \in \mathcal{F}L^{s,p}$ then the sum of these operators converges in $\mathcal{F}L^{s,p}$ for small time t, when s and p satisfy the conditions of Theorem 1.1. Furthermore, this gives a weak solution of (1.2), justifying our formal derivation.

2. Multilinear operators

We rewrite (1.2) as a system of ordinary differential equations of the spatial Fourier series of u (see [10, formula (1.9)], and [1, Lemma 8.16]).

$$\begin{aligned}
&\frac{d\hat{u}(n,t)}{dt} - in^3\hat{u}(n,t)\\
&= -i\sum_{n_1+n_2+n_3=n}\hat{u}(n_1,t)\hat{u}(n_2,t)n_3\hat{u}(n_3,t) + i\sum_{n_1}\hat{u}(n_1,t)\hat{u}(-n_1,t)n\hat{u}(n,t)\\
&= \frac{-in}{3}\sum^{*}_{n_1+n_2+n_3=n}\hat{u}(n_1,t)\hat{u}(n_2,t)\hat{u}(n_3,t) + in\hat{u}(n,t)\hat{u}(-n,t)\hat{u}(n,t),
\end{aligned}\tag{2.1}$$

where the star means the sum is taken over the triples satisfying $n_j \neq n$, $j = 1,2,3$. We note that these are precisely the triples with $\sigma(n_1,n_2,n_3) \neq 0$.

Let $a(n,t) = e^{-in^3t}\hat{u}(n,t)$, then $a_n(t)$ satisfy

$$\begin{aligned}
\frac{da(n,t)}{dt} = -\frac{in}{3}&\sum^{*}_{n_1+n_2+n_3=n} e^{i\sigma(n_1,n_2,n_3)t}a(n_1,t)a(n_2,t)a(n_3,t)\\
&+ ina(n,t)a(-n,t)a(n,t),
\end{aligned}$$

where

$$\sigma(n_1,n_2,n_3) = n_1^3+n_2^3+n_3^3-(n_1+n_2+n_3)^3 = -3(n_1+n_2)(n_2+n_3)(n_3+n_1).$$

Or, in integral form,

$$\begin{aligned}
a(n,t) = a(n,0) - \frac{in}{3}\int_0^t &\sum^{*}_{n_1+n_2+n_3=n} e^{i\sigma(n_1,n_2,n_3)s}a(n_1,s)a(n_2,s)a(n_3,s)ds\\
&+ in\int_0^t |a(n,s)|^2a(n,s)ds.
\end{aligned}\tag{2.2}$$

If, a is sufficiently nice, say $a \in C([0,T],l^1)$ (which is the case if $u \in C([0,T],H^s(\mathbb{T}))$ for large s) then we can exchange the order of the integration and summation to obtain

$$\begin{aligned}
a(n,t) = a(n,0) - \frac{in}{3}&\sum^{*}_{n_1+n_2+n_3=n}\int_0^t e^{i\sigma(n_1,n_2,n_3)s}a(n_1,s)a(n_2,s)a(n_3,s)ds\\
&+ in\int_0^t |a(n,s)|^2a(n,s)ds.
\end{aligned}\tag{2.3}$$

Replacing the $a(n_j,s)$ in the right hand side by their equations obtained using (2.3), we get

$$\begin{aligned}
a(n,t) &= a(n,0) - \frac{in}{3}\sum^{*}_{n_1+n_2+n_3=n} a(n_1,0)a(n_2,0)a(n_3,0)\int_0^t e^{i\sigma(n_1,n_2,n_3)s}ds\\
&\quad + in|a(n,0)|^2a(n,0)\int_0^t ds + \text{ additional terms}\\
&= a(n,0) - \frac{n}{3}\sum^{*}_{n_1+n_2+n_3=n}\frac{a(n_1,0)a(n_2,0)a(n_3,0)}{\sigma(n_1,n_2,n_3)}(e^{i\sigma(n_1,n_2,n_3)t}-1)\\
&\quad + int|a(n,0)|^2a(n,0) + \text{ additional terms}
\end{aligned}\tag{2.4}$$

The additional terms are those which depend not only on $a(\cdot,0)$. An example of the additional terms is

$$-\frac{nn_3}{9}\sum^{*}_{n_1+n_2+n_3=n} a(n_1,0)a(n_2,0)\sum^{*}_{m_1+m_2+m_3=n_3}\int_0^t e^{i\sigma(n_1,n_2,n_3)s}\int_0^s e^{i\sigma(m_1,m_2,m_3)s'} \times a(m_1,s')a(m_2,s')a(m_3,s')ds'ds$$

Then we can again use (2.3) for each appearance of $a(m,\cdot)$ in the additional terms, and obtain more and more complicated additional terms. We refer to section 2 of [3] for more detailed description of these additional terms. Continuing this process indefinitely, we get a formal expansion of $a(n,t)$ as a sum of multilinear operators of $a(\cdot,0)$.

We will now describe these operators and then show that their sum converges. Again, we refer to section 3 of [3] for more detailed explanations. Each of our multilinear operators will be associated to a tree, which has the property that each of its node has either zero or three children. We will only consider trees with this property. If a node v of T has three children, they will be denoted by v_1, v_2, v_3. We denote by T^0 the set of non-terminal nodes of T, and T^∞ the set of terminal nodes of T. Clearly, if $|T| = 3k+1$ then $|T^0| = k$ and $|T^\infty| = 2k+1$.

Definition 2.1. Let T be a tree. Then $\mathcal{J}(T)$ is the set of $j \in \mathbb{Z}^T$ such that if $v \in T^0$ then

$$j_v = j_{v_1} + j_{v_2} + j_{v_3},$$

and either $j_{v_i} \neq j_v$ for all i, or $j_{v_1} = -j_{v_2} = j_{v_3} = j_v$. We will denote by $v(T)$ be the root of T and $j(T) = j(v(T))$. For $j \in \mathcal{J}(T)$ and $v \in T^0$,

$$\sigma(j,v) := \sigma(j(v_1), j(v_2), j(v_3)).$$

Also define

$$\mathcal{R}(T,t) = \{s \in \mathbb{R}_+^{T^0} : \text{ if } v < w \text{ then } 0 \le s_v \le s_w \le t\}.$$

Using the above definitions, we can rewrite (2.4) as

$$\begin{aligned} a(n,t) = a(n,0) + \sum_{|T|=4}\omega_T \sum_{j\in\mathcal{J}(T), j(T)=n} & na(j(v(T)_1),0)a(j(v(T)_2),0) \\ & \times a(j(v(T)_3),0)\int_{\mathcal{R}(T,t)} c(j,v(T),s)ds + \text{additional terms}, \end{aligned}$$

here $c(j,v,s) = e^{i\sigma(j,v)s}$, and ω_T is a constant with $|\omega_T| \le 1$.

Continuing this replacement process, it leads to

$$\begin{aligned} a(n,t) = a(n,0) + \sum_{|T|<3k+1}\omega_T \sum_{j\in\mathcal{J}(T), j(T)=n}\prod_{u\in T^0} j_u \prod_{v\in T^\infty} a(j_v,0)\int_{\mathcal{R}(T,t)} c(j,s)ds \\ + \text{additional terms} \end{aligned}$$

where

$$c(j,s) = \prod_{v\in T^0} c(j,v,s_v)$$

We will show that the series

$$a(n,0) + \sum_T \omega_T \sum_{j\in\mathcal{J}(T), j(T)=n}\prod_{u\in T^0} j_u \prod_{v\in T^\infty} a(j_v,0)\int_{\mathcal{R}(T,t)} c(j,s)ds$$

converges in $C([0,T], l^p)$ when $a(\cdot,0) \in l^p$.

3. l^p convergence

Let T be a tree and $j \in \mathcal{J}(T)$. We define

$$I_T(t,j) = \int_{\mathcal{R}(T,t)} c(j,s)ds,$$

and

$$S_T(t)(a_v)_{v\in T^\infty}(n) = \omega_T \sum_{j\in\mathcal{J}(T),j(T)=n} \prod_{u\in T^0} j_u \prod_{v\in T^\infty} a_v(j_v) I_T(t,j).$$

We first give an estimate for $I_T(t,j)$ which allows us to bound S_T.

Lemma 3.1. *For $0 \le t \le 1$,*

$$|I_T(j,t)| \le (Ct)^{|T^0|/2} \prod_{v\in T^0} \langle \sigma(j,v)\rangle^{-1/2}.$$

Proof. For $v \in T^0$, define the level of v, denoted $l(v)$, to be the length of the unique path connecting $v(T)$ and v. Let O be the set of $v \in T^0$ for which $l(v)$ is odd, and E those v for which $l(v)$ is even.

First we fix the variables s_v with $v \in E$, and take the integration in the variables s_v with $v \in O$. For each $v \in O$, the result of the integration is

$$\frac{1}{\sigma(j,v)} \left(e^{i\sigma(j,v)s_{\tilde{v}}} - e^{i\sigma(j,v)\max\{s_{v(1)},s_{v(2)},s_{v(3)}\}} \right)$$

if $\sigma(j,v) \neq 0$, and

$$s_{\tilde{v}} - \max\{s_{v(1)}, s_{v(2)}, s_{v(3)}\}.$$

if $\sigma(j,v) = 0$. Here $\widetilde{v}$ is the parent of v. Thus, we obtain the factor

$$\prod_{v\in O} \langle \sigma(j,v\rangle^{-1}$$

and an integral in s_v, $v \in E$ where the integrand is bounded by $2^{|O|}$. As the domain of integration in s_v with $v \in E$ has measure less than $t^{|E|}$, we see that

$$|I_T(j,t)| \le 2^{|T^0|} t^{|E|} \prod_{v\in O} \langle \sigma(j,v)\rangle^{-1}.$$

By switching the role of O and E, we get

$$|I_T(j,t)| \le 2^{|T^0|} t^{|O|} \prod_{v\in E} \langle \sigma(j,v)\rangle^{-1}.$$

Combining these two estimates, we obtain the lemma. □

By Lemma 3.1,

$$|S_T(t)(a_v)_{v\in T^\infty}(n)| \le (Ct)^{|T^0|/2} \sum_{j\in\mathcal{J}(T):j(T)=n} \prod_{u\in T^0} \langle \sigma(j,u)\rangle^{-1/2} |j_u| \prod_{v\in T^\infty} |a_v(j_v)|.$$

Let

$$\widetilde{S}_T(a_v)_{v\in T^\infty}(n) = \sum_{j\in\mathcal{J}(T):j(T)=n} \prod_{u\in T^0} \langle \sigma(j,u)\rangle^{-1/2} |j_u| \prod_{v\in T^\infty} |a_v(j_v)|,$$

and

$$\widetilde{S}(a_1,a_2,a_3)(n) = \sum_{n_1+n_2+n_3=n}^{*} |n| \langle \sigma(n_1,n_2,n_3)\rangle^{-1/2} \prod_{i=1}^{3} |a_i(n_i)| + |n| \prod_{i=1}^{3} |a_i(n)|.$$

It is clear that
$$\widetilde{S}_T(a_v)_{v\in T^\infty} = \widetilde{S}(\widetilde{S}_{T_1}(a_v)_{v\in T_1^\infty}, \widetilde{S}_{T_2}(a_v)_{v\in T_2^\infty}, \widetilde{S}_{T_3}(a_v)_{v\in T_3^\infty}).$$
where T_i is the subtree of T that contains all nodes u such that $u \le v(T)_i$ (recall that $v(T)$ is the root of T). Hence, to bound S_T, it suffices to bound $\widetilde{S}$. For this purpose, we will use the following simple lemma.

Lemma 3.2. *Let S be the multilinear operator defined by*
$$S(a_1,a_2,a_3)(n) = \sum_{n_1+n_2+n_3=n} m(n_1,n_2,n_3)\prod_{j=1}^3 a_j(n_j),$$
Let $1 \le p \le \infty$. Then for any pair of indices $i \ne j \in \{1,2,3\}$,
$$\|S(a_1,a_2,a_3)\|_{l^p} \le \sup_n \|m(n_1,n_2,n_3)\|_{l^{p'}_{i,j}} \prod_{k=1}^3 \|a_k\|_{l^p}.$$

Proof. By Hölder's inequality, for any n,
$$\begin{aligned}|S(a_1,a_2,a_3)(n)| &\le \|m(n_1,n_2,n_3)\|_{l^{p'}_{i,j}} \|\prod_{k=1}^3 a_k\|_{l^p_{i,j}}\\ &\le \sup_n \|m(n_1,n_2,n_3)\|_{l^{p'}_{i,j}} \|\prod_{k=1}^3 a_k\|_{l^p_{i,j}}\end{aligned}$$
Taking l^p-norm in n we obtain the lemma. □

Showing that $\widetilde{S}$ is a bounded multilinear map on $l^{s,p} := \{a : \langle\cdot\rangle^s a \in l^p\}$ is equivalent to showing that S is bounded on l^p where S is the operator with kernel
$$m(n_1,n_2,n_3) = \frac{\langle n\rangle^s |n|}{\langle\sigma(n_1,n_2,n_3)\rangle^{1/2}\prod_{k=1}^3\langle n_k\rangle^s}$$
where $n_1+n_2+n_3=n$. We split S into sum of two operators S_1 and S_2 where S_1 has kernel
$$m_1(n_1,n_2,n_3) = \frac{\langle n\rangle^s|n|}{\prod_{k=1}^3\langle n_k\rangle^s\langle n-n_k\rangle^{1/2}} \quad \text{if } n=n_1+n_2+n_3,\ n_i\ne n$$
and S_2 has kernel
$$m_2(n_1,n_2,n_3) = n/\langle n\rangle^{2s} \quad \text{if } n_1=-n_2=n_3=n.$$
Clearly, S_2 is bounded on l^p if and only if $s \ge 1/2$.

It remains to bound S_1, for which we have the following result.

Proposition 3.3. *S_1 is bounded in $l^p\times l^p\times l^p$ to l^p when $s\ge 1/4$ and $p'(s+\frac14)>1$.*

Proof. In the proof, all the sums are taken over the triples (n_1,n_2,n_3) that satisfy the additional property that $n_i\ne n$, for all $1\le i\le 3$. Clearly, we can assume $n>0$. Note that if say $|n_1|\ge 5n$ then as $|n_2+n_3| = |n-n_1|\ge 4n$, at least one of n_2 and n_3 has absolute value bigger than $2n$. Also, we cannot have $|n_i|\le n/4$ for all i. Thus, up to permutation, there are four cases.

(1) $|n_1|,|n_2|,|n_3|\in[n/4,5n]$
(2) $|n_1|,|n_2|\in[n/4,5n]$, $|n_3|\le n/4$
(3) $|n_1|\in[n/4,5n]$, $|n_2|,|n_3|\le n/4$
(4) $|n_1|,|n_2|\ge 2n$

By Lemma 3.2, it suffices to show that in each of these four regions, for some $i \neq j$ the $l^{p'}_{i,j}$-norm of m is bounded.

Case 1. As $3n = \sum(n-n_i)$ for some index i, say $i=3$, we must have $|n-n_3| \sim n$. Since we also have $|n_1|, |n_2| \gtrsim n$,

$$|m(n_1,n_2,n_3)| \lesssim \frac{\langle n\rangle^{1/2-s}}{\langle n_3\rangle^s |(n-n_1)(n-n_2)|^{1/2}}.$$

We will use the inequality

$$|\frac{1}{n_3(n-n_2)}| = |\frac{1}{n_1}\Big(\frac{1}{n_3} - \frac{1}{n-n_2}\Big)| \leq \frac{1}{|n_1|}\Big(\frac{1}{|n_3|} + \frac{1}{|n-n_2|}\Big).$$

(1) If $1/4 \leq s \leq 1/2$: then $\langle n_3\rangle^{p'(1/2-s)} \lesssim \langle n\rangle^{p'(1/2-s)}$, so

$$\begin{aligned}
\|m\|^{p'}_{l^{p'}_{1,2}} &\lesssim \sum_{|n_1|\leq 5n} \frac{\langle n\rangle^{p'(1/2-s)}}{|n-n_1|^{p'/2}} \sum_{|n_2|\leq 5n} \frac{\langle n_3\rangle^{p'(1/2-s)}}{(\langle n_3\rangle |n-n_2|)^{p'/2}}\\
&\lesssim \sum_{|n_1|\leq 5n} \frac{\langle n\rangle^{p'(1/2-s)}}{|n-n_1|^{p'/2}} \sum_{|n_2|\leq 5n} \frac{\langle n\rangle^{p'(1/2-s)}}{|n_1|^{p'/2}}\Big(\frac{1}{|n-n_2|^{p'/2}} + \frac{1}{|n-n_1-n_2|^{p'/2}}\Big)\\
&\lesssim \sum_{|n_1|\leq 5n} \frac{\langle n\rangle^{p'(1-2s)} A_n}{|(n-n_1)n_1|^{p'/2}}\\
&\lesssim \langle n\rangle^{p'(1-2s)} A_n \sum_{|n_1|\leq 5n} \Big(\frac{1}{n}(\frac{1}{|n-n_1|} + \frac{1}{|n_1|})\Big)^{p'/2}\\
&\lesssim \langle n\rangle^{p'(1/2-2s)} A_n^2.
\end{aligned}$$

where $\sum_{0<j<5n} j^{-p'/2} = A_n$. As

$$A_n \lesssim \begin{cases} n^{1-p'/2} & \text{if } p' < 2\\ \log\langle n\rangle & \text{if } p' = 2\\ 1 & \text{if } p' > 2 \end{cases}$$

we easily check that $\langle n\rangle^{(1/2-2s)p'} A_n^2$ is bounded by a constant, under our hypothesis on s and p'.

(2) If $s > 1/2$: then $\langle n-n_2\rangle^{p'(s-1/2)} \lesssim \langle n\rangle^{p'(s-1/2)}$, so

$$\begin{aligned}
\|m\|^{p'}_{l^{p'}_{1,2}} &\lesssim \sum_{|n_1|\leq 5n} \frac{\langle n\rangle^{p'(1/2-s)}}{|n-n_1|^{p'/2}} \sum_{|n_2|\leq 5n} \frac{\langle n-n_2\rangle^{p'(s-1/2)}}{(\langle n_3\rangle |n-n_2|)^{p's}}\\
&\lesssim \sum_{|n_1|\leq 5n} \frac{\langle n\rangle^{p'(1/2-s)}}{|n-n_1|^{p'/2}} \sum_{|n_2|\leq 5n} \frac{\langle n\rangle^{p'(s-1/2)}}{|n_1|^{p's}}\Big(\frac{1}{|n-n_2|^{p's}} + \frac{1}{|n-n_1-n_2|^{p's}}\Big)\\
&\lesssim \sum_{|n_1|\leq 5n} \frac{B_n}{|n-n_1|^{p'/2}|n_1|^{p's}}\\
&\lesssim B_n \sum_{|n_1|\leq 5n} |n-n_1|^{p'(s-1/2)} \Big(\frac{1}{n}(\frac{1}{|n-n_1|} + \frac{1}{|n_1|})\Big)^{p's}\\
&\lesssim \langle n\rangle^{-p'/2} B_n^2.
\end{aligned}$$

where $B_n = \sum_{0<j<5n} j^{-p's}$. As

$$B_n \lesssim \begin{cases} n^{1-p's} & \text{if } p's < 1 \\ \log\langle n\rangle & \text{if } p's = 1 \\ 1 & \text{if } p's > 1 \end{cases}$$

we easily check that $\langle n\rangle^{-p'/2} B_n^2$ is bounded by a constant, under our hypothesis on s and p'.

Case 2 This case can be treated in exactly the same way as the first case, except when $n_3 = 0$. In the region $n_3 = 0$,

$$\begin{aligned} \|m\|_{l^{p'}_{1,3}}^{p'} &\lesssim \sum_{n_1} \frac{\langle n\rangle^{p'(1/2-s)}}{|n_1(n-n_1)|^{p'/2}} \le \sum_{n_1} \langle n\rangle^{-p's}\Big(\frac{1}{|n_1|^{p'/2}} + \frac{1}{|n-n_1|^{p'/2}}\Big) \\ &\lesssim \langle n\rangle^{-p's} A_n \lesssim 1 \end{aligned}$$

Case 3 As $|n_1|, |n-n_2|, |n-n_3| \sim n$,

$$|m(n_1,n_2,n_3)| \lesssim \frac{1}{\langle n_2\rangle^s \langle n_3\rangle^s |n_2+n_3|^{1/2}}.$$

Without loss of generality, we assume that $|n_3| \ge |n_2|$.

(1) If $|n_2| < |n_3|/2$:

$$\begin{aligned} \|m\|_{l^{p'}_{2,3}}^{p'} &\lesssim \sum_{0\le|n_2|\le n/4} \frac{1}{\langle n_2\rangle^{p's}} \sum_{n/4\ge|n_3|>2n_2} \frac{1}{\langle n_3\rangle^{p'(s+1/2)}} \\ &\lesssim \sum_{0\le|n_2|\le n/4} \frac{1}{\langle n_2\rangle^{p'(2s+1/2)-1}} \lesssim 1 \end{aligned}$$

if $(s+1/4)p' > 1$.

(2) If $|n_2| \ge |n_3|/2$:

$$\begin{aligned} \|m\|_{l^{p'}_{2,3}}^{p'} &\lesssim \sum_{|n_3|\le n/4} \frac{1}{\langle n_3\rangle^{2p's}} \sum_{|n_3|\ge n_2\ge|n_3|/2} \frac{1}{\langle n_3+n_2\rangle^{p'/2}} \\ &\lesssim \sum_{|n_3|\le n/4} \frac{1}{\langle n_3\rangle^{2p's}} \max\{\log\langle n_3\rangle, \langle n_3\rangle^{-p'/2+1}\} \\ &\lesssim \sum_{|n_3|\le n/4} \frac{\log\langle n_3\rangle}{\langle n_3\rangle^{2p's}} + \sum_{|n_3|\le n/4} \frac{1}{\langle n_3\rangle^{p'(2s+1/2)-1}} \lesssim 1 \end{aligned}$$

as $2p's \ge p'(s+1/4) > 1$.

Case 4 $|n_1|, |n_2| > 2n$: Note that in this case, $|n_1| \sim |n-n_1|$ and $|n_2| \sim |n-n_3|$.

(1) If $|n_3|, |n-n_3| \ge n/2$: we have

$$|m(n_1,n_2,n_3)| \lesssim \frac{\langle n\rangle^{1/2}}{\langle n_1\rangle^{s+1/2}\langle n_2\rangle^{s+1/2}},$$

hence

$$\begin{aligned} \|m\|_{l^{p'}_{1,2}}^{p'} &\lesssim \langle n\rangle^{p'/2} \sum_{|n_1|,|n_2|>2n} \frac{1}{\langle n_1\rangle^{p'(s+1/2)}\langle n_2\rangle^{p'(s+1/2)}} \\ &\lesssim \frac{\langle n\rangle^{p'/2}}{\langle 2n\rangle^{p'(2s+1)-2}} \lesssim 1. \end{aligned}$$

(2) If $|n_3| < n/2$: then $|n_1| \sim |n_2|$ and $|n - n_3| \geq n/2$, so

$$|m(n_1, n_2, n_3)| \lesssim \frac{n^{s+1/2}}{\langle n_1 \rangle^{2s+1} \langle n_3 \rangle^s},$$

hence

$$\|m\|_{l^{p'}_{1,3}}^{p'} \lesssim B_n \sum_{|n_1|>2n} \frac{n^{p'(s+1/2)}}{\langle n_1 \rangle^{p'(2s+1)}} \lesssim \frac{B_n}{n^{p'(s+1/2)-1}} \lesssim 1$$

(3) If $|n - n_3| < n/2$: then $|n_1| \sim |n_2|$ and $|n_3| \sim n$. Hence,

$$|m(n_1, n_2, n_3)| \lesssim \frac{n}{\langle n_1 \rangle^{2s+1} \langle n - n_3 \rangle^{1/2}}.$$

Therefore,

$$\begin{aligned}
\|m\|_{l^{p'}_{1,3}}^{p'} &\lesssim \sum_{|n_1|\geq 2n} \sum_{n/2<n_3<3n/2} \frac{n^{p'}}{\langle n_1 \rangle^{p'(2s+1)} \langle n - n_3 \rangle^{p'/2}} \\
&\lesssim \sum_{|n_1|\geq 2n} \frac{A_n n^{p'}}{\langle n_1 \rangle^{p'(2s+1)}} \lesssim \frac{A_n}{n^{2p's-1}} \lesssim 1
\end{aligned}$$

This concludes the proof of the proposition. □

Proof of Theorem 1.1. Let $u_0 \in \mathcal{F}L^{s,p}$ and $a(n) = \widehat{u_0}(n)$. By Proposition 3.3,

$$\|S_T((a_v)_{v\in T^\infty})\|_{l^{s,p}} \leq C^{|T^0|} t^{|T^0|/2} \prod_{v\in T^\infty} \|a_v\|_{l^{s,p}}.$$

Hence,

$$\begin{aligned}
&\|a(n) + \sum_T \omega_T \sum_{j\in\mathcal{J}(T), j(T)=n} \prod_{u\in T^0} j_u \prod_{v\in T^\infty} a(j_v) \int_{\mathcal{R}(T,t)} c(j,s)ds\|_{l^{s,p}} \\
&\leq \|a\|_{l^{s,p}} + \sum_T \|S_T(a, \dots, a)\|_{l^{s,p}} \\
&\leq \sum_{k=0}^{\infty} (Ct)^{k/2} \|a\|_{l^{s,p}}^{2k+1} = \frac{\|u_0\|_{\mathcal{F}L^{s,p}}}{1 - \sqrt{Ct}\|u_0\|_{\mathcal{F}L^{s,p}}^2}
\end{aligned} \tag{3.1}$$

for all $t \lesssim \min\{1, \|u_0\|_{\mathcal{F}L^{s,p}}^{-4}\}$.

Let $T \sim \min\{1, \|u_0\|_{\mathcal{F}L^{s,p}}^{-4}\}$, then for $t \in [0, T]$ we can define

$$a(n,t) = a(n) + \sum_T \omega_T \sum_{j\in\mathcal{J}(T), j(T)=n} \prod_{u\in T^0} j_u \prod_{v\in T^\infty} a(j_v) \int_{\mathcal{R}(T,t)} c(j,s)ds$$

and the solution map $u = Wu_0$ by

$$\widehat{u}(n,t) = e^{-in^3 t} a(n,t).$$

It follows from (3.1) that W is uniformly continuous. The same argument as that of [3] shows that u is limit of classical solutions. □

The proof of Proposition 1.2 is basically the same as that of [3, Proposition 1.4], hence we omit it.

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