

TERMINAL VALUE PROBLEMS FOR FIRST AND SECOND ORDER NONLINEAR EQUATIONS ON TIME SCALES

ROMAN HILSCHER, CHRISTOPHER C. TISDELL

ABSTRACT. In this paper we examine “terminal” value problems for dynamic equations on time scales – that is, a dynamic equation whose solutions are asymptotic at infinity. We present a number of new theorems that guarantee the existence and uniqueness of solutions, as well as some comparison-type results. The methods we employ feature dynamic inequalities, weighted norms, and fixed-point theory.

1. INTRODUCTION

The theory of “terminal” value problems, where the problem consists of a dynamic equation coupled with asymptotic behavior of the solution at ∞ , forms an interesting and more challenging field of research than the theory of initial value problems. This is due to even the basic results and methods known for initial value problems, such as the perturbation technique, are unavailable for use in the setting of terminal value problems. For example, the existence of a solution to the terminal value problem $x' = f(t, x)$, $x(\infty) = x_0$, need not imply that the terminal value problem $x' = f(t, x) \pm \frac{1}{n}$, $x(\infty) = x_0 \pm \frac{1}{n}$ has a solution, see [1, pg. 1173].

In this work, we examine terminal value problems for “dynamic equations on time scales”, which is a new and versatile area of mathematics that is more general than the fields of differential equations and difference equations. The area of time scales originates in the work of Hilger in [23]. Such investigations reveal the bonds and distinctions between the two areas and also provide a framework with which to more accurately model stop-start processes.

Our main interest herein is in the qualitative properties of solutions to terminal value problems on time scales, including the existence, uniqueness, and comparison theorems. The methods that we employ involve dynamic inequalities, weighted norms, and fixed point theory. The motivation for using the weighted (or Bielecki) norms originates in [31] and the references quoted therein, where this method was used in order to prove the existence and uniqueness results for nonlinear initial value problems on bounded time scales. The existence of bounded solutions to *initial* value problems for second order dynamic equations and inequalities on unbounded time scales was studied in [3], while in [2] results of this type are given for certain

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first order dynamic equations. In the latter three references, as well as in the present paper, the fixed point theory is utilized.

Our results extend some of the ideas in [1] and, more recently, those of [20]. More specifically, we provide some extensions of the comparison results in [1], which were formulated for terminal value problems involving ordinary differential equations, to the time scale environment. Furthermore, compared to [20] we allow in Section 4 the nonlinearity $f(t, x^\sigma)$ or $f(t, x^\sigma, x^{\Delta\sigma})$ to be vector valued and we pose no restriction on the sign of its entries. Also, we assume in those results that the leading coefficient $r(t)$ is merely nonzero as opposed to the assumption of its positivity in [20]. In addition, for the case of positive $r(t)$ and nonnegative nonlinearity f we extend in Section 5 the ideas in [20] from the scalar case to the matrix/vector case. Some of the main results (e.g., Theorems 3.5, 3.6, 4.2, 4.5, 5.1 and 5.2) appear to be new even for the special case $\mathbb{T} = \mathbb{Z}$, that is, for difference equations.

For additional papers that contain comparison and existence and uniqueness results for first-order terminal value problems involving ordinary differential equations, we refer the reader to [21, 22, 27, 32]. For papers dealing with second-order terminal value problems, the reader is referred to [22, 29, 30]. The methods used in the range of the aforementioned papers involve differential inequalities and the fixed-point theorems of Banach or Schauder.

The setup of the paper is the following. In Section 2 we introduce necessary notation and terminology as well as some preparatory results about the time scale exponential function. In Section 3 we derive an existence and uniqueness theorem for the terminal value problem of the first order. Then we continue in deriving comparison results for solutions of first order dynamic inequalities. In Section 4 we consider terminal value problems for second order dynamic equations with scalar leading coefficient, while in Section 5 we deal with such equations with matrix leading coefficient and with nonnegative nonlinearity. In Section 6 we present examples illustrating the applicability of the obtained results. Finally, in Section 7 we discuss further applications and extensions, in particular to nabla dynamic terminal value problems.

2. PREREQUISITES AND NOTATION

Let $n \in \mathbb{N}$ be a fixed natural number. For a real symmetric $n \times n$ matrix A we write $A > 0$ or $A \geq 0$ for A being a positive definite or positive semidefinite matrix, respectively. Moreover, if B is a real symmetric $n \times n$ matrix, then we write $A < B$ or $A \leq B$ if $B - A > 0$ or $B - A \geq 0$, respectively.

In this paper we will use the vector norm $|\cdot|_\infty$ on \mathbb{R}^n denoted for simplicity by $|x| := |x|_\infty = \max\{|x_i|, i = 1, \dots, n\}$. Given a number $0 < q \leq \infty$, we use the notation $\Omega_q := \{x \in \mathbb{R}^n, |x| < q\}$ for the open q -ball in \mathbb{R}^n . Then we can identify Ω_∞ with \mathbb{R}^n .

Let \mathbb{T} be a time scale, i.e., a nonempty closed subset of \mathbb{R} , which is bounded below and unbounded above. Then $a := \min \mathbb{T}$ exists and we may identify \mathbb{T} with the time scale interval $[a, \infty)_\mathbb{T}$. We shall use the common time scale notation and terminology e.g. from the books [10, 11]. In particular, the forward and backward jump operators are denoted by σ and ρ , and the graininess is $\mu(t) := \sigma(t) - t$. As it is common, the sets of all continuous, rd-continuous, or rd-continuously Δ -differentiable functions (on a given interval) will be denoted by C , C_{rd} , and C_{rd}^1 , respectively. The sup norm in the space of bounded n -vector functions $x \in C$ on

$[a, \infty)_{\mathbb{T}}$ will be denoted by $\|x\|_0 := \sup_{t \in [a, \infty)_{\mathbb{T}}} |x(t)|$. The sup norm in the space of bounded n -vector functions $x \in C_{\text{rd}}^1$ on $[a, \infty)_{\mathbb{T}}$ such that x^Δ is also bounded will be denoted by $\|x\|_1 := \max\{\|x\|_0, \|x^\Delta\|_0\}$. The improper integrals used in this paper are defined in the traditional way as $\int_a^\infty f(t) \Delta t := \lim_{b \rightarrow \infty} \int_a^b f(t) \Delta t$, see e.g. [8, Section 5.6] and [9, Section 4]. In addition, motivated by [26, Definition 3], we adopt the following terminology.

Definition 2.1. Let $0 < q \leq \infty$ and $f : [a, \infty)_{\mathbb{T}} \times \Omega_q \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a function. We write $f \in C_{\text{rd}} \times C \times C$ and say that f is $C_{\text{rd}} \times C \times C$ -continuous on its domain if for any $(t_0, x_0, v_0) \in [a, \infty)_{\mathbb{T}} \times \Omega_q \times \mathbb{R}^n$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |(t - t_0, x - x_0, v - v_0)| < \delta$ implies

$$|F(t, x, v) - F(t_0, x_0, v_0)| < \varepsilon. \quad (2.1)$$

When the point t_0 is left-dense and right-scattered at the same time, then replace t_0 in (2.1) by t_0^- (the left-hand limit).

In other words, $f \in C_{\text{rd}} \times C \times C$ means that f is continuous at any point $(t_0, y_0, v_0) \in [a, \infty)_{\mathbb{T}} \times \Omega_q \times \mathbb{R}^n$ when t_0 is right-dense, and that f is *jointly regulated*, that is, $\lim_{n \rightarrow \infty} f(t_n, x_n, v_n)$ exists (finite) whenever $t_n \rightarrow t_0^-$ or $t_n \rightarrow t_0^+$, and $(x_n, v_n) \rightarrow (x_0, v_0)$.

The following result is a minor modification of [26, Proposition 1]. It shows that the above continuity concept is the right one when considering time scale delta integrals involving a $C_{\text{rd}} \times C \times C$ -continuous function f in the composition with a C_{rd}^1 function x .

Proposition 2.2. Let $x \in C_{\text{rd}}^1[a, \infty)_{\mathbb{T}}$ and assume that $f \in C_{\text{rd}} \times C \times C$ on $[a, \infty)_{\mathbb{T}} \times \Omega_q \times \mathbb{R}^n$ with $0 < q \leq \infty$. Then $f(\cdot, x^\sigma(\cdot), x^{\Delta\sigma}(\cdot)) \in C_{\text{rd}}$.

Similarly, when the function f is defined only on $[a, \infty)_{\mathbb{T}} \times \Omega_q$ we have the following statement.

Proposition 2.3. Let $x \in C[a, \infty)_{\mathbb{T}}$ and assume that $f \in C_{\text{rd}} \times C$ on $[a, \infty)_{\mathbb{T}} \times \Omega_q$ with $0 < q \leq \infty$. Then $f(\cdot, x^\sigma(\cdot)) \in C_{\text{rd}}$.

When considering terminal value problems, we shall use the abbreviation

$$x(\infty) := \lim_{t \rightarrow \infty} x(t).$$

The vector space of all real n -vector functions defined on $[a, \infty)_{\mathbb{T}}$ will be denoted throughout the paper by \mathcal{F} .

For completeness we recall the statement of the Banach fixed point theorem adjusted to the setting of this paper.

Proposition 2.4. Let X be a Banach space (i.e., a complete normed space) with norm $\|\cdot\|_X$ and let $U \subseteq X$ be its nonempty and closed subset. If a mapping $F : U \rightarrow U$ is a contraction, i.e., if there exists $L \in (0, 1)$ such that $\|Fx - Fy\|_X \leq L\|x - y\|_X$ for all $x, y \in U$, then F has a unique fixed point, i.e., there exists a unique element $x \in U$ such that $x = Fx$. Furthermore, if $x_0 \in U$ is arbitrary, and if we set $x_{i+1} := Fx_i$ for all $i \in \mathbb{N}$, then the sequence $\{x_i\}_{i=0}^\infty$ converges in X to the fixed point x , and the error between the i -th iteration x_i and the fixed point x satisfies the estimate

$$\|x_i - x\|_X \leq \frac{L^i}{1 - L} \|x_1 - x_0\|_X, \quad i \in \mathbb{N}.$$

Next we present some important properties of the time scale exponential function. By definition, see [10, Definition 2.30], for an rd-continuous and regressive function $p : [a, \infty)_{\mathbb{T}} \rightarrow \mathbb{R}$, the time scale exponential function $e_{p(\cdot)}(t, a)$ is defined to be the unique solution of the initial value problem $u^\Delta = p(t)u$, $u(a) = 1$. In this paper we will utilize the time scale exponential functions corresponding to the initial value problem

$$u^\Delta = -p(t)u^\sigma, \quad t \in [a, \infty)_{\mathbb{T}}, \quad u(a) = 1. \quad (2.2)$$

By expanding u^σ in (2.2) with the formula $u^\sigma = u + \mu(t)u^\Delta$ and using the regressivity of $p(\cdot)$, the dynamic equation in (2.2) is equivalent to the equation $u^\Delta = (\ominus p)(t)u$, where $\ominus p(t) := [-p(t)]/[1 + \mu(t)p(t)]$. Thus, the time scale exponential function $e_{\ominus p(\cdot)}(t, a)$ is the unique solution of the initial value problem (2.2). Motivated by [20, Lemma 3.1], we have the following result.

Lemma 2.5. *Assume that $p : [a, \infty)_{\mathbb{T}} \rightarrow (0, \infty)$, $p \in C_{\text{rd}}$, and*

$$\int_a^\infty p(s) \Delta s < \infty. \quad (2.3)$$

Let $u(t) := e_{\ominus p(\cdot)}(t, a)$ be the time scale exponential function corresponding to the initial value problem (2.2). Then $u(\cdot)$ is positive and decreasing on $[a, \infty)_{\mathbb{T}}$, and $\lim_{t \rightarrow \infty} u(t) =: u_0 \in (0, 1)$. Furthermore,

$$\sup_{t \in [a, \infty)_{\mathbb{T}}} \frac{1}{u(t)} \int_t^\infty [-u^\Delta(s)] \Delta s = 1 - u_0. \quad (2.4)$$

Proof. Since $p(\cdot)$ is positive, $\ominus p(\cdot)$ is a negative function. Hence, we have $1 + \mu(t)(\ominus p)(t) = 1/[1 + \mu(t)p(t)] > 0$ on $[a, \infty)_{\mathbb{T}}$, i.e., $\ominus p(\cdot)$ is an rd-continuous and positively regressive function. By [10, Theorem 2.44], we get that $u(t) > 0$ on $[a, \infty)_{\mathbb{T}}$. Consequently, $u^\Delta(t) < 0$, the function $u(\cdot)$ is decreasing on $[a, \infty)_{\mathbb{T}}$, the limit u_0 exists, and $u_0 \in [0, 1)$. The fact that actually $u_0 > 0$ follows from the assumption (2.3). We refer to the proof of [20, Lemma 3.1] for the details. For the proof of (2.4), we have

$$\sup_{t \in [a, \infty)_{\mathbb{T}}} \frac{1}{u(t)} \int_t^\infty [-u^\Delta(s)] \Delta s = \sup_{t \in [a, \infty)_{\mathbb{T}}} \left(1 - \frac{u_0}{u(t)}\right) = 1 - u_0,$$

because the function u attains its maximum value $u(a) = 1$. \square

3. FIRST ORDER EQUATIONS

Consider the first order time scale dynamic equation

$$x^\Delta + f(t, x^\sigma) = 0, \quad t \in [a, \infty)_{\mathbb{T}}. \quad (3.1)$$

For a given positive number N we define the set

$$X_N := \{x \in C[a, \infty)_{\mathbb{T}}, \|x\|_0 \leq N\}, \quad (3.2)$$

Then X_N is a closed subset of the Banach space $(C[a, \infty)_{\mathbb{T}}, \|\cdot\|_0)$.

Remark 3.1. Given a function $\psi : [a, \infty)_{\mathbb{T}} \rightarrow [c, d]$, $0 < c \leq d < \infty$, we introduce on the space $C[a, \infty)_{\mathbb{T}}$ another norm

$$\|x\|_\psi := \|x/\psi\|_0 = \sup_{t \in [a, \infty)_{\mathbb{T}}} \frac{|x(t)|}{\psi(t)}.$$

The norm $\|\cdot\|_\psi$ is on $C[a, \infty)_\mathbb{T}$ clearly equivalent to the norm $\|\cdot\|_0$, so that $(C[a, \infty)_\mathbb{T}, \|\cdot\|_\psi)$ is also a Banach space, compare with [31, Lemma 3.3].

The following theorem is then our first result.

Theorem 3.2. *Assume that $f : [a, \infty)_\mathbb{T} \times \Omega_q \rightarrow \mathbb{R}^n$ with $0 < q \leq \infty$, $f \in C_{\text{rd}} \times C$, is a function satisfying the Lipschitz condition*

$$|f(t, x) - f(t, y)| \leq k(t) |x - y|, \quad \text{for all } t \in [a, \infty)_\mathbb{T}, \quad x, y \in \Omega_q, \quad (3.3)$$

where $k : [a, \infty)_\mathbb{T} \rightarrow (0, \infty)$, $k \in C_{\text{rd}}$, and

$$\int_a^\infty k(s) \Delta s < \infty. \quad (3.4)$$

Let $A \in \mathbb{R}^n$ be a given vector. If there exists a number $N \in \mathbb{R}$, $|A| \leq N < q$, such that

$$\int_a^\infty |f(s, x^\sigma(s))| \Delta s \leq N - |A|, \quad \text{for all } x \in X_N, \quad (3.5)$$

where X_N is defined by (3.2), then the problem (3.1) has a unique solution $x(t)$ on $[a, \infty)_\mathbb{T}$ satisfying $x(\infty) = A$.

Proof. We will apply the Banach fixed point theorem in the space $(C[a, \infty)_\mathbb{T}, \|\cdot\|_\psi)$ for a suitably chosen function ψ . Define the operator $F : X_N \rightarrow \mathcal{F}$ (the space of n -vector functions) by

$$[Fx](t) := A + \int_t^\infty f(s, x^\sigma(s)) \Delta s, \quad t \in [a, \infty)_\mathbb{T}.$$

It follows from Proposition 2.3 and assumption (3.5) that $[Fx](t)$ is well-defined for all $t \in [a, \infty)_\mathbb{T}$. Furthermore,

$$\begin{aligned} |[Fx](t)| &\leq |A| + \int_t^\infty |f(s, x^\sigma(s))| \Delta s \leq |A| + \int_a^\infty |f(s, x^\sigma(s))| \Delta s \\ &\leq |A| + N - |A| = N, \quad t \in [a, \infty)_\mathbb{T}. \end{aligned}$$

Hence, $\|Fx\|_0 \leq N$. Since Fx is Δ -differentiable, hence continuous, it follows that $Fx \in X_N$, and

$$[Fx]^\Delta(t) = -f(t, x^\sigma(t)), \quad t \in [a, \infty)_\mathbb{T}. \quad (3.6)$$

Next, motivated by its introduction and use in [31], choose the function $\psi(t)$ to be the time scale exponential function $e_{\ominus k(\cdot)}(t, a)$. Then, by Lemma 2.5, we have $0 < \psi_0 \leq \psi(t) \leq 1$ for all $t \in [a, \infty)_\mathbb{T}$ with $\psi_0 \in (0, 1)$, where $\psi_0 := \lim_{t \rightarrow \infty} \psi(t)$. Thus, by Remark 3.1, $(C[a, \infty)_\mathbb{T}, \|\cdot\|_\psi)$ is a Banach space. By using (3.3) and (2.4) with $u := \psi$ and $u_0 := \psi_0$, we have for $x, y \in X_N$

$$\begin{aligned} \|Fx - Fy\|_\psi &\leq \sup_{t \in [a, \infty)_\mathbb{T}} \frac{1}{\psi(t)} \int_t^\infty |f(s, x^\sigma(s)) - f(s, y^\sigma(s))| \Delta s \\ &\leq \sup_{t \in [a, \infty)_\mathbb{T}} \frac{1}{\psi(t)} \int_t^\infty k(s) |x^\sigma(s) - y^\sigma(s)| \Delta s \\ &\leq \|x - y\|_\psi \sup_{t \in [a, \infty)_\mathbb{T}} \frac{1}{\psi(t)} \int_t^\infty [-\psi^\Delta(s)] \Delta s = (1 - \psi_0) \|x - y\|_\psi. \end{aligned}$$

Hence, the mapping F is a contraction in X_N . By Proposition 2.4, there is a unique function $x \in X_N$ such that $x = Fx$, i.e.,

$$x(t) = A + \int_t^\infty f(s, x^\sigma(s)) \Delta s, \quad t \in [a, \infty)_\mathbb{T}. \quad (3.7)$$

By (3.6) and Proposition 2.3, $Fx \in C_{\text{rd}}^1$, and x satisfies equation (3.1). Finally, from assumption (3.5) it follows that

$$\lim_{t \rightarrow \infty} \int_t^\infty f(s, x^\sigma(s)) \Delta s = 0,$$

and hence, identity (3.7) yields that $x(\infty) = A$. \square

In Theorem 3.2 the solution $x(t)$ approaches the *a priori* given limit A , and the number N defining the set X_N then depends on $|A|$. It may be hard to find such number N . On the other hand, in the following result the solution $x(t)$ approaches a limit M , and at the same time the vector M determines the set in which the contraction mapping F is defined. This type of result is then of the same fashion as e.g. the results in [20].

Corollary 3.3. *Assume that $f : [a, \infty)_\mathbb{T} \times \Omega_q \rightarrow \mathbb{R}^n$ with $0 < q \leq \infty$, $f \in C_{\text{rd}} \times C$, is a function satisfying the Lipschitz condition (3.3), where $k : [a, \infty)_\mathbb{T} \rightarrow (0, \infty)$, $k \in C_{\text{rd}}$, and (3.4) holds. If there exists a vector $M \in \mathbb{R}^n$, $|M| < q$, such that*

$$\int_a^\infty |f(s, x^\sigma(s))| \Delta s \leq |M|, \quad \text{for all } x \in X_{2|M|},$$

then the problem (3.1) has a unique solution $x(t)$ on $[a, \infty)_\mathbb{T}$ satisfying $x(\infty) = M$.

Proof. We let $A := M$ and $N := 2|M|$ in Theorem 3.2. \square

Our attention now turns to the following dynamic equation

$$x^\Delta + f(t, x) = 0, \quad t \in [a, \infty)_\mathbb{T}, \quad (3.8)$$

where f is scalar-valued. Our interest is in obtaining comparison-type theorems for solutions x to (3.8) subject to

$$x(\infty) = A. \quad (3.9)$$

Lemma 3.4. *Assume that $f : [a, \infty)_\mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$, $f \in C_{\text{rd}} \times C$, and there exist functions $u, v : [a, \infty)_\mathbb{T} \rightarrow \mathbb{R}$ such that: $u(\infty), v(\infty)$ both exist,*

$$u^\Delta(t) + f(t, u(t)) \geq 0, \quad \text{for all } t \in [a, \infty)_\mathbb{T}, \quad (3.10)$$

$$v^\Delta(t) + f(t, v(t)) \leq 0, \quad \text{for all } t \in [a, \infty)_\mathbb{T}, \quad (3.11)$$

$$f(t, p) \leq f(t, q), \quad \text{for all } q \leq p. \quad (3.12)$$

If $u(\infty) < v(\infty)$, then $u(t) < v(t)$ for all $t \in [a, \infty)_\mathbb{T}$.

Proof. Argue by contradiction by assuming that there exists a point $t_1 \in [a, \infty)_\mathbb{T}$ such that

$$u(t_1) \geq v(t_1), \quad \text{and} \quad (3.13)$$

$$u(t) < v(t), \quad \text{for all } t \in (t_1, \infty)_\mathbb{T}. \quad (3.14)$$

There are two cases to discuss: (a) the point t_1 is right-scattered; (b) the point t_1 is right-dense.

(a) If t_1 is right-scattered, then $u^\Delta(t_1) < v^\Delta(t_1)$ and so from (3.10) and (3.11) we obtain

$$-f(t_1, u(t_1)) \leq u^\Delta(t_1) < v^\Delta(t_1) \leq -f(t_1, v(t_1)),$$

which is a contradiction to (3.12).

(b) If t_1 is right-dense, then (3.13) is forced to become $u(t_1) = v(t_1)$ by (3.14) and the intermediate value theorem. As per part (a) we see

$$-f(t_1, u(t_1)) < -f(t_1, v(t_1)),$$

which is impossible since $u(t_1) = v(t_1)$.

Thus, in either case we have $u(t) < v(t)$ for all $t \in [a, \infty)_\mathbb{T}$. \square

We will now apply Lemma 3.4 to obtain comparison results for solutions to (3.8), (3.9).

Theorem 3.5. *Assume that $f : [a, \infty)_\mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$, $f \in C_{\text{rd}} \times C$, satisfying (3.12) and there exists a function $u : [a, \infty)_\mathbb{T} \rightarrow \mathbb{R}$ such that $u(\infty)$ exists and (3.10) holds. If x is a solution to (3.8), (3.9) and $u(\infty) < A$, then $x(t) > u(t)$ for all $t \in [a, \infty)_\mathbb{T}$.*

Proof. Take $v := x$ in Lemma 3.4 and the result follows. \square

Similarly, we have the following result by taking $u := x$ in Lemma 3.4.

Theorem 3.6. *Assume that $f : [a, \infty)_\mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$, $f \in C_{\text{rd}} \times C$, satisfying (3.12) and there exists a function $v : [a, \infty)_\mathbb{T} \rightarrow \mathbb{R}$ such that $v(\infty)$ exists and (3.11) holds. If x is a solution to (3.8), (3.9) and $v(\infty) > A$, then $x(t) < v(t)$ for all $t \in [a, \infty)_\mathbb{T}$.*

4. SECOND ORDER EQUATIONS WITH SCALAR LEADING COEFFICIENT

The methods used in Section 3 to derive the existence and uniqueness results for the first order equations can be naturally used in order to derive similar results for the second order dynamic equations. For the second order setting there are two cases depending on whether the nonlinearity f involves the Δ -derivative of x or does not. As we shall see, these two cases differ in the assumption on the leading coefficient $r(t)$. Note that as in the previous section the function f can take both positive and negative values. Consider first the equation

$$(r(t)x^\Delta)^\Delta + f(t, x^\sigma) = 0, \quad t \in [a, \infty)_\mathbb{T}. \quad (4.1)$$

We note that while the functions f and x in (4.1) are n -vector valued, the function r will be (in this section) assumed to be scalar valued. Furthermore, compared with some recent oscillation and asymptotic results for second order dynamic equations [3, 4, 5, 12, 13, 18, 19, 20, 28] in which $r(t) > 0$ on $[a, \infty)_\mathbb{T}$, in this paper we assume (if not otherwise stated) that $r(t) \neq 0$ only. This type of assumption is common in the oscillation theory of difference equations, see e.g. [14, 16], and have also been adopted in some papers in the time scale setting [15, 17, 24, 25].

The results in this section directly generalize [20, Theorems 4.2 and 4.5] to vector valued nonlinearity f which can take negative values and the leading coefficient $r(t)$ is assumed to be nonzero only.

Remark 4.1. Given a function $r : [a, \infty)_\mathbb{T} \rightarrow \mathbb{R}$, $r \in C_{\text{rd}}$, such that

$$\inf_{t \in [a, b]_\mathbb{T}} |r(t)| > 0, \quad \text{for all } b \in [a, \infty)_\mathbb{T}, \quad (4.2)$$

it follows that $r(t) \neq 0$ for all $t \in [a, \infty)_{\mathbb{T}}$, and the function $\frac{1}{r}$ also belongs to C_{rd} on $[a, \infty)_{\mathbb{T}}$. Hence, the integrals

$$R(t, s) := \int_s^t \frac{1}{r(\tau)} \Delta\tau, \quad \bar{R}(t, s) := \int_s^t \frac{1}{|r(\tau)|} \Delta\tau, \quad t, s \in [a, \infty)_{\mathbb{T}},$$

are well-defined. Obviously, for a fixed $s \in [a, \infty)_{\mathbb{T}}$ both $R(\cdot, s)$ and $\bar{R}(\cdot, s)$ belong to C_{rd}^1 with (Δ -differentiating with respect to the first argument) $R^\Delta(t, s) = \frac{1}{r(t)}$ and $\bar{R}^\Delta(t, s) = \frac{1}{|r(t)|} > 0$ for all $t \in [a, \infty)_{\mathbb{T}}$. Consequently, the function $\bar{R}(t, s)$ is increasing as t increases or, for the same reason, as s decreases. Moreover, we have

$$|R(t, s)| \leq \bar{R}(t, s) \leq \bar{R}(t, a), \quad t, s \in [a, \infty)_{\mathbb{T}}, \quad t \geq s. \quad (4.3)$$

In connection with these functions we shall frequently use the identities

$$R(\sigma(s), t) = R(\sigma(s), a) - R(t, a), \quad \bar{R}(\sigma(s), t) = \bar{R}(\sigma(s), a) - \bar{R}(t, a), \quad (4.4)$$

for $t, s \in [a, \infty)_{\mathbb{T}}$. Next we present the first main result of this section.

Theorem 4.2. *Assume that $f : [a, \infty)_{\mathbb{T}} \times \Omega_q \rightarrow \mathbb{R}^n$ with $0 < q \leq \infty$, $f \in C_{\text{rd}} \times C$, and $r : [a, \infty)_{\mathbb{T}} \rightarrow \mathbb{R}$, $r \in C_{\text{rd}}$, are given functions satisfying condition (4.2) and the Lipschitz condition (3.3), in which $k : [a, \infty)_{\mathbb{T}} \rightarrow (0, \infty)$, $k \in C_{\text{rd}}$, and*

$$\int_a^\infty \bar{R}(\sigma(s), a) k(s) \Delta s < \infty. \quad (4.5)$$

Let $A \in \mathbb{R}^n$ be a given vector. If there exists a number $N \in \mathbb{R}$, $|A| \leq N < q$, such that

$$\int_a^\infty \bar{R}(\sigma(s), a) |f(s, x^\sigma(s))| \Delta s \leq N - |A|, \quad \text{for all } x \in X_N, \quad (4.6)$$

where X_N is defined by (3.2), then the problem (4.1) has a unique solution $x(t)$ on $[a, \infty)_{\mathbb{T}}$ satisfying $x(\infty) = A$ and $(rx^\Delta)(\infty) = 0$.

Before proving Theorem 4.2 we need to establish an auxiliary lemma.

Lemma 4.3. *Let $g : [a, \infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$, $g \in C_{\text{rd}}$, and $r : [a, \infty)_{\mathbb{T}} \rightarrow \mathbb{R}$, $r \in C_{\text{rd}}$, be given functions such that condition (4.2) holds and*

$$\int_a^\infty \bar{R}(\sigma(s), a) |g(s)| \Delta s < \infty. \quad (4.7)$$

Define the function

$$G(t) := \int_t^\infty R(\sigma(s), t) g(s) \Delta s, \quad t \in [a, \infty)_{\mathbb{T}}.$$

Then $G(t)$ is well-defined, $G \in C_{\text{rd}}^1$ on $[a, \infty)_{\mathbb{T}}$, and

$$G^\Delta(t) = -\frac{1}{r(t)} \int_t^\infty g(s) \Delta s, \quad t \in [a, \infty)_{\mathbb{T}}. \quad (4.8)$$

Moreover,

$$\lim_{t \rightarrow \infty} G(t) = 0, \quad \lim_{t \rightarrow \infty} r(t) G^\Delta(t) = 0. \quad (4.9)$$

Proof. First note that, since $\bar{R}(\sigma(s), t) \leq \bar{R}(\sigma(s), a)$ for $t \geq a$, we have for all $t \in [a, \infty)_{\mathbb{T}}$

$$|G(t)| \leq \int_t^\infty \bar{R}(\sigma(s), t) |g(s)| \Delta s \leq \int_t^\infty \bar{R}(\sigma(s), a) |g(s)| \Delta s =: \bar{G}(t) \leq \bar{G}(a).$$

Hence, by assumption (4.7), $G(t)$ is well defined for all $t \in [a, \infty)_{\mathbb{T}}$. Next, since $\bar{R}(t, a) \leq \bar{R}(\sigma(s), a)$ for $\sigma(s) \geq t$, the estimate

$$\left| R(t, a) \int_t^\infty g(s) \Delta s \right| = \bar{R}(t, a) \left| \int_t^\infty g(s) \Delta s \right| \leq \bar{G}(t), \quad t \in [a, \infty)_{\mathbb{T}},$$

and the fact that $\bar{R}(t, a) > 0$ for $t > a$ show that $\left| \int_t^\infty g(s) \Delta s \right|$ and hence $\int_t^\infty g(s) \Delta s$ are finite for any $t \in (a, \infty)_{\mathbb{T}}$. Fix any $t_0 \in (a, \infty)_{\mathbb{T}}$. Since $g \in C_{\text{rd}}$, the integral $\int_a^{t_0} g(s) \Delta s$ exists, and then with respect to the previous conclusion we get that $\int_a^\infty g(s) \Delta s = \left\{ \int_a^{t_0} + \int_{t_0}^\infty \right\} g(s) \Delta s$ exists finite. The latter then implies that

$$\lim_{t \rightarrow \infty} \int_t^\infty g(s) \Delta s = 0. \quad (4.10)$$

Thus, by using the first expression in (4.4), we may write

$$G(t) = \int_t^\infty R(\sigma(s), a) g(s) \Delta s - R(t, a) \int_t^\infty g(s) \Delta s, \quad t \in [a, \infty)_{\mathbb{T}}, \quad (4.11)$$

in which both improper integrals exist finite. This shows that G is a C_{rd}^1 function on $[a, \infty)_{\mathbb{T}}$. Using the time scale product rule when Δ -differentiating the second term in (4.11) we obtain formula (4.8). Finally, the first limit in (4.9) follows from the fact that (for example) $G(a)$ is finite, while the second limit in (4.9) is a consequence of formula (4.8) in combination with the limit (4.10). \square

We are now ready to derive Theorem 4.2.

Proof of Theorem 4.2. We will apply the Banach fixed point theorem in the space $(C[a, \infty)_{\mathbb{T}}, \|\cdot\|_\psi)$ for a suitably chosen function ψ . Define the operator $F : X_N \rightarrow \mathcal{F}$ (the space of n -vector functions) by

$$[Fx](t) := A - \int_t^\infty R(\sigma(s), t) f(s, x^\sigma(s)) \Delta s, \quad t \in [a, \infty)_{\mathbb{T}}.$$

Set $g(t) := f(t, x^\sigma(t))$ on $[a, \infty)_{\mathbb{T}}$. Then Proposition 2.3 yields that $g \in C_{\text{rd}}$. By assumption (4.6) and Lemma 4.3, we have that $[Fx](t)$ is well-defined for all $t \in [a, \infty)_{\mathbb{T}}$, $Fx \in C_{\text{rd}}^1$, and

$$[Fx]^\Delta(t) = \frac{1}{r(t)} \int_t^\infty g(s) \Delta s, \quad t \in [a, \infty)_{\mathbb{T}}, \quad (4.12)$$

with the limits

$$\lim_{t \rightarrow \infty} [Fx](t) = A, \quad \lim_{t \rightarrow \infty} r(t) [Fx]^\Delta(t) = 0. \quad (4.13)$$

Furthermore, inequality (4.3) and assumption (4.6) yield that $\|[Fx](t)\| \leq N$ for all $t \in [a, \infty)_{\mathbb{T}}$, so that $\|Fx\|_0 \leq N$ and $Fx \in X_N$.

Next, similarly to the proof of Theorem 3.2, we choose the function $\psi(t)$ to be the time scale exponential function $e_{\ominus p(\cdot)}(t, a)$, where $p(t) := \bar{R}(\sigma(t), a) k(t)$ for $t \in [a, \infty)_{\mathbb{T}}$. That is, $\psi^\Delta(t) = -p(t) \psi^\sigma(t)$ on $[a, \infty)_{\mathbb{T}}$. Then, by Lemma 2.5, we have $0 < \psi_0 \leq \psi(t) \leq 1$ for all $t \in [a, \infty)_{\mathbb{T}}$ with $\psi_0 \in (0, 1)$, where $\psi_0 := \lim_{t \rightarrow \infty} \psi(t)$.

Thus, by Remark 3.1, $(C[a, \infty)_{\mathbb{T}}, \|\cdot\|_{\psi})$ is a Banach space. By using the Lipschitz condition (3.3), we get for any $x, y \in X_N$

$$\begin{aligned} \|Fx - Fy\|_{\psi} &\leq \sup_{t \in [a, \infty)_{\mathbb{T}}} \frac{1}{\psi(t)} \int_t^{\infty} \bar{R}(\sigma(s), t) |f(s, x^{\sigma}(s)) - f(s, y^{\sigma}(s))| \Delta s \\ &\leq \sup_{t \in [a, \infty)_{\mathbb{T}}} \frac{1}{\psi(t)} \int_t^{\infty} \bar{R}(\sigma(s), t) k(s) |x^{\sigma}(s) - y^{\sigma}(s)| \Delta s. \end{aligned} \quad (4.14)$$

Now if $t = a$, then for $s \geq t = a$ we have $\bar{R}(\sigma(s), t) k(s) = \frac{-\psi^{\Delta}(s)}{\psi^{\sigma}(s)}$. If $t > a$, then for $s \geq t$ the quantity $\bar{R}(\sigma(s), a) > 0$ and $0 \leq \bar{R}(\sigma(s), t) < \bar{R}(\sigma(s), a)$. In this case we have

$$\bar{R}(\sigma(s), t) k(s) = \frac{\bar{R}(\sigma(s), t)}{\bar{R}(\sigma(s), a)} \bar{R}(\sigma(s), a) k(s) \leq \frac{-\psi^{\Delta}(s)}{\psi^{\sigma}(s)}, \quad (4.15)$$

and in combination with the previous case we see that inequality (4.15) holds for any $t \in [a, \infty)_{\mathbb{T}}$. Thus, we get from (4.14) by using (4.15), the definition of $\|x - y\|_{\psi}$, and condition (2.4) with $u := \psi$ and $u_0 := \psi_0$ that

$$\|Fx - Fy\|_{\psi} \leq \|x - y\|_{\psi} \sup_{t \in [a, \infty)_{\mathbb{T}}} \frac{1}{\psi(t)} \int_t^{\infty} [-\psi^{\Delta}(s)] \Delta s = (1 - \psi_0) \|x - y\|_{\psi}.$$

Hence, the mapping F is a contraction in X_N . By Proposition 2.4, there is a unique function $x \in X_N$ such that $x = Fx$, i.e.,

$$x(t) = A - \int_t^{\infty} R(\sigma(s), t) g(s) \Delta s, \quad t \in [a, \infty)_{\mathbb{T}}. \quad (4.16)$$

From the limits in (4.13) we get that

$$x(\infty) = [Fx](\infty) = A \quad \text{and} \quad (rx^{\Delta})(\infty) = \lim_{t \rightarrow \infty} r(t) [Fx]^{\Delta}(t) = 0.$$

Moreover, equations (4.12) and (4.16) show that the function x satisfies

$$r(t) x^{\Delta}(t) = \int_t^{\infty} g(s) \Delta s, \quad t \in [a, \infty)_{\mathbb{T}}. \quad (4.17)$$

While the right-hand side of (4.17) is Δ -differentiable, it follows that

$$(r(t) x^{\Delta}(t))^{\Delta} = -g(t), \quad t \in [a, \infty)_{\mathbb{T}},$$

i.e., the function x satisfies the dynamic equation (4.1). The proof is complete. \square

Next we turn our attention to the more general dynamic equation

$$(r(t) x^{\Delta})^{\Delta} + f(t, x^{\sigma}, x^{\Delta\sigma}) = 0, \quad t \in [a, \infty)_{\mathbb{T}}. \quad (4.18)$$

As we saw in Theorem 4.2, the problem (4.1) which does not involve x^{Δ} in f can be treated within the set X_N consisting of certain continuous functions x . On the other hand, the problem (4.18) must be considered in a narrower space, because it is implicitly assumed in the form of this equation that x^{Δ} exists throughout the interval $[a, \infty)_{\mathbb{T}}$. Therefore, we introduce the set

$$X_N^1 := \{x \in C_{\text{rd}}^1[a, \infty)_{\mathbb{T}}, \|x\|_1 < \infty, \|x\|_0 \leq N\}. \quad (4.19)$$

Then X_N^1 is a closed subset of the Banach space $(C_{\text{rd}}^1[a, \infty)_{\mathbb{T}}, \|\cdot\|_1)$.

Remark 4.4. Given a function $\varphi : [a, \infty)_{\mathbb{T}} \rightarrow [c, d]$, $0 < c \leq d < \infty$, we introduce on the space $C_{\text{rd}}^1[a, \infty)_{\mathbb{T}}$ another norm

$$\|x\|_{\varphi} := \max \{ \|x/\varphi\|_0, \|x^{\Delta}/\varphi\|_0 \}.$$

Since $c \|x\|_{\varphi} \leq \|x\|_1 \leq d \|x\|_{\varphi}$, the norm $\|\cdot\|_{\varphi}$ is on $C_{\text{rd}}^1[a, \infty)_{\mathbb{T}}$ equivalent to the norm $\|\cdot\|_1$. Hence, $(C_{\text{rd}}^1[a, \infty)_{\mathbb{T}}, \|\cdot\|_{\varphi})$ is also a Banach space.

Our main result regarding equation (4.18) is the following.

Theorem 4.5. *Assume that $f : [a, \infty)_{\mathbb{T}} \times \Omega_q \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $0 < q \leq \infty$, $f \in C_{\text{rd}} \times C \times C$, and $r : [a, \infty)_{\mathbb{T}} \rightarrow \mathbb{R}$, $r \in C_{\text{rd}}$, are functions satisfying*

$$\inf_{t \in [a, \infty)_{\mathbb{T}}} |r(t)| \geq r_0 > 0 \quad (4.20)$$

for some number $r_0 > 0$, the Lipschitz condition

$$|f(t, x, u) - f(t, y, v)| \leq k(t) [|x - y| + |u - v|] \quad (4.21)$$

for all $t \in [a, \infty)_{\mathbb{T}}$, $x, y \in \Omega$, $u, v \in \mathbb{R}^n$, where $k : [a, \infty)_{\mathbb{T}} \rightarrow (0, \infty)$, $k \in C_{\text{rd}}$, and condition

$$\int_a^{\infty} [\bar{R}(\sigma(s), a) + 1] k(s) \Delta s < \infty. \quad (4.22)$$

Let $A \in \mathbb{R}^n$ be a given vector. If there exists a number $N \in \mathbb{R}$, $|A| \leq N < q$, such that

$$\int_a^{\infty} \bar{R}(\sigma(s), a) |f(s, x^{\sigma}(s), x^{\Delta\sigma}(s))| \Delta s \leq N - |A|, \quad \text{for all } x \in X_N^1, \quad (4.23)$$

$$\sup_{t \in [a, \infty)_{\mathbb{T}}} \left| \int_t^{\infty} f(s, x^{\sigma}(s), x^{\Delta\sigma}(s)) \Delta s \right| < \infty, \quad \text{for all } x \in X_N^1, \quad (4.24)$$

where X_N^1 is defined by (4.19), then the problem (4.18) has a unique solution $x(t)$ on $[a, \infty)_{\mathbb{T}}$ satisfying $x(\infty) = A$ and $(rx^{\Delta})(\infty) = 0$.

Let us briefly comment on the main differences between the above Theorems 4.5 and 4.2.

Remark 4.6. (i) In Theorem 4.5, the assumption (4.20) on the function $r(\cdot)$ is stronger than the assumption (4.2) in Theorem 4.2. Therefore, functions $r(\cdot)$ decaying to zero at infinity are not allowed in Theorem 4.5 while they are still admissible for Theorem 4.2.

(ii) It is a part of the proof of Theorem 4.5 that assumption (4.23) implies the finiteness of the improper integral $\int_t^{\infty} f(s, x^{\sigma}(s), x^{\Delta\sigma}(s)) \Delta s$ in condition (4.24) for every $t \in [a, \infty)_{\mathbb{T}}$ and every $x \in X_N^1$.

(iii) Another difference between Theorems 4.5 and 4.2 is the presence of the additional condition (4.24) in Theorem 4.5. Note that this condition is satisfied e.g. when the improper integral

$$\int_a^{\infty} |f(s, x^{\sigma}(s), x^{\Delta\sigma}(s))| \Delta s < \infty \quad (4.25)$$

for any $x \in X_N^1$. The latter condition is satisfied, in particular, when $n = 1$ and $f(\cdot, \cdot, \cdot) \geq 0$ as in [20]. Assuming (4.20), condition (4.24) in fact means that $\|[Fx]^{\Delta}\|_0$ is finite, which is needed in order to show that $Fx \in X_N^1$. On the other hand, the finiteness of $\|[Fx]^{\Delta}\|_0$ is not required in the proof of Theorem 4.2, since there we work in the set X_N only.

Proof of Theorem 4.5. The proof is similar to the proof of Theorem 4.2 and we shall include only the main differences. We will apply the Banach fixed point theorem in the space $(C_{\text{rd}}^1[a, \infty)_{\mathbb{T}}, \|\cdot\|_{\varphi})$ for a suitably chosen function φ . Define the operator $F : X_N^1 \rightarrow \mathcal{F}$ by

$$[Fx](t) := A - \int_t^{\infty} R(\sigma(s), t) f(s, x^{\sigma}(s), x^{\Delta\sigma}(s)) \Delta s, \quad t \in [a, \infty)_{\mathbb{T}},$$

and set $g(t) := f(t, x^{\sigma}(t), x^{\Delta\sigma}(t))$ on $[a, \infty)_{\mathbb{T}}$. Then Proposition 2.2 yields that $g \in C_{\text{rd}}$. As in the proof of Theorem 4.2 we get that $[Fx](t)$ is well defined for all $t \in [a, \infty)_{\mathbb{T}}$, and formulas (4.12) and (4.13) hold. In particular, $\int_t^{\infty} g(s) \Delta s$ exists finite for all $t \in [a, \infty)_{\mathbb{T}}$. Assumption (4.23) now yields that $|[Fx](t)| \leq N$ for all $t \in [a, \infty)_{\mathbb{T}}$, so that $\|Fx\|_0 \leq N$. Furthermore, from (4.12) and assumption (4.24) we get

$$\|[Fx]^{\Delta}\|_0 = \sup_{t \in [a, \infty)_{\mathbb{T}}} \frac{1}{|r(t)|} \left| \int_t^{\infty} g(s) \Delta s \right| \leq \frac{1}{r_0} \sup_{t \in [a, \infty)_{\mathbb{T}}} \left| \int_t^{\infty} g(s) \Delta s \right| < \infty.$$

This yields that $\|Fx\|_1 = \max\{\|Fx\|_0, \|[Fx]^{\Delta}\|_0\}$ is finite, and thus $Fx \in X_N^1$.

Choose the function $\varphi(t)$ to be the time scale exponential function $e_{\ominus p(\cdot)}(t, a)$, where $p(t) := \frac{2}{r_0} [\bar{R}(\sigma(t), a) + 1] k(t)$ for $t \in [a, \infty)_{\mathbb{T}}$. Then, by Lemma 2.5, we have $0 < \varphi_0 \leq \varphi(t) \leq 1$ for all $t \in [a, \infty)_{\mathbb{T}}$ with $\varphi_0 \in (0, 1)$, where $\varphi_0 := \lim_{t \rightarrow \infty} \varphi(t)$. Thus, by Remark 4.4, $(C_{\text{rd}}^1[a, \infty)_{\mathbb{T}}, \|\cdot\|_{\varphi})$ is a Banach space. Similarly to the proof of Theorem 4.2 we now deduce that for any $x, y \in X_N^1$

$$\|(Fx - Fy)/\varphi\|_0 \leq (1 - \varphi_0) \|x - y\|_{\varphi}. \quad (4.26)$$

Furthermore, by assumption (4.20) and the Lipschitz condition (4.21),

$$\begin{aligned} & \|([Fx]^{\Delta} - [Fy]^{\Delta})/\varphi\|_0 \\ &= \sup_{t \in [a, \infty)_{\mathbb{T}}} \frac{1}{\varphi(t)} \left| \frac{-1}{r(t)} \int_t^{\infty} [g(s) - f(s, y^{\sigma}(s), y^{\Delta\sigma}(s))] \Delta s \right| \\ &\leq \frac{1}{r_0} [\|(x - y)/\varphi\|_0 + \|(x^{\Delta} - y^{\Delta})/\varphi\|_0] \sup_{t \in [a, \infty)_{\mathbb{T}}} \frac{1}{\varphi(t)} \int_t^{\infty} k(s) \varphi^{\sigma}(s) \Delta s \\ &\leq \frac{2}{r_0} \|x - y\|_{\varphi} \sup_{t \in [a, \infty)_{\mathbb{T}}} \frac{1}{\varphi(t)} \int_t^{\infty} k(s) \varphi^{\sigma}(s) \Delta s. \end{aligned}$$

Now the choice of φ , the fact that $\bar{R}(\sigma(s), a) + 1 \geq 1$ for any $s \geq t$, and condition (2.4) with $u := \varphi$ and $u_0 := \varphi_0$ yield that

$$\begin{aligned} \|([Fx]^{\Delta} - [Fy]^{\Delta})/\varphi\|_0 &= \|x - y\|_{\varphi} \sup_{t \in [a, \infty)_{\mathbb{T}}} \frac{1}{\varphi(t)} \int_t^{\infty} \frac{1}{\bar{R}(\sigma(s), a) + 1} [-\varphi^{\Delta}(s)] \Delta s \\ &\leq \|x - y\|_{\varphi} \sup_{t \in [a, \infty)_{\mathbb{T}}} \frac{1}{\varphi(t)} \int_t^{\infty} [-\varphi^{\Delta}(s)] \Delta s \\ &= (1 - \varphi_0) \|x - y\|_{\varphi}. \end{aligned} \quad (4.27)$$

The estimates (4.26) and (4.27) now yield that $\|Fx - Fy\|_{\varphi} \leq (1 - \varphi_0) \|x - y\|_{\varphi}$, that is, the mapping F is a contraction in X_N^1 . Hence, by Proposition 2.4, there is a unique function $x \in X_N^1$ such that $x = Fx$. The rest of the proof is the same as in Theorem 4.2, namely we conclude that the function x satisfies the equation (4.18), and the limits $x(\infty) = A$ and $(rx^{\Delta})(\infty) = 0$. \square

Similarly as in Corollary 3.3, upon the choice $A := M$ and $N := 2|M|$ in Theorems 4.2 and 4.5 we can now derive results on the solvability of the terminal value problems for the equations (4.1) and (4.18) with the limits $x(\infty) = M$ and $(rx^\Delta)(\infty) = 0$.

Corollary 4.7. *Assume that $f : [a, \infty)_\tau \times \Omega_q \rightarrow \mathbb{R}^n$ with $0 < q \leq \infty$, $f \in C_{\text{rd}} \times C$, and $r : [a, \infty)_\tau \rightarrow \mathbb{R}$, $r \in C_{\text{rd}}$, are given functions satisfying condition (4.2) and the Lipschitz condition (3.3), in which $k : [a, \infty)_\tau \rightarrow (0, \infty)$, $k \in C_{\text{rd}}$, and (4.5) holds. If there exists a vector $M \in \mathbb{R}^n$, $|M| < q$, such that*

$$\int_a^\infty \bar{R}(\sigma(s), a) |f(s, x^\sigma(s))| \Delta s \leq |M|, \quad \text{for all } x \in X_{2|M|},$$

then the problem (4.1) has a unique solution $x(t)$ on $[a, \infty)_\tau$ which satisfies $x(\infty) = M$ and $(rx^\Delta)(\infty) = 0$.

Corollary 4.8. *Assume that $f : [a, \infty)_\tau \times \Omega_q \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $0 < q \leq \infty$, $f \in C_{\text{rd}} \times C \times C$, and $r : [a, \infty)_\tau \rightarrow \mathbb{R}$, $r \in C_{\text{rd}}$, are functions satisfying (4.20) for some number $r_0 > 0$, the Lipschitz condition (4.21) in which $k : [a, \infty)_\tau \rightarrow (0, \infty)$, $k \in C_{\text{rd}}$, and condition (4.22) holds. If there exists a vector $M \in \mathbb{R}^n$, $|M| < q$, such that*

$$\int_a^\infty \bar{R}(\sigma(s), a) |f(s, x^\sigma(s), x^{\Delta\sigma}(s))| \Delta s \leq |M|, \quad \text{for all } x \in X_{2|M|}^1,$$

$$\sup_{t \in [a, \infty)_\tau} \left| \int_t^\infty f(s, x^\sigma(s), x^{\Delta\sigma}(s)) \Delta s \right| < \infty, \quad \text{for all } x \in X_{2|M|}^1,$$

then the problem (4.18) has a unique solution $x(t)$ on $[a, \infty)_\tau$ which satisfies $x(\infty) = M$ and $(rx^\Delta)(\infty) = 0$.

5. SECOND ORDER EQUATIONS WITH MATRIX LEADING COEFFICIENT

In this section we present existence and uniqueness results for second order n -vector dynamic equations of the form (4.1) and (4.18), but in which the leading coefficient is a nonnegative $n \times n$ matrix function and f is a nonnegative n -vector function. The method for proving such results is a combination of the approach from [20], where scalar valued dynamic equations with nonnegative f were considered, with the methods presented in Section 4.

Since in this section the nonlinearity f and solutions x will have nonnegative entries, we modify the notation accordingly. If a vector $x \in \mathbb{R}^n$ has nonnegative entries, we shall denote it by $x \geq 0$. Similarly, for two vectors $x, y \in \mathbb{R}^n$ we write $x \leq y$ provided $y - x \geq 0$, i.e., their entries are compared componentwise.

For a real $n \times n$ matrix A we shall use any norm $\|\cdot\|$ compatible with (e.g., induced by) the vector norm $|\cdot|$, for example, the maximum row sum matrix norm $\|A\| := \|A\|_\infty = \max\{\sum_{j=1}^n |a_{ij}|, i = 1, \dots, n\}$. We also require, that the matrix norm is monotone; that is,

$$\begin{aligned} &\text{if } A \text{ and } B \text{ are symmetric with nonnegative} \\ &\text{entries and } 0 < A \leq B, \text{ then } \|A\| \leq \|B\|. \end{aligned} \tag{5.1}$$

We refer to [7] for more properties of matrix norms. Throughout this section we will denote by $\mathbb{R}_+ := [0, \infty)$ the set of all nonnegative real numbers. Similarly, the set of all such n -tuples will be denoted by $\mathbb{R}_+^n := [0, \infty)^n$. Accordingly with the

notation introduced in Section 2, we denote by Ω_q^+ the intersection of the open q -ball Ω_q with \mathbb{R}_+^n , that is $\Omega_q^+ := \{x \in \mathbb{R}_+^n, |x| < q\}$.

For any vector $M \in \mathbb{R}_+^n$ we define the set

$$Y_M := \{x \in C[a, \infty)_{\mathbb{T}}, 0 \leq x(t) \leq M, \text{ for all } t \in [a, \infty)_{\mathbb{T}}\}. \quad (5.2)$$

Then Y_M is a closed subset of the Banach space $(C[a, \infty)_{\mathbb{T}}, \|\cdot\|_0)$. Consider the second order dynamic equation

$$(P(t)x^\Delta)^\Delta + f(t, x^\sigma) = 0, \quad t \in [a, \infty)_{\mathbb{T}}. \quad (5.3)$$

In this section we denote by $\lambda_0(t)$ the smallest eigenvalue of the symmetric matrix $P(t)$. Similarly to Remark 4.1, for $t, s \in [a, \infty)_{\mathbb{T}}$ we define the symmetric $n \times n$ matrix

$$Q(t, s) := \int_s^t P^{-1}(\tau) \Delta\tau.$$

We emphasize that in this section the matrix $P(t)$ and the n -vectors $f(t, x)$ or $f(t, x, y)$ have only nonnegative entries.

The following two results generalize [20, Theorems 4.2 and 4.5] to vector valued nonlinearity f and matrix valued leading coefficient $r(t)$.

Theorem 5.1. *Assume that $f : [a, \infty)_{\mathbb{T}} \times \Omega_q^+ \rightarrow \mathbb{R}_+^n$ with $0 < q \leq \infty$, $f \in C_{\text{rd}} \times C$, and $P : [a, \infty)_{\mathbb{T}} \rightarrow \mathbb{R}_+^{n \times n}$, $P \in C_{\text{rd}}$, $P(t)$ positive definite for all $t \in [a, \infty)_{\mathbb{T}}$, are given functions satisfying*

$$\inf_{t \in [a, b]_{\mathbb{T}}} \lambda_0(t) > 0, \quad \text{for all } b \in [a, \infty)_{\mathbb{T}},$$

and the Lipschitz condition (3.3), in which $k : [a, \infty)_{\mathbb{T}} \rightarrow (0, \infty)$, $k \in C_{\text{rd}}$, and

$$\int_a^\infty \|Q(\sigma(s), a)\| k(s) \Delta s < \infty,$$

where the matrix norm $\|\cdot\|$ is compatible with the vector norm $|\cdot|$ and monotone in the sense of condition (5.1). If there exists a vector $M \in \mathbb{R}_+^n$, $|M| < q$, such that

$$\int_a^\infty Q(\sigma(s), a) f(s, x^\sigma(s)) \Delta s \leq M, \quad \text{for all } x \in Y_M, \quad (5.4)$$

where Y_M is defined by (5.2), then the problem (5.3) has a unique solution $x(t)$ on $[a, \infty)_{\mathbb{T}}$ satisfying $x(\infty) = M$ and $(rx^\Delta)(\infty) = 0$. Furthermore,

$$x(t) \geq Q(t, a) P(t) x^\Delta(t), \quad t \in [a, \infty)_{\mathbb{T}}. \quad (5.5)$$

Proof. The proof is a combination of the proofs of Theorem 4.2 and [20, Theorem 4.2]. With the notation $g(s) := f(s, x^\sigma(s))$, the operator

$$[Tx](t) := M - \int_t^\infty Q(\sigma(s), t) g(s) \Delta s, \quad t \in [a, \infty)_{\mathbb{T}},$$

maps the set Y_M into itself. Indeed, $Tx \in C_{\text{rd}}^1[a, \infty)_{\mathbb{T}} \subseteq C[a, \infty)_{\mathbb{T}}$ and since the functions P and f have nonnegative entries, we have

$$[Tx]^\Delta(t) = P^{-1}(t) \int_t^\infty g(s) \Delta s \geq 0, \quad t \in [a, \infty)_{\mathbb{T}}, \quad (5.6)$$

and $[Tx](t) \leq M$ on $[a, \infty)_{\mathbb{T}}$ and $[Tx](a) \geq 0$. This yields that $[Tx](\cdot)$ is nondecreasing so that $[Tx](t) \geq 0$ on $[a, \infty)_{\mathbb{T}}$. Furthermore,

$$0 \leq \int_t^\infty Q(\sigma(s), t) g(s) \Delta s = \int_t^\infty Q(\sigma(s), a) g(s) \Delta s - Q(t, a) \int_t^\infty g(s) \Delta s \quad (5.7)$$

implies that

$$0 \leq Q(t, a) \int_t^\infty g(s) \Delta s \leq \int_t^\infty Q(\sigma(s), a) g(s) \Delta s,$$

so that, by assumption (5.4),

$$\lim_{t \rightarrow \infty} \int_t^\infty Q(\sigma(s), a) g(s) \Delta s = 0 = \lim_{t \rightarrow \infty} Q(t, a) \int_t^\infty g(s) \Delta s. \quad (5.8)$$

Equations (5.7) and (5.8) now yield that

$$\lim_{t \rightarrow \infty} \int_t^\infty Q(\sigma(s), t) g(s) \Delta s = 0,$$

Thus, the definition of Tx implies that $[Tx](\infty) = M$. The contraction property of T is proven similarly as in Theorem 4.2 but with $p(t) := [\|Q(\sigma(t), a)\| + 1] k(t)$. Here $\|\cdot\|$ is the previously discussed matrix norm, for which we have $\|Q(\sigma(s), t)\| \leq \|Q(\sigma(s), a)\|$ for $t \in [a, \infty)_{\mathbb{T}}$ and $s \geq t$.

Thus, the Banach Theorem (Proposition 2.4) yields a unique function $x \in Y_M$ with $Tx = x$, i.e., $x \in C[a, \infty)_{\mathbb{T}}$ and $0 \leq x(t) \leq M$ (componentwise) for all $t \in [a, \infty)_{\mathbb{T}}$. From (5.6) we get $x^\Delta = [Tx]^\Delta \in C_{\text{rd}}[a, \infty)_{\mathbb{T}}$ which implies that $x \in C_{\text{rd}}^1[a, \infty)_{\mathbb{T}}$. Moreover, since $Q(\cdot, a)$ is positive definite and increasing, it follows that either some of its eigenvalues tend monotonically to ∞ or all its eigenvalues are bounded and in this case $Q(t, a)$ converges to some constant matrix $Q_0 > 0$ as $t \rightarrow \infty$. But in any of these two cases the second limit in (5.8) implies that

$$\lim_{t \rightarrow \infty} \int_t^\infty g(s) \Delta s = 0. \quad (5.9)$$

Hence, by (5.6) and (5.9),

$$(Px^\Delta)(\infty) = \lim_{t \rightarrow \infty} P(t) [Tx]^\Delta(t) = \int_t^\infty g(s) \Delta s = 0.$$

Furthermore, from $x = Tx$ we get

$$x(t) \geq M - \int_a^\infty Q(\sigma(s), a) g(s) \Delta s + Q(t, a) P(t) P^{-1}(t) \int_t^\infty g(s) \Delta s,$$

so that by assumption (5.4) and by using formula (5.6) in $x^\Delta(t) = [Tx]^\Delta(t)$ we have inequality (5.5). The proof is now complete. \square

Next we define the set

$$Y_M^1 := \{x \in C_{\text{rd}}^1[a, \infty)_{\mathbb{T}}, \quad 0 \leq x(t) \leq M, \quad x^\Delta(t) \geq 0 \\ \text{for all } t \in [a, \infty)_{\mathbb{T}}, \quad \|x^\Delta\|_0 < \infty\}. \quad (5.10)$$

Then Y_M^1 is a closed subset of the Banach space $(C_{\text{rd}}^1[a, \infty)_{\mathbb{T}}, \|\cdot\|_1)$. Consider the second order dynamic equation

$$(P(t) x^\Delta)^\Delta + f(t, x^\sigma, x^{\Delta\sigma}) = 0, \quad t \in [a, \infty)_{\mathbb{T}}. \quad (5.11)$$

Theorem 5.2. Assume that $f : [a, \infty)_{\mathbb{T}} \times \Omega_q^+ \times \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ with $0 < q \leq \infty$, $f \in C_{\text{rd}} \times C \times C$, and $P : [a, \infty)_{\mathbb{T}} \rightarrow \mathbb{R}_+^{n \times n}$, $P \in C_{\text{rd}}$, $P(t)$ positive definite for all $t \in [a, \infty)_{\mathbb{T}}$, are given functions satisfying

$$\inf_{t \in [a, \infty)_{\mathbb{T}}} \lambda_0(t) \geq r_0 > 0$$

for some number $r_0 > 0$, and the Lipschitz condition (4.21), in which $k : [a, \infty)_{\mathbb{T}} \rightarrow (0, \infty)$, $k \in C_{\text{rd}}$, and

$$\int_a^\infty [\|Q(\sigma(s), a)\| + 1] k(s) \Delta s < \infty,$$

where the matrix norm $\|\cdot\|$ is compatible with the vector norm $|\cdot|$ and monotone in the sense of condition (5.1). If there exists a vector $M \in \mathbb{R}_+^n$, $|M| < q$, such that

$$\begin{aligned} \int_a^\infty Q(\sigma(s), a) f(s, x^\sigma(s), x^{\Delta\sigma}(s)) \Delta s &\leq M, \quad \text{for all } x \in Y_M^1, \\ \sup_{t \in [a, \infty)_{\mathbb{T}}} \left| \int_t^\infty f(s, x^\sigma(s), x^{\Delta\sigma}(s)) \Delta s \right| &< \infty, \quad \text{for all } x \in Y_M^1, \end{aligned}$$

where Y_M^1 is defined by (5.10), then the problem (5.11) has a unique solution $x(t)$ on $[a, \infty)_{\mathbb{T}}$ satisfying $x(\infty) = M$ and $(rx^\Delta)(\infty) = 0$. Furthermore, inequality (5.5) holds.

Proof. The proof is a combination of the proofs of Theorem 4.5 and [20, Theorem 4.5]. The proof of the contraction now uses the function

$$p(t) := \frac{2}{r_0} [\|Q(\sigma(t), a)\| + 1] k(t).$$

The details are omitted. □

6. EXAMPLES

We now present some examples to illustrate how to apply the main ideas of this paper.

Example 6.1. Consider the scalar-valued terminal value problem

$$x^\Delta = \frac{(x^\sigma)^2}{9e^\sigma(t, a)}, \quad t \in [a, \infty)_{\mathbb{T}}.$$

and $x(\infty) = 1$. We claim that this problem has a unique solution x on $[a, \infty)_{\mathbb{T}}$.

Proof. Our objective is to show that the conditions of Theorem 3.2 are satisfied. We choose $q := 3$ to form Ω_q . Next, we see for all $t \in [a, \infty)_{\mathbb{T}}$ and all $u, v \in \Omega_3$ we have

$$\frac{|u^2 - v^2|}{9e^\sigma(t, a)} \leq \frac{6|u - v|}{9e^\sigma(t, a)}$$

and so (3.3) holds for

$$k(t) := \frac{2}{3e^\sigma(t, a)}, \quad t \in [a, \infty)_{\mathbb{T}}.$$

Furthermore, it is not difficult to show that the left-hand side of (3.4) is $\frac{2}{3}$ and so inequality (3.4) holds for the above defined function $k(\cdot)$.

Now choose $N := 2$ to form X_N . See that for all $x \in X_2$ we then have

$$\int_a^\infty \frac{[x^\sigma(s)]^2}{9e^{\sigma(s,a)}} \Delta s \leq \int_a^\infty \frac{4}{9e^{\sigma(s,a)}} \Delta s = \frac{4}{9}$$

and so (3.5) holds. Thus, the claim follows from Theorem 3.2. □

Example 6.2. In this example we illustrate the applicability of Theorem 4.2. Let $n = 1, q = \infty, \mathbb{T} = \mathbb{Z}, a = 0$, and

$$r(t) := \frac{(-1)^t}{t+1}, \quad f(t, x) := \frac{1}{(t+1)^{3+\beta}} \frac{x}{1+x^2}, \quad \text{with } \beta > 0.$$

Then we claim that the assumptions of Theorem 4.2 are satisfied.

Proof. First note that $r(t)$ changes its sign but

$$\inf_{t \in [0, b]_{\mathbb{Z}}} |r(t)| = \inf_{t \in [0, b]_{\mathbb{Z}}} \frac{1}{t+1} = \frac{1}{b+1} > 0 \quad \text{for every } b \in [0, \infty)_{\mathbb{Z}},$$

hence condition (4.2) holds. It follows that

$$R(t, s) = \sum_{i=s}^{t-1} \frac{i+1}{(-1)^i} = \sum_{i=s+1}^t (-1)^{i-1} i, \quad \bar{R}(t, s) = \sum_{i=s+1}^t i = \frac{t(t+1)}{2} - \frac{s(s+1)}{2}.$$

Moreover, $f(t, x)$ is Lipschitz with $k(t) := 1/(t+1)^{3+\beta}$, i.e., condition (3.3) holds. With the estimate $(i+1)(i+2) \leq 2(i+1)^2$, a simple calculation shows that

$$\int_0^\infty \bar{R}(s+1, 0) k(s) \Delta s = \sum_{i=0}^\infty \frac{(i+1)(i+2)}{2(i+1)^{3+\beta}} \leq \sum_{i=0}^\infty \frac{1}{(i+1)^{1+\beta}} < \infty,$$

by the integral criterion for infinite series. Thus, condition (4.5) holds. Finally, since the function $x/(1+x^2)$ is bounded on \mathbb{R} (by $\frac{1}{2}$), we get for any sequence $x \in X_N$ that

$$\begin{aligned} \int_0^\infty \bar{R}(s+1, 0) |f(s, x(s+1))| \Delta s &\leq \frac{1}{2} \sum_{i=0}^\infty \frac{1}{(i+1)^{\beta+1}} = \frac{1}{2} + \frac{1}{2} \int_1^\infty \frac{1}{\tau^{\beta+1}} d\tau \\ &\leq \frac{1}{2} + \frac{1}{2\beta} = \frac{\beta+1}{2\beta}. \end{aligned}$$

Since the above estimate is independent of $x(\cdot)$, we may choose $N := |A| + \frac{\beta+1}{2\beta}$, and then inequality (4.6) holds. Hence, by Theorem 4.2, for any $A \in \mathbb{R}$ the terminal value problem

$$\begin{aligned} \Delta \left(\frac{(-1)^t}{(t+1)^{3+\beta}} \Delta x(t) \right) + \frac{1}{(t+1)^{3+\beta}} \frac{x(t+1)}{1+x^2(t+1)} &= 0, \quad t \in [0, \infty)_{\mathbb{Z}}, \\ \lim_{t \rightarrow \infty} x(t) = A, \quad \lim_{t \rightarrow \infty} \frac{(-1)^t}{(t+1)^{3+\beta}} \Delta x(t) &= 0, \end{aligned}$$

has a unique solution $x(\cdot)$ on $[0, \infty)_{\mathbb{Z}}$. □

Example 6.3. Note that the leading coefficient $r(t)$ from Example 6.2 is not allowed in Theorem 4.5, since

$$\inf_{t \in [0, \infty)_{\mathbb{Z}}} |r(t)| = \inf_{t \in [0, \infty)_{\mathbb{Z}}} \frac{1}{t+1} = 0,$$

contradicting condition (4.20). However, one can consider a leading coefficient such as $r(t) = (-1)^t$ on $[0, \infty)_{\mathbb{Z}}$, which is admissible in Theorem 4.5.

Example 6.4. In this example we illustrate the applicability of Theorem 4.5. Let $n = 1$, $q = \infty$, $\mathbb{T} = \mathbb{R}$, $a = 0$, and

$$r(t) := 1, \quad f(t, x, y) := \frac{1}{(t+1)^{2+\beta}} \left(\cos x + \frac{\sin y}{1 + \sin^2 y} \right), \quad \text{with } \beta > 0.$$

Then we claim that the assumptions of Theorem 4.5 are satisfied.

Proof. We have $R(t, s) = \bar{R}(t, s) = \int_s^t 1 \, d\tau = t - s$, and the Lipschitz condition (4.21) is satisfied with the function $k(t) := 1/(t+1)^{2+\beta}$. Conditions (4.22) and (4.23) are verified similarly as in Example 6.2. Condition (4.24) follows from (4.25) in Remark 4.6(iii) and from the estimate

$$\int_0^\infty |f(s, x(s), x'(s))| \, ds \leq \frac{3}{2} \int_0^\infty \frac{1}{(s+1)^{2+\beta}} \, ds = \frac{3}{2(\beta+1)} < \infty$$

for every $x \in X_N^1$, since the function $\cos x + \frac{\sin y}{1 + \sin^2 y}$ is bounded on \mathbb{R} (by $\frac{3}{2}$). Hence, by Theorem 4.5, for any $A \in \mathbb{R}$ the terminal value problem

$$x'' + \frac{1}{(t+1)^{2+\beta}} \left(\cos x + \frac{\sin x'}{1 + \sin^2 x'} \right) = 0, \quad t \in [0, \infty),$$

$$\lim_{t \rightarrow \infty} x(t) = A, \quad \lim_{t \rightarrow \infty} x'(t) = 0,$$

has a unique solution $x(\cdot)$ on $[0, \infty)$. □

The two examples above motivate the following corollaries of Theorems 4.2 and 4.5, in which the existence of the number N is guaranteed from the assumed estimates on the data.

Corollary 6.5. *Assume that $g : \Omega_q \rightarrow \mathbb{R}^n$ with $0 < q \leq \infty$, $g \in C^1$, and $r : [a, \infty)_{\mathbb{T}} \rightarrow \mathbb{R}$, $r \in C_{rd}$, and $k : [a, \infty)_{\mathbb{T}} \rightarrow (0, \infty)$, $k \in C_{rd}$, are given functions satisfying condition (4.2),*

$$\int_a^\infty \bar{R}(\sigma(s), a) k(s) \Delta s \leq k_1 < \infty, \tag{6.1}$$

$|g(x)| \leq M_1$ on Ω_q , and $g'(x)$ is bounded on Ω_q . Then for any $A \in \mathbb{R}^n$ such that $|A| < q - M_1 k_1$ (in particular, for any $A \in \mathbb{R}^n$ if $q = \infty$) the problem

$$(r(t) x^\Delta)^\Delta + k(t) g(x^\sigma) = 0, \quad t \in [a, \infty)_{\mathbb{T}},$$

has a unique solution $x(\cdot)$ on $[a, \infty)_{\mathbb{T}}$ satisfying $x(\infty) = A$ and $(rx^\Delta)(\infty) = 0$.

Proof. Upon taking $f(t, x) := k(t) g(x)$ and $N := |A| + M_1 k_1$ we show that these data satisfy the assumptions of Theorem 4.2. □

Corollary 6.6. *Assume that $g : \Omega_q \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $0 < q \leq \infty$, $g \in C^1$, and $r : [a, \infty)_{\mathbb{T}} \rightarrow \mathbb{R}$, $r \in C_{rd}$, and $k : [a, \infty)_{\mathbb{T}} \rightarrow (0, \infty)$, $k \in C_{rd}$, are given functions satisfying conditions (4.20), (6.1), (3.4), and $|g(x, y)| \leq M_1$ on $\Omega_q \times \mathbb{R}^n$, and $g_x(x, y)$ and $g_y(x, y)$ are bounded on $\Omega_q \times \mathbb{R}^n$. Then for any $A \in \mathbb{R}^n$ such that $|A| < q - M_1 k_1$ (in particular, for any $A \in \mathbb{R}^n$ if $q = \infty$) the problem*

$$(r(t) x^\Delta)^\Delta + k(t) g(x^\sigma, x^{\Delta\sigma}) = 0, \quad t \in [a, \infty)_{\mathbb{T}},$$

has a unique solution $x(\cdot)$ on $[a, \infty)_{\mathbb{T}}$ satisfying $x(\infty) = A$ and $(rx^\Delta)(\infty) = 0$.

Proof. Upon taking $f(t, x, y) := k(t)g(x, y)$ and $N := |A| + M_1 k_1$ we show that these data satisfy the assumptions of Theorem 4.5. Note that condition (4.25) from Remark 4.6(iii) is used in order to verify condition (4.24). \square

In Example 6.2 we had $k(t) = 1/(t+1)^{3+\beta}$, $k_1 = \frac{\beta+1}{\beta}$, $g(x) = x/(1+x^2)$, and $M_1 = \frac{1}{2}$, while in Example 6.4 we had $k(t) = 1/(t+1)^{2+\beta}$, $k_1 = \frac{1}{\beta}$, $g(x, y) = \cos x + \frac{\sin y}{1+\sin^2 y}$, and $M_1 = \frac{3}{2}$.

7. FURTHER APPLICATIONS AND EXTENSIONS

Nabla dynamic equations, see e.g. [6], can be considered as a dual version of Δ -differential equations. The backward graininess function is denoted by $\nu(t) := t - \rho(t)$. The spaces of ld-continuous and ld-continuously ∇ -differentiable functions are accordingly denoted by C_{ld} and C_{ld}^1 . Similarly to Definition 2.1 we have the notion of $f \in C_{\text{ld}} \times C \times C$, and results corresponding to Propositions 2.2 and 2.3 now hold true for the ∇ -setting. The ∇ -exponential function having the properties concluded in Lemma 2.5 corresponds to the dynamic equation $u^\nabla = -p(t)u$, see [6, Section 3.2]. Consequently, all the results of this paper extend directly to the corresponding results for the ∇ -differential equations. As examples of such results we have the following.

Theorem 7.1. *Assume that $f : [a, \infty)_{\mathbb{T}} \times \Omega_q \rightarrow \mathbb{R}^n$ with $0 < q \leq \infty$ is a function satisfying $f \in C_{\text{ld}} \times C$ and the Lipschitz condition (3.3), where $k : [a, \infty)_{\mathbb{T}} \rightarrow (0, \infty)$, $k \in C_{\text{ld}}$, and*

$$\int_a^\infty k(s) \nabla s < \infty. \quad (7.1)$$

Let $A \in \mathbb{R}^n$ be a given vector. If there exists a number $N \in \mathbb{R}$, $|A| \leq N < q$, such that

$$\int_a^\infty |f(s, x(s))| \nabla s \leq N - |A|, \quad \text{for all } x \in X_N,$$

where X_N is defined by (3.2), then the problem

$$x^\nabla + f(t, x) = 0, \quad t \in [a, \infty)_{\mathbb{T}}. \quad (7.2)$$

has a unique solution $x(t)$ on $[a, \infty)_{\mathbb{T}}$ satisfying $x(\infty) = A$.

Corollary 7.2. *Assume that $f : [a, \infty)_{\mathbb{T}} \times \Omega_q \rightarrow \mathbb{R}^n$ with $0 < q \leq \infty$ is a function satisfying $f \in C_{\text{ld}} \times C$ and the Lipschitz condition (3.3), where $k : [a, \infty)_{\mathbb{T}} \rightarrow (0, \infty)$, $k \in C_{\text{ld}}$, and (7.1) holds. If there exists a vector $M \in \mathbb{R}^n$, $|M| < q$, such that*

$$\int_a^\infty |f(s, x(s))| \nabla s \leq |M|, \quad \text{for all } x \in X_{2|M|},$$

then the problem (7.2) has a unique solution $x(t)$ on $[a, \infty)_{\mathbb{T}}$ satisfying $x(\infty) = M$.

The corresponding results to Theorems 4.2, 4.5, 5.1, 5.2 and Corollaries 6.5, 6.6 now hold under appropriate assumptions for the second order ∇ -dynamic equations

$$\begin{aligned}(r(t)x^\nabla)^\nabla + f(t, x) &= 0, & t \in [a, \infty)_\mathbb{T}, \\ (r(t)x^\nabla)^\nabla + f(t, x, x^\nabla) &= 0, & t \in [a, \infty)_\mathbb{T}, \\ (P(t)x^\nabla)^\nabla + f(t, x) &= 0, & t \in [a, \infty)_\mathbb{T}, \\ (P(t)x^\nabla)^\nabla + f(t, x, x^\nabla) &= 0, & t \in [a, \infty)_\mathbb{T}, \\ (r(t)x^\nabla)^\nabla + k(t)g(x) &= 0, & t \in [a, \infty)_\mathbb{T}, \\ (r(t)x^\nabla)^\nabla + k(t)g(x, x^\nabla) &= 0, & t \in [a, \infty)_\mathbb{T},\end{aligned}$$

respectively.

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ROMAN HILSCHER

DEPARTMENT OF MATHEMATICS AND STATISTICS, FACULTY OF SCIENCE, MASARYK UNIVERSITY,
JANÁČKOVO NÁM. 2A, CZ-60200 BRNO, CZECH REPUBLIC

E-mail address: hilscher@math.muni.cz

CHRISTOPHER C. TISDELL

SCHOOL OF MATHEMATICS AND STATISTICS, THE UNIVERSITY OF NEW SOUTH WALES, SYDNEY
NSW 2052, AUSTRALIA

E-mail address: cct@unsw.edu.au