Electronic Journal of Differential Equations, Vol. 2008(2008), No. 62, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

ON BOUNDARY-VALUE PROBLEMS FOR HIGHER-ORDER DIFFERENTIAL INCLUSIONS

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Abstract. We show the existence of solutions to boundary-value problems for higher-order differential inclusion $x^{(n)}(t) \in F(t, x(t))$, where $F(.,.)$ is a closed multifunction, measurable in t and Lipschitz continuous in x. We use the fixed point theorem introduced by Covitz and Nadler for contraction multivalued maps.

1. Introduction

The aim of this paper is to establish the existence of solutions of the following higher-order boundary-value problems:

- For $n \geq 2$

$$\begin{gathered} x^{(n)}(t) \in F(t, x(t)) \quad \text{a.e. on } [0,1]; \\ x^{(i)}(0) = 0, \quad 0 \leq i \leq n-2; \\ x(\eta) = x(1). \end{gathered} \tag{1.1}$$

- For $n \geq 2$

$$\begin{gathered} x^{(n)}(t) \in F(t, x(t)) \quad \text{a.e. on } [0,1]; \\ x(0) = x'(\eta); \quad x(1) = x(\tau). \end{gathered} \tag{1.2}$$

- For $n \geq 4$

$$\begin{gathered} x^{(n)}(t) \in F(t, x(t)) \quad \text{a.e. on } [0,1]; \\ x^{(i)}(0) = x^{(i+1)}(\eta), \quad 2 \leq i \leq n-2; \\ x(0) = x'(\eta); \quad x(1) = x(\tau). \end{gathered} \tag{1.3}$$

- For $n \geq 2$

$$\begin{gathered} x^{(n)}(t) \in F(t, x(t)) \quad \text{a.e. on } [0,1]; \\ x^{(i)}(0) = x^{(i+1)}(\eta), \quad 0 \leq i \leq n-2. \end{gathered} \tag{1.4}$$

where $F : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}}$ is a closed multivalued map, measurable with respect to the first argument and Lipschitz with respect to the second argument, and $(\eta, \tau) \in]0,1[^2$.

2000 *Mathematics Subject Classification.* 34A60, 34B10, 34B15.
Key words and phrases. Boundary value problems; contraction; measurability; multifunction.

Submitted March 14, 2007. Published April 22, 2008.

Three and four-point boundary-value problems for second-order differential inclusions was initiated by Benchohra and Ntouyas, see [4, 5, 6]. The authors investigate the existence of solutions on compact intervals for the problems (1.1) and (1.2) in the particular case $n = 2$. In order to obtain solutions of (1.1) and (1.2) when F is not necessarily convex values, Benchohra and Ntouyas (see [6]) reduce the existence of solutions to the search for fixed points of a suitable multivalued map on the Banach space $\mathcal{C}([0,1],\mathbb{R})$. Indeed, they used the fixed point theorem for contraction multivalued maps, due to Covitz and Nadler [3].

In this paper, we give an extension of the Benchohra and Ntouyas's result [6] to the $n-$order non-convex boundary-value problems and we prove the existence of solutions for (1.3) and (1.4). We shall adopt the technique used by Benchohra and Ntouyas in the previous paper.

2. Preliminaries and statement of the main results

Let (E,d) be a complete metric space. We denote by $\mathcal{C}([0,1],E)$ the Banach space of continuous functions from $[0,1]$ to E with the norm $\|x(.)\|_\infty := \sup\big\{\|x(t)\|; t \in [0,1]\big\}$, where $\|\cdot\|$ is the norm of E. For $x \in E$ and for nonempty sets A, B of E we denote $d(x,A) = \inf\{d(x,y); y \in A\}$, $e(A,B) := \sup\{d(x,B); x \in A\}$ and $H(A,B) := \max\{e(A,B), e(B,A)\}$. A multifunction is said to be measurable if its graph is measurable. For more detail on measurability theory, we refer the reader to the book of Castaing and Valadier [2].

Definition 2.1. Let $T : E \to 2^E$ be a multifunction with closed values.

(1) T is k-Lipschitz if and only if
$$H\big(T(x), T(y)\big) \le kd(x,y), \quad \text{for each } x, y \in E.$$
(2) T is a contraction if and only if it is k-Lipschitz with $k < 1$.
(3) T has a fixed point if there exists $x \in E$ such that $x \in T(x)$.

Let us recall the following results that will be used in the sequel.

Lemma 2.2. [3] *If $T : E \to 2^E$ is a contraction with nonempty closed values, then it has a fixed point.*

Lemma 2.3. [7] *Assume that $F : [a,b] \times \mathbb{R} \to 2^{\mathbb{R}}$ is a multifunction with nonempty closed values satisfying:*

- *For every $x \in \mathbb{R}$, $F(.,x)$ is measurable on $[a,b]$;*
- *For every $t \in [a,b]$, $F(t,.)$ is (Hausdorff) continuous on $\mathbb{R}$.*

Then for any measurable function $x(.) : [a,b] \to \mathbb{R}$, the multifunction $F(.,x(.))$ is measurable on $[a,b]$.

Definition 2.4. A function $x(.) : [0,1] \to \mathbb{R}$ is said to be a solution of (1.1) (resp. (1.2), (1.3), (1.4)) if $x(.)$ is $(n-1)$-times differentiable, $x^{(n-1)}(.)$ is absolutely continuous and $x(.)$ satisfies the conditions of (1.1) (resp. (1.2), (1.3), (1.4)).

Let $\eta \in \mathbb{R}$ and $n \in \mathbb{N} \setminus \{0,1\}$. Define a sequence of functions $(\varphi_p(.))_{2 \le p \le n}$ by: For all $t \in [0,1]$
$$\varphi_2(t) = 1;$$
$$\varphi_3(t) = t + \varphi_2(\eta);$$

$$\varphi_p(t) = \frac{t^{p-2}}{(p-2)!} + \sum_{k=3}^{p-1} \varphi_{k-1}(\eta)\frac{t^{p-k}}{(p-k)!} + \varphi_{p-1}(\eta).$$

We remark that

(a) For $t \in [0,1]$ and $k \in \{0,\dots,n-2\}$, $\varphi_n^{(k)}(t) = \varphi_{n-k}(t)$;
(b) For $k \in \{0,\dots,n-3\}$, $\varphi_{n-k}(0) = \varphi_{n-k-1}(\eta)$;
(c) For $k \in \{0,\dots,n-2\}$ the function $\varphi_n^{(k)}(.)$ is increasing.

Assumptions. We will use the following hypotheses:

(H1) $F : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}}$ is a multivalued map with nonempty closed values satisfying
  (i) For each $x \in \mathbb{R}$, $t \mapsto F(t,x)$ is measurable;
  (ii) There exists a function $m(.) \in L^1([0,1],\mathbb{R}^+)$ such that for all $t \in [0,1]$ and for all $x_1, x_2 \in \mathbb{R}$,
$$H\big(F(t,x_1), F(t,x_2)\big) \;\le\; m(t)|x_1 - x_2|.$$

(H2) For $\eta \in]0,1[$,
$$\frac{1}{(n-1)!}\Big(L(1) + \frac{L(\eta)+L(1)}{1-\eta^{n-1}}\Big) \;<\; 1$$
where $L(t) = \int_0^t m(s)ds$ for all $t \in [0,1]$;

(H3) For $(\eta,\tau) \in]0,1[^2$,
$$\frac{(3-\tau)L(1) + 2L(\tau)}{(1-\tau)(n-1)!} + \sum_{k=0}^{n-2} \frac{L(\eta)}{(1-\tau)k!}\big[(3-\tau)\varphi_n^{(k)}(1) + 2\varphi_n^{(k)}(\tau)\big] \;<\; 1;$$

(H4) For $\eta \in]0,1[$,
$$\frac{L(1)}{(n-1)!} + L(\eta)\sum_{k=0}^{n-2} \frac{\varphi_n^{(k)}(1)}{k!} \;<\; 1.$$

Main results. We shall prove the following results.

Theorem 2.5. *If assumptions* (H1) *and* (H2) *are satisfied, then problem* (1.1) *has at least one solution on* $[0,1]$.

Theorem 2.6. *If assumptions* (H1) *and* (H3) *are satisfied, then problems* (1.2) *and* (1.3) *have at least one solution on* $[0,1]$.

Theorem 2.7. *If assumptions* (H1) *and* (H4) *are satisfied, then problem* (1.4) *has at least one solution on* $[0,1]$.

3. Proof of the main results

Proof of Theorem 2.5. For $y(.) \in \mathcal{C}([0,1],\mathbb{R})$, set
$$S_{F,y(.)} := \big\{g \in L^1([0,1],\mathbb{R}) : g(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,1]\big\}.$$
By Lemma 2.3, for $y(.) \in \mathcal{C}([0,1],\mathbb{R})$, $F(.,y(.))$ is closed and measurable, then it has a selection. Thus $S_{F,y(.)}$ is nonempty. Let us transform the problem into a fixed point problem. Consider the multivalued map $T : \mathcal{C}([0,1],\mathbb{R}) \to 2^{\mathcal{C}([0,1],\mathbb{R})}$ defined

as follows: for $y(.) \in L^1([0,1],\mathbb{R})$, $T(y(.))$ is the set of all $z(.) \in \mathcal{C}([0,1],\mathbb{R})$, such that

$$\begin{aligned} z(t) = &\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} g(s)ds + \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} g(s)ds \\ &- \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} g(s)ds, \end{aligned}$$

where $g \in S_{F,y(.)}$.

We shall show that T satisfies the assumptions of Lemma 2.2. The proof will be given in two steps:

Step 1: *T has non-empty closed values.* Indeed, let $(y_p(.))_{p\geq 0} \in T(y(.))$ converges to $\bar{y}(.)$ in $\mathcal{C}([0,1],\mathbb{R})$. Then $\bar{y}(.) \in \mathcal{C}([0,1],\mathbb{R})$ and for each $t \in [0,1]$,

$$\begin{aligned} y_p(t) \in &\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} F(s,y(s))ds + \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} F(s,y(s))ds \\ &- \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} F(s,y(s))ds. \end{aligned}$$

Since the sets

$$\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} F(s,y(s))ds\,, \quad \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} F(s,y(s))ds\,,$$
$$\frac{t^{n-1}}{1-\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} F(s,y(s))ds$$

are closed for all $t \in [0,1]$, we have

$$\begin{aligned} \bar{y}(t) \in &\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} F(s,y(s))ds + \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} F(s,y(s))ds \\ &- \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} F(s,y(s))ds. \end{aligned}$$

Then $\bar{y}(.) \in T(y(.))$. So $T(y(.))$ is closed for each $y(.) \in \mathcal{C}([0,1],\mathbb{R})$.

Step 2: *T is a contraction.* Indeed, let $y_1(.), y_2(.) \in \mathcal{C}([0,1],\mathbb{R})$ and $z_1(.) \in T(y_1(.))$. Then

$$\begin{aligned} z_1(t) = &\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} g_1(s)ds + \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} g_1(s)ds \\ &- \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} g_1(s)ds, \end{aligned}$$

where $g_1 \in S_{F,y_1(.)}$. Consider the multivalued map $U : [0,1] \to 2^{\mathbb{R}}$, defined by

$$U(t) = \big\{x \in \mathbb{R} : |g_1(t) - x| \le m(t)|y_1(t) - y_2(t)|\big\}.$$

For each $t \in [0,1]$, $U(t)$ is nonempty. Indeed, let $t \in [0,1]$, from (H1) we have

$$H\big(F(t,y_1(t)), F(t,y_2(t))\big) \le m(t)|y_1(t) - y_2(t)|.$$

Hence, there exists $x \in F(t, y_2(t))$, such that

$$|g_1(t) - x| \le m(t)|y_1(t) - y_2(t)|.$$

By [2, Proposition III.4], the multifunction

$$V : t \to U(t) \cap F(t, y_2(t)) \tag{3.1}$$

is measurable. Then there exists a measurable selection of V denoted g_2 such that

$$g_2(t) \in F(t, y_2(t)) \quad \text{and} \quad |g_1(t) - g_2(t)| \le m(t)|y_1(t) - y_2(t)|, \quad \text{for each } t \in [0,1].$$

Now, for $t \in [0,1]$ set

$$\begin{aligned} z_2(t) = &\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} g_2(s)ds + \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} g_2(s)ds \\ &- \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} g_2(s)ds. \end{aligned}$$

Then

$$\begin{aligned} |z_1(t) - z_2(t)| \le &\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} |g_1(t) - g_2(s)|ds \\ &+ \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} |g_1(s) - g_2(s)|ds \\ &+ \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} |g_1(s) - g_2(s)|ds \\ \le &\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} m(s)|y_1(s) - y_2(s)|ds \\ &+ \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} m(s)|y_1(s) - y_2(s)|ds \\ &+ \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} m(s)|y_1(s) - y_2(s)|ds \\ \le &\frac{1}{(n-1)!} \|y_1(.) - y_2(.)\|_\infty \int_0^1 m(s)ds \\ &+ \frac{1}{(1-\eta^{n-1})(n-1)!} \|y_1(.) - y_2(.)\|_\infty \int_0^\eta m(s)ds \\ &+ \frac{1}{(1-\eta^{n-1})(n-1)!} \|y_1(.) - y_2(.)\|_\infty \int_0^1 m(s)ds \\ \le &\frac{1}{(n-1)!} \Big(L(1) + \frac{L(\eta) + L(1)}{1-\eta^{n-1}}\Big) \|y_1(.) - y_2(.)\|_\infty. \end{aligned}$$

So, we conclude that

$$\|z_1(.) - z_2(.)\|_\infty \le \frac{1}{(n-1)!} \Big(L(1) + \frac{L(\eta) + L(1)}{1-\eta^{n-1}}\Big) \|y_1(.) - y_2(.)\|_\infty.$$

By the analogous relation, obtained by interchanging the roles of $y_1(.)$ and $y_2(.)$, it follows that

$$H\big(T(y_1(.)), T(y_2(.))\big) \le \frac{1}{(n-1)!} \Big(L(1) + \frac{L(\eta) + L(1)}{1-\eta^{n-1}}\Big) \|y_1(.) - y_2(.)\|_\infty.$$

Consequently, T is a contraction. Hence, by Lemma 2.2, T has a fixed point $y(.)$.

Proposition 3.1. *$y(.)$ is a solution of* (1.1).

Proof. We have

$$y(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} g(s)ds + \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} g(s)ds$$

$$- \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} g(s)ds,$$

where $g \in S_{F,y(.)}$. Then

$$\begin{aligned} y(\eta) &= \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} g(s)ds + \frac{\eta^{n-1}}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} g(s)ds \\ &\quad - \frac{\eta^{n-1}}{1-\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} g(s)ds \\ &= \frac{1}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} g(s)ds - \frac{\eta^{n-1}}{1-\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} g(s)ds \end{aligned}$$

and

$$\begin{aligned} y(1) &= \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} g(s)ds + \frac{1}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} g(s)ds \\ &\quad - \frac{1}{1-\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} g(s)ds \\ &= \frac{1}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} g(s)ds - \frac{\eta^{n-1}}{1-\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} g(s)ds, \end{aligned}$$

hence $y(1) = y(\eta)$. On the other hand, for $0 \le i \le n-2$, we have

$$\begin{aligned} y^{(i)}(t) &= \int_0^t \frac{(t-s)^{n-i-1}}{(n-i-1)!} g(s)ds + \frac{(n-1)\dots(n-i)t^{n-i-1}}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} g(s)ds \\ &\quad - \frac{(n-1)\dots(n-i)t^{n-i-1}}{1-\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} g(s)ds, \end{aligned}$$

hence $y^{(i)}(0) = 0$. Finally, it is clear that $y^{(n)}(t) = g(t)$, so $y^{(n)}(t) \in F(t, y(t))$. $\square$

Proof of Theorem 2.6. We transform the problem into a fixed point problem. For $t \in [0,1]$, set

$$\psi_n^g(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} g(s)ds + \sum_{k=0}^{n-2} \varphi_n^{(k)}(t) \int_0^\eta \frac{(\eta-s)^k}{k!} g(s)ds,$$

where $g \in S_{F,y(.)}$. Consider the multivalued map, $T : \mathcal{C}([0,1],\mathbb{R}) \to 2^{\mathcal{C}([0,1],\mathbb{R})}$ defined as follows: for $y(.) \in \mathcal{C}([0,1],\mathbb{R})$,

$$T(y(.)) := \big\{ z(.) \in \mathcal{C}([0,1],\mathbb{R}) : z(t) = \psi_n^g(t) + \frac{1+t}{1-\tau}\big(\psi_n^g(\tau) - \psi_n^g(1)\big)\big\}.$$

We shall show that T satisfies the assumptions of Lemma 2.2. The proof will be given in two steps:

Step 1: *T has non-empty closed values.* Indeed, let $(y_p(.))_{p\ge 0} \in T(y(.))$ converges to $\bar{y}(.)$ in $\mathcal{C}([0,1],\mathbb{R})$. Then $\bar{y}(.) \in \mathcal{C}([0,1],\mathbb{R})$ and for each $t \in [0,1]$,

$$\begin{aligned} y_p(t) &\in \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} F(s,y(s))ds + \sum_{k=0}^{n-2} \varphi_n^{(k)}(t) \int_0^\eta \frac{(\eta-s)^k}{k!} F(s,y(s))ds \\ &\quad + \frac{1+t}{1-\tau}\Big[\int_0^\tau \frac{(\tau-s)^{n-1}}{(n-1)!} F(s,y(s))ds \end{aligned}$$

$$+\sum_{k=0}^{n-2}\varphi_n^{(k)}(\tau)\int_0^\eta \frac{(\eta-s)^k}{k!}F(s,y(s))ds$$
$$-\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}F(s,y(s))ds-\sum_{k=0}^{n-2}\varphi_n^{(k)}(1)\int_0^\eta \frac{(\eta-s)^k}{k!}F(s,y(s))ds\Big].$$

Since the set

$$\int_0^t \frac{(t-s)^k}{k!}F(s,y(s))ds$$

is closed for all $t\in[0,1]$ and $0\le k\le n-1$, we have

$$\begin{aligned}\bar{y}(t)\in&\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}F(s,y(s))ds+\sum_{k=0}^{n-2}\varphi_n^{(k)}(t)\int_0^\eta \frac{(\eta-s)^k}{k!}F(s,y(s))ds\\&+\frac{1+t}{1-\tau}\Big[\int_0^\tau \frac{(\tau-s)^{n-1}}{(n-1)!}F(s,y(s))ds\\&+\sum_{k=0}^{n-2}\varphi_n^{(k)}(\tau)\int_0^\eta \frac{(\eta-s)^k}{k!}F(s,y(s))ds\\&-\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}F(s,y(s))ds-\sum_{k=0}^{n-2}\varphi_n^{(k)}(1)\int_0^\eta \frac{(\eta-s)^k}{k!}F(s,y(s))ds\Big].\end{aligned}$$

Then $\bar{y}(.)\in T(y(.))$. So $T(y(.))$ is closed for each $y(.)\in\mathcal{C}([0,1],\mathbb{R})$.
Step 2: *T is a contraction.* Indeed, let $y_1(.),y_2(.)\in\mathcal{C}([0,1],\mathbb{R})$ and $z_1(.)\in T(y_1(.))$. Then

$$z_1(t)=\psi_n^{g_1}(t)+\frac{1+t}{1-\tau}\big(\psi_n^{g_1}(\tau)-\psi_n^{g_1}(1)\big),$$

where $g_1\in S_{F,y_1(.)}$. By (3.1), there exists g_2 such that

$$g_2(t)\in F(t,y_2(t))\quad\text{and}\quad |g_1(t)-g_2(t)|\le m(t)|y_1(t)-y_2(t)|,\quad\text{for each } t\in[0,1].$$

Now, set for all $t\in[0,1]$,

$$z_2(t)=\psi_n^{g_2}(t)+\frac{1+t}{1-\tau}\big(\psi_n^{g_2}(\tau)-\psi_n^{g_2}(1)\big).$$

On the other hand, we have

$$\begin{aligned}|\psi_n^{g_2}(t)-\psi_n^{g_1}(t)|&\le\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}|g_1(s)-g_2(s)|ds\\&\quad+\sum_{k=0}^{n-2}\varphi_n^{(k)}(t)\int_0^\eta \frac{(\eta-s)^k}{k!}|g_1(s)-g_2(s)|ds\\&\le\frac{1}{(n-1)!}\int_0^t m(s)|y_1(s)-y_2(s)|ds\\&\quad+\sum_{k=0}^{n-2}\varphi_n^{(k)}(t)\frac{1}{k!}\int_0^\eta m(s)|y_1(s)-y_2(s)|ds\\&\le\frac{1}{(n-1)!}\|y_1(.)-y_2(.)\|_\infty\int_0^t m(s)ds\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=0}^{n-2}\varphi_n^{(k)}(t)\frac{1}{k!}\|y_1(.)-y_2(.)\|_\infty\int_0^\eta m(s)ds\\
&\le\Big(\frac{L(1)}{(n-1)!}+L(\eta)\sum_{k=0}^{n-2}\frac{\varphi_n^{(k)}(1)}{k!}\Big)\|y_1(.)-y_2(.)\|_\infty .
\end{aligned}
$$

Then, by (c)

$$
\begin{aligned}
|z_2(t)-z_1(t)| &\le |\psi_n^{g_2}(t)-\psi_n^{g_1}(t)|+\frac{1+t}{1-\tau}\Big[|\psi_n^{g_2}(\tau)-\psi_n^{g_1}(\tau)|+|\psi_n^{g_2}(1)-\psi_n^{g_1}(1)|\\
&\le\Big(\frac{L(1)}{(n-1)!}+L(\eta)\sum_{k=0}^{n-2}\frac{\varphi_n^{(k)}(1)}{k!}\Big)\|y_1(.)-y_2(.)\|_\infty\\
&\quad+\frac{2}{1-\tau}\Big[\Big(\frac{L(\tau)}{(n-1)!}+L(\eta)\sum_{k=0}^{n-2}\frac{\varphi_n^{(k)}(\tau)}{k!}\Big)\|y_1(.)-y_2(.)\|_\infty\\
&\quad+\Big(\frac{L(1)}{(n-1)!}+L(\eta)\sum_{k=0}^{n-2}\frac{\varphi_n^{(k)}(1)}{k!}\Big)\|y_1(.)-y_2(.)\|_\infty\Big]\\
&\le\Big[\frac{(3-\tau)L(1)+2L(\tau)}{(1-\tau)(n-1)!}\\
&\quad+\sum_{k=0}^{n-2}\frac{L(\eta)}{(1-\tau)k!}\big[(3-\tau)\varphi_n^{(k)}(1)+2\varphi_n^{(k)}(\tau)\big]\Big]\|y_1(.)-y_2(.)\|_\infty .
\end{aligned}
$$

By the analogous relation, obtained by interchanging the roles of $y_1(.)$ and $y_2(.)$, it follows that

$$
\begin{aligned}
H\big(T(y_1(.)),T(y_2(.))\big)\le\Big[\frac{(3-\tau)L(1)+2L(\tau)}{(1-\tau)(n-1)!}+\sum_{k=0}^{n-2}\frac{L(\eta)}{(1-\tau)k!}\big[(3-\tau)\varphi_n^{(k)}(1)\\
+2\varphi_n^{(k)}(\tau)\big]\Big]\|y_1(.)-y_2(.)\|_\infty .
\end{aligned}
$$

Consequently, T is a contraction. Thus, by Lemma 2.2, T has a fixed point $y(.)$.

Proposition 3.2. *$y(.)$ is a solution of* (1.2) *and* (1.3).

Proof. We have

$$
y(t)=\psi_n^g(t)+\frac{1+t}{1-\tau}\big(\psi_n^g(\tau)-\psi_n^g(1)\big),
$$

where $g\in S_{F,y(.)}$. Then

$$
y(1)=\psi_n^g(1)+\frac{2}{1-\tau}\big(\psi_n^g(\tau)-\psi_n^g(1)\big)=\frac{-1-\tau}{1-\tau}\psi_n^g(1)+\frac{2}{1-\tau}\psi_n^g(\tau)
$$

and

$$
y(\tau)=\psi_n^g(\tau)+\frac{1+\tau}{1-\tau}\big(\psi_n^g(\tau)-\psi_n^g(1)\big)=\frac{-1-\tau}{1-\tau}\psi_n^g(1)+\frac{2}{1-\tau}\psi_n^g(\tau),
$$

hence $y(1)=y(\tau)$. On the other hand, for $0\le i\le n-2$ and $t\in[0,1]$, we have

$$
[\psi_n^g]^{(i)}(t)=\int_0^t\frac{(t-s)^{n-i-1}}{(n-i-1)!}g(s)ds+\sum_{k=0}^{n-2}\varphi_n^{(k+i)}(t)\int_0^\eta\frac{(\eta-s)^k}{k!}g(s)ds
$$

$$
\begin{aligned}
&= \int_0^t \frac{(t-s)^{n-i-1}}{(n-i-1)!} g(s)ds + \sum_{l=i}^{n+i-2} \varphi_n^{(l)}(t) \int_0^\eta \frac{(\eta-s)^{l-i}}{(l-i)!} g(s)ds \\
&= \int_0^t \frac{(t-s)^{n-i-1}}{(n-i-1)!} g(s)ds + \sum_{l=i}^{n-2} \varphi_n^{(l)}(t) \int_0^\eta \frac{(\eta-s)^{l-i}}{(l-i)!} g(s)ds.
\end{aligned}
$$

Then, by (a) and (b)

$$
\begin{aligned}
[\psi_n^g]^{(i)}(0) &= \int_0^\eta \frac{(\eta-s)^{n-i-2}}{(n-i-2)!} g(s)ds + \sum_{l=i}^{n-3} \varphi_n^{(l)}(0) \int_0^\eta \frac{(\eta-s)^{l-i}}{(l-i)!} g(s)ds \\
&= \int_0^\eta \frac{(\eta-s)^{n-i-2}}{(n-i-2)!} g(s)ds + \sum_{l=i}^{n-3} \varphi_{n-l-1}(\eta) \int_0^\eta \frac{(\eta-s)^{l-i}}{(l-i)!} g(s)ds
\end{aligned}
$$

and by (a)

$$
\begin{aligned}
[\psi_n^g]^{(i+1)}(\eta) &= \int_0^\eta \frac{(\eta-s)^{n-i-2}}{(n-i-2)!} g(s)ds + \sum_{l=i}^{n-3} \varphi_n^{(l+1)}(\eta) \int_0^\eta \frac{(\eta-s)^{l-i}}{(l-i)!} g(s)ds \\
&= \int_0^\eta \frac{(\eta-s)^{n-i-2}}{(n-i-2)!} g(s)ds + \sum_{l=i}^{n-3} \varphi_{n-l-1}(\eta) \int_0^\eta \frac{(\eta-s)^{l-i}}{(l-i)!} g(s)ds,
\end{aligned}
$$

consequently

$$
[\psi_n^g]^{(i+1)}(\eta) = [\psi_n^g]^{(i)}(0), \tag{3.2}
$$

which implies that $y(0) = y'(\eta)$ and $y^{(i)}(0) = y^{(i+1)}(\eta)$ for $2 \le i \le n-2$ whenever if $n \ge 4$. Finally, it is clear that $y^{(n)}(t) = g(t)$, hence $y^{(n)}(t) \in F(t, y(t))$. $\square$

Proof of Theorem 2.7. Consider the multivalued map $T : \mathcal{C}([0,1],\mathbb{R}) \to 2^{\mathcal{C}([0,1],\mathbb{R})}$ defined as follows: for $y(.) \in \mathcal{C}([0,1],\mathbb{R})$,

$$
T(y(.)) := \big\{ z(.) \in \mathcal{C}([0,1],\mathbb{R}) : z(t) = \psi_n^g(t) \big\}.
$$

We shall show that T satisfies the assumptions of Lemma 2.2. The proof will be given in two steps:

Step 1: *T has non-empty closed values.* Indeed, let $(y_p(.))_{p\ge 0} \in T(y(.))$ converges to $\bar{y}(.)$ in $\mathcal{C}([0,1],\mathbb{R})$. Then $\bar{y}(.) \in \mathcal{C}([0,1],\mathbb{R})$ and for each $t \in [0,1]$,

$$
y_p(t) \in \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} F(s, y(s))ds + \sum_{k=0}^{n-2} \varphi_n^{(k)}(t) \int_0^\eta \frac{(\eta-s)^k}{k!} F(s, y(s))ds.
$$

Since the set

$$
\int_0^t \frac{(t-s)^k}{k!} F(s, y(s))ds
$$

is closed for all $t \in [0,1]$ and $0 \le k \le n-1$, we have

$$
\bar{y}(t) \in \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} F(s, y(s))ds + \sum_{k=0}^{n-2} \varphi_n^{(k)}(t) \int_0^\eta \frac{(\eta-s)^k}{k!} F(s, y(s))ds.
$$

Then $\bar{y}(.) \in T(y(.))$. So $T(y(.))$ is closed for each $y(.) \in \mathcal{C}([0,1],\mathbb{R})$.

Step 2: *T is a contraction.* Indeed, let $y_1(.), y_2(.) \in \mathcal{C}([0,1],\mathbb{R})$ and $z_1(.) \in T(y_1(.))$. Then

$$
z_1(t) = \psi_n^{g_1}(t),
$$

where $g_1 \in S_{F,y_1(.)}$. By (3.1), there exists g_2 such that

$$g_2(t) \in F(t, y_2(t)) \quad \text{and} \quad |g_1(t) - g_2(t)| \le m(t)|y_1(t) - y_2(t)|, \quad \text{for each } t \in [0,1].$$

Now, for $t \in [0,1]$, we set $z_2(t) = \psi_n^{g_2}(t)$.

On the other hand, we have

$$\begin{aligned} |\psi_n^{g_2}(t) - \psi_n^{g_1}(t)| &\le \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}|g_1(s) - g_2(s)|ds \\ &\quad + \sum_{k=0}^{n-2} \varphi_n^{(k)}(t) \int_0^\eta \frac{(\eta - s)^k}{k!}|g_1(s) - g_2(s)|ds \\ &\le \frac{1}{(n-1)!} \int_0^t m(s)|y_1(s) - y_2(s)|ds \\ &\quad + \sum_{k=0}^{n-2} \varphi_n^{(k)}(t) \frac{1}{k!} \int_0^\eta m(s)|y_1(s) - y_2(s)|ds \\ &\le \frac{1}{(n-1)!} \|y_1(.) - y_2(.)\|_\infty \int_0^t m(s)ds \\ &\quad + \sum_{k=0}^{n-2} \varphi_n^{(k)}(t) \frac{1}{k!} \|y_1(.) - y_2(.)\|_\infty \int_0^\eta m(s)ds \\ &\le \Big(\frac{L(t)}{(n-1)!} + L(\eta) \sum_{k=0}^{n-2} \frac{\varphi_n^{(k)}(t)}{k!} \Big) \|y_1(.) - y_2(.)\|_\infty. \end{aligned}$$

Then, by (c)

$$|z_2(t) - z_1(t)| \le \Big(\frac{L(1)}{(n-1)!} + L(\eta) \sum_{k=0}^{n-2} \frac{\varphi_n^{(k)}(1)}{k!} \Big) \|y_1(.) - y_2(.)\|_\infty.$$

By the analogous relation, obtained by interchanging the roles of $y_1(.)$ and $y_2(.)$, it follows that

$$H\big(T(y_1(.)), T(y_2(.))\big) \le \Big(\frac{L(1)}{(n-1)!} + L(\eta) \sum_{k=0}^{n-2} \frac{\varphi_n^{(k)}(1)}{k!} \Big) \|y_1(.) - y_2(.)\|_\infty.$$

Consequently, T is a contraction. Hence, by Lemma 2.2, T has a fixed point $y(.)$. □

Proposition 3.3. *$y(.)$ is a solution of (1.4).*

Proof. By (3.2), we have $y^{(i)}(0) = y^{(i+1)}(\eta)$, for $0 \le i \le n-2$. Since $y^{(n)}(t) = g(t)$, we have $y^{(n)}(t) \in F(t, y(t))$. □

References

[1] A. Boucherif and S. M. Bouguima; *Nonlinear second order ordinary differential equations with nonlocal boundary conditions*, Commu. Appl. Nonl. Anal., 5(2), (1998), 73-85.

[2] C. Castaing and M. Valadier; *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics 580, Springer-Verlag, Berlin-Heidelberg-New York, (1977).

[3] H. Covitz and S. B. Jr. Nadler; *Multivalued contraction mappings in generalized metric spaces*, Israel J. Math., 8, (1970), 5-11.

[4] M. Benchohra and S. K. Ntouyas; *A note on a three-point boundary-value problem for second order differential inclusions*, Mathematical Notes, Miskolc. 2, (2001), 39-47.

[5] M. Benchohra and S. K. Ntouyas; *Multi-point boundary-value problems for second order differential inclusions*, Math. Vesnik, to appear.

[6] M. Benchohra and S. K. Ntouyas; *On three and for point boundary-value problems for second order differential inclusions*, Mathematical Notes, Miskolc. 2, (2001), 93-101.
[7] Q. Zhu; *On the solution set differential inclusions in Banach spaces*, J. Diff. eqs., No. 41, (2001), 1-8.
[8] S. A. Marano; *A remark on a second order three-point boundary-value problem*, J. Math. Anal. Appl., 183, (1994), 518-522.

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