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A NOTE ON LOCAL SMOOTHING EFFECTS FOR THE UNITARY GROUP ASSOCIATED WITH THE KDV EQUATION

XAVIER CARVAJAL

Abstract. In this note we show interesting local smoothing effects for the unitary group associated to Korteweg-de Vries type equation. Our main tools are the Hardy-Littlewood-Sobolev and Hausdorff-Young inequalities. Using our local smoothing effect and a dual version, we estimate the growth of the norm of solutions of the complex modified KdV equation.

1. Introduction

In this note we describe some results on local smoothing effects for solutions of the initial value problem (IVP)

$$\begin{gathered} \partial_t u + b\partial_x^3 u = 0, \\ u(x,0) = u_0(x). \end{gathered} \tag{1.1}$$

We define the unitary group $U(t)u_0$ as the solution of the linear initial-value problem (1.1), in this way

$$\widehat{U(t)u_0}(\xi) = e^{it(b\xi^3)}\widehat{u_0}(\xi). \tag{1.2}$$

Kenig et al. [3] (see also [1] and [4]) proved the following local smoothing effect

$$\|\partial_x U(t')u_0\|_{L_x^\infty \mathcal{L}_t^2} \le \|\partial_x U(t')u_0\|_{L_x^\infty L_t^2} \le c\|u_0\|_{L^2}. \tag{1.3}$$

They also proved that

$$\big\|\partial_x^2 \int_0^t U(t-t')f(t',x)dt'\big\|_{L_x^\infty L_t^2} \le c\|f\|_{L_x^1 L_t^2}. \tag{1.4}$$

In this work we obtain a local smoothing effect (Theorem 1.1), more general than local smoothing effect (1.3). We also consider the IVP for the complex modified Korteweg-de Vries type equation:

$$\begin{gathered} \partial_t u + b\partial_x^3 u + \gamma\partial_x(|u|^2 u) = 0, \\ u(x,0) = u_0(x), \end{gathered} \tag{1.5}$$

where u is a complex valued function and b, γ are real parameters with $b\gamma \neq 0$.

Using our local smoothing effect we also proved an interesting result on growth norms (Theorem 1.2).

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The flow associated with (1.5) leads to the quantity

$$I_1(u) = \int_{\mathbb{R}} |u(x,t)|^2 dx, \tag{1.6}$$

which is conserved in time. Also, when $b \cdot \gamma \neq 0$ we have the time invariant quantity

$$I_2(u) = k_1 \int_{\mathbb{R}} |\partial_x u(x,t)|^2 dx + k_2 \int_{\mathbb{R}} |u(x,t)|^4 dx, \tag{1.7}$$

where $k_1 = 3b\gamma$ and $k_2 = -3\gamma^2/2$. The main results in this work are stated as follows.

Theorem 1.1. *Let $U(t)u_0$ be the solution of the linear problem associated to* (1.1) *and let $p \geq 2$ and $1/p + 1/q = 1$.*
If $2 < p < \infty$ and $4/q - 2 < s < 1/q + 1$ then

$$\|\partial_x U(t')u_0\|_{L_x^\infty \mathcal{L}_t^p} \leq c_{p,s}(1+t)^{1/p} \|D^s u_0\|_{L^q}.$$

If $p = 2$ and $0 \leq s < 3/2$, then

$$\|\partial_x U(t')u_0\|_{L_x^\infty \mathcal{L}_t^2} \leq c_s t^{s/3} \|D^s u_0\|_{L^2}. \tag{1.8}$$

If $p = \infty$ and $3/2 < s$, then

$$\|\partial_x U(t')u_0\|_{L_x^\infty \mathcal{L}_t^\infty} \leq c_s \|u_0\|_{H^s}. \tag{1.9}$$

Theorem 1.2. *Let $u \in \mathcal{C}(\mathbb{R}, H^2(\mathbb{R}))$ be solution of* (1.5) *and $T > 0$. Then for all $t \in (0,T)$ there exist a function $\delta = \delta(\|u\|_{L_x^2 \mathcal{L}_T^\infty}, \|u\|_{\mathcal{L}_T^\infty \dot{H}^{1/4}})$ such that*

$$\|u(t)\|_{\dot{H}^\theta} \leq \|u_0\|_{\dot{H}^\theta} + \delta t \|u_0\|_{L^2}^3, \tag{1.10}$$

where $0 \leq \theta \leq 1$.

The notation used here is standard in partial differential equations. We will use the Lebesgue space-time $L_x^p \mathcal{L}_\tau^q$ endowed with the norm

$$\|f\|_{L_x^p \mathcal{L}_\tau^q} = \big\| \|f\|_{\mathcal{L}_\tau^q} \big\|_{L_x^p} = \Big(\int_{\mathbb{R}} \Big(\int_0^\tau |f(x,t)|^q dt \Big)^{p/q} dx \Big)^{1/p}.$$

We will use the notation $\|f\|_{L_x^p L_t^q}$ when the integration in the time variable is on the whole real line. The notation $\|u\|_{L^p}$ is used when there is no doubt about the variable of integration.

2. Smoothing Local Effects

In this section we prove new smoothing local effects for the unitary group associated with the Korteweg-de Vries equation (Theorem 1.1), which will be fundamental in the proof of Theorem 1.2.

Linear Estimates. The next lemma is a preliminary result to be used in the proof of Theorem 1.2.

Lemma 2.1. *Let $u(x,t') = U(t')u_0(x)$ be the solution of* (1.1)*. We have the maximal function estimates*

$$\|U(t')u_0\|_{L_x^4 L_t^\infty} \leq c\|D^{1/4} u_0\|_{L^2}, \tag{2.1}$$

and for $s > 3/4$ and $\rho > 3/4$

$$\|U(t')u_0\|_{L_x^2 \mathcal{L}_t^\infty} \leq c(1+t)^\rho \|u_0\|_{H^s}. \tag{2.2}$$

and

$$\big\|\partial_x^2 \int_0^t U(t-t')f(t',x)dt'\big\|_{L_x^\infty \mathcal{L}_\tau^2} \leq c\|f\|_{L_x^1 \mathcal{L}_\tau^2}. \tag{2.3}$$

Proof. The proof of (2.1) and (2.2) can be found in [3]. To prove (2.3), let $\tau > 0$ and $g(t',\tau,x) = f(t',x)\chi_{[0,\tau]}(t')$. Then

$$\begin{aligned}
\big\|\partial_x^2 \int_0^t U(t-t')f(t',x)dt'\big\|_{L_x^\infty \mathcal{L}_\tau^2} &= \Big\|\Big(\int_0^\tau \big|\partial_x^2 \int_0^t U(t-t')g(t',\tau,x)dt'\big|^2 dt\Big)^{1/2}\Big\|_{L_x^\infty} \\
&\leq \Big\|\Big(\int_{\mathbb{R}} \big|\partial_x^2 \int_0^t U(t-t')g(t',\tau,x)dt'\big|^2 dt\Big)^{1/2}\Big\|_{L_x^\infty} \\
&= \big\|\partial_x^2 \int_0^t U(t-t')g(t',\tau,x)dt'\big\|_{L_x^\infty L_t^2},
\end{aligned}$$

and by inequality (1.4) we obtain (2.3). □

Proof of Theorem 1.1. Let $\varphi \in \mathcal{C}_0^\infty$ with $\varphi(t') = 1$ in $[-t,t]$, $0 \leq \varphi(t') \leq 1$ and $\operatorname{supp}\varphi \subset [-2t, 2t]$, then

$$\|\partial_x U(t')u_0\|_{L_x^\infty \mathcal{L}_t^p} \leq \|\varphi(t')\partial_x U(t')u_0\|_{L_x^\infty L_t^p}.$$

Using duality, we consider $g \in L^q$, $\|g\|_{L^q} = 1$ and the expression

$$I(x,t) := \big|\int_{\mathbb{R}} g(t')\varphi(t')\partial_x U(t')u_0 dt'\big|.$$

Now using the change of variable $t' = -t'$ we can assume that

$$I(x,t) := \big|\int_{\mathbb{R}} g(t')\varphi(t')\partial_x U(-t')u_0 dt'\big|.$$

Fubinni Theorem and the definition of group $U(t)$, shows that

$$\begin{aligned}
I(x,t) &= \big|\int_{\mathbb{R}} g(t')\varphi(t')\int_{\mathbb{R}} e^{ix\xi - i\xi^3 t'} i\xi\widehat{u_0}(\xi)d\xi dt'\big| \\
&= \big|\int_{\mathbb{R}} e^{ix\xi}\xi\widehat{u_0}(\xi)\Big(\int_{\mathbb{R}} g(t')\varphi(t')e^{-i\xi^3 t'}dt'\Big)d\xi\big| \\
&= \big|\int_{\mathbb{R}} \widehat{u_0}(\xi)\xi e^{ix\xi}\widehat{\varphi g}(\xi^3)d\xi\big|,
\end{aligned} \tag{2.4}$$

and by Plancherel's equality, Hölder inequality and Hausdorff-Young inequality we have

$$\begin{aligned}
I(x,t) &= \big|\int_{\mathbb{R}} |\xi|^s\widehat{u_0}(\xi)\frac{\xi e^{ix\xi}}{|\xi|^s}\widehat{\varphi g}(\xi^3)d\xi\big| \\
&= \big|\int_{\mathbb{R}} D^s u_0(y)\mathcal{F}\Big(\frac{\xi e^{ix\xi}}{|\xi|^s}\widehat{\varphi g}(\xi^3)\Big)(y)dy\big| \\
&\leq \|D^s u_0\|_{L^q}\big\|\mathcal{F}\Big(\frac{\xi e^{ix\xi}}{|\xi|^s}\widehat{\varphi g}(\xi^3)\Big)(y)\big\|_{L^p} \\
&\leq \|D^s u_0\|_{L^q}\big\|\frac{\xi e^{ix\xi}}{|\xi|^s}\widehat{\varphi g}(\xi^3)\big\|_{L^q}.
\end{aligned} \tag{2.5}$$

Now, we make the change of variable $y = \xi^3$ to obtain:

$$\Big\|\frac{\xi e^{ix\xi}}{|\xi|^s}\widehat{\varphi g}(\xi^3)\Big\|_{L^q}^q = \frac{1}{3}\int_{\mathbb{R}} \frac{|\widehat{\varphi g}(y)|^q dy}{|y|^\alpha}, \tag{2.6}$$

where $\alpha = (2-(1-s)q)/3$. Note that if $p = q = 2$ and $s = 0$, then $\alpha = 0$, therefore in this case

$$I(x,t) \leq c\|u_0\|_{L^2}\|\varphi g\|_{L^2} \leq c\|u_0\|_{L^2}\|g\|_{L^2} = c\|u_0\|_{L^2},$$

and in this case we obtain (1.8).

If $p = q = 2$ and $0 < s < 3/2$, then $0 < \alpha = 2s/3 < 1$, using properties of the Fourier transform and the Hardy-Littlewood-Sobolev inequality it is not hard to deduce the following string of inequalities

$$\begin{aligned} \int_{\mathbb{R}} \frac{|\widehat{\varphi g}(y)|^2}{|y|^{2s/3}} dy &= \int_{\mathbb{R}} |\widehat{\varphi g}(y)|^2 \Big|\widehat{\frac{1}{|x|^{1-s/3}}}(y)\Big|^2 dy \\ &\leq \Big\|(\varphi g) * \frac{1}{|x|^{1-s/3}}\Big\|_{L^2}^2 \\ &\leq c_s \|\varphi g\|_{L^{6/(3+2s)}}^2 \\ &\leq c_s \|\varphi\|_{L^{3/s}}^2 \|g\|_{L^2}^2 \\ &\leq c_s t^{2s/3} \|g\|_{L^2}^2. \end{aligned} \tag{2.7}$$

If $p > 2$ and $4/q - 2 < s < 1/q + 1$, then $0 < \alpha < 1$ (observe that $4/q - 2 > 1 - 2/q$), we can write the integral in (2.6) as follows

$$\int_{\mathbb{R}} \frac{|\widehat{\varphi g}(y)|^q dy}{|y|^\alpha} = \int_{|y|\leq 1} \frac{|\widehat{\varphi g}(y)|^q dy}{|y|^\alpha} + \int_{|y|>1} \frac{|\widehat{\varphi g}(y)|^q dy}{|y|^\alpha} := I_1^q + I_2^q,$$

hence

$$I_1^q \leq c_{s,q}\|\widehat{\varphi g}\|_{L^\infty}^q \leq c_{s,q}\|\varphi g\|_{L^1}^q \leq c_{s,q}\|\varphi\|_{L^p}^q \|g\|_{L^q}^q \leq c_{s,q} t^{q/p},$$

note that $s > 4/q - 2$ implies $\alpha p/(p-q) > 1$, therefore using Hölder inequality and Hausdorf-Young inequality in I_2^q we obtain

$$I_2^q \leq \|\widehat{\varphi g}\|_{L^p}^q \Big(\int_{|y|>1} \frac{dy}{|y|^{\alpha p/(p-q)}}\Big)^{1-q/p} \leq c_{s,q}\|\varphi g\|_{L^q}^q \leq c_{s,q}\|g\|_{L^q}^q.$$

If $p = \infty$ and $s > 3/2$, then (2.4) gives

$$I(x,t) \leq \|\widehat{\varphi g}\|_{L^\infty} \|\widehat{u_0}(\xi)\xi\|_{L^1} \leq c_s\|g\|_{L^1}\|u_0\|_{H^s}.$$

Note that, for $s > 1/2$ using immersion we also have

$$\|\partial_x U(t')u_0\|_{L_t^\infty L_x^\infty} \leq c_s\|\partial_x U(t')u_0\|_{H^s} \leq c_s\|u_0\|_{H^{s+1}}.$$

Hence we have finished the proof of Theorem 1.1. □

Corollary 2.2. *Let* $0 \leq s \leq 1$ *and* $u_0 \in L^2$. *Then*

$$\|D_x^s U(t')u_0\|_{L_x^\infty \mathcal{L}_t^2} \leq c_s t^{(1-s)/3}\|u_0\|_{L^2}. \tag{2.8}$$

The proof of the above corollary follows from (1.8).

Corollary 2.3. *Let* $f \in L_x^1\mathcal{L}_t^2$ *and* $U(t')$ *be as in* (1.2). *Then for* $0 \leq s \leq 1$ *we have*

$$\Big\|D_x^s \int_0^t U(t-t')f(x,t')dt'\Big\|_{L_x^2} \leq c_s t^{(1-s)/3}\|f\|_{L_x^1\mathcal{L}_t^2}. \tag{2.9}$$

Proof. Inequality (2.9) follows from (2.8) and a duality argument. In fact, by Plancherel identity, definition of the group $U(t)$ and (2.8), we have for $\|g\|_{L^2} = 1$:

$$\begin{aligned}\int_{\mathbb{R}} \Big(D_x^s \int_0^t U(-t')f(x,t')dt'\Big)\overline{g(x)}dx &= \int_0^t \int_{\mathbb{R}} f(x,t')\overline{D_x^s U(t')g(x)}dxdt' \\ &\leq \|f\|_{L_x^1\mathcal{L}_t^2}\|D_x^s U(t')g(x)\|_{L_x^\infty\mathcal{L}_t^2} \\ &\leq ct^{(1-s)/3}\|f\|_{L_x^1\mathcal{L}_t^2}\|g\|_{L^2}.\end{aligned}$$

□

Proof of Theorem 1.2. The next lemma is used in the proof.

Lemma 2.4. *Let $u \in \mathcal{C}(\mathbb{R}, H^2)$ be the solution of* (1.5)*. Then*

$$\begin{aligned}\|u\|_{L_x^2\mathcal{L}_t^\infty} \leq c(1+t)^{3/4+}\|u(0)\|_{H^{3/4+}} + c(1+t)^{3/4+}\int_0^t (\|u(t')\|_{H^{1/2+}}\|u(t')\|_{H^2}^2 \\ + \|u(t')\|_{H^{1/2+}}^2\|u(t')\|_{H^2})dt'.\end{aligned} \tag{2.10}$$

Proof. To prove the first inequality we rely on the integral equation form

$$u(t) = U(t)u_0 - \gamma\int_0^t U(t-\tau)\left(\partial_x(|u|^2u)\right)(\tau),$$

the linear estimate (2.2) show that if $u(0) \in H^2$ then for any $t > 0$,

$$\begin{aligned}\|u\|_{L_x^2\mathcal{L}_t^\infty} &\leq c(1+t)^{3/4+}\|u(0)\|_{H^{3/4+}} \\ &\quad + c(1+t)^{3/4+}\int_0^t (\||u|^2u(t')\|_{L_x^2} + \|\partial_x^2(|u|^2u)(t')\|_{L_x^2})dt',\end{aligned} \tag{2.11}$$

using the immersions $\|u(t)\|_{L_x^\infty} \leq c\|u(t)\|_{H^{1/2+}}$, $\|u(t)\|_{L_x^4} \leq c\|u(t)\|_{\dot{H}^{1/4}}$ it follows that

$$\||u|^2u(t')\|_{L_x^2} \leq \|u(t')\|_{L_x^\infty}\|u^2(t')\|_{L_x^2} \leq c\|u(t')\|_{H^{1/2+}}\|u(t')\|_{L_x^4}^2 < \infty, \tag{2.12}$$

and using Leibniz rule, it is easy to see that

$$\begin{aligned}\|\partial_x^2(|u|^2u)(t')\|_{L_x^2} &\leq c\|uu_x^2(t')\|_{L_x^2} + c\|u^2u_{xx}(t')\|_{L_x^2} \\ &\leq c\|u(t')\|_{H^{1/2+}}\|u(t')\|_{H^2}^2 + c\|u(t')\|_{H^{1/2+}}^2\|u(t')\|_{H^2} < \infty.\end{aligned}$$

Hence combining this inequality and (2.11), we obtain (2.10). □

Lemma 2.5. *Let $u \in \mathcal{C}(\mathbb{R}, H^2(\mathbb{R}))$ be solution of* (1.5) *and $0 \leq s \leq 1$. Then*

$$\begin{aligned}\|D_x^s u(t)\|_{L_x^2} &\leq \|D^s u_0\|_{L^2} \\ &\quad + ct^{(1-s)/3}\|u\|_{L_x^2\mathcal{L}_t^\infty}^2\Big(\|u_0\|_{L^2} + t^{1/2}\|u\|_{\mathcal{L}_t^\infty\dot{H}^{1/4}}^2\|u\|_{L_x^2\mathcal{L}_t^\infty}\Big).\end{aligned} \tag{2.13}$$

Proof. Without loss of generality we restrict our attention to the real case $u \in \mathbb{R}$. The equivalent integral equation is

$$u(t) = U(t)u_0 - \gamma\int_0^t U(t-\tau)\left(\partial_x(u^3)\right)(\tau)d\tau =: U(t)u_0 + z(t). \tag{2.14}$$

Let $\Gamma(t)=\|u\|_{L^2_x\mathcal{L}^\infty_t}$. From (2.14), Corollary 2.3 and Hölder inequality, we have

$$\begin{aligned}\|D^s_x u(t)\|_{L^2_x} &\le \|D^s_x U(t)u_0\|_{L^2_x}+\|D^s_x z(t)\|_{L^2_x}\\ &\le \|D^s u_0\|_{L^2}+ct^{(1-s)/3}\|u^2u_x\|_{L^1_x\mathcal{L}^2_t}\\ &\le \|D^s u_0\|_{L^2}+ct^{(1-s)/3}\Gamma(t)^2\|u_x\|_{L^\infty_x\mathcal{L}^2_t}.\end{aligned}\tag{2.15}$$

Using (1.3), (2.3) and Hölder inequality, we obtain

$$\begin{aligned}\|\partial_x u\|_{L^\infty_x\mathcal{L}^2_t} &\le \|\partial_x U(t')u_0\|_{L^\infty_x\mathcal{L}^2_t}+\|\partial_x z\|_{L^\infty_x\mathcal{L}^2_t}\\ &\le c\|u_0\|_{L^2}+c\|u^3\|_{L^1_x\mathcal{L}^2_t}\\ &\le c\|u_0\|_{L^2}+c\|u\|^2_{L^4_x\mathcal{L}^4_t}\Gamma(t)\\ &\le c\|u_0\|_{L^2}+ct^{1/2}\|u\|^2_{\mathcal{L}^\infty_t L^4_x}\Gamma(t)\\ &\le c\|u_0\|_{L^2}+ct^{1/2}\|u\|^2_{\mathcal{L}^\infty_T\dot H^{1/4}}\Gamma(t),\end{aligned}\tag{2.16}$$

where in the last inequality we use immersion $\|u\|_{L^4_x}\le\|u\|_{\dot H^{1/4}}$. As a consequence of (2.15) and (2.16) we have (2.13). Thus the proof is complete. $\square$

Proof of Theorem 1.2. Let $T>0$. Then there is a $\delta_0=\delta_0(T)>0$ such that

$$\|u\|_{L^2_xL^\infty([\tau_1,\tau_2])}<2\|u_0\|_{L^2},\quad \text{for all } \tau_1,\tau_2\in[0,T],\ |\tau_1-\tau_2|\le\delta_0.\tag{2.17}$$

To verify this we use contradiction, we suppose that for all n there exist $\tau_1^n,\tau_2^n\in[0,T]$, $|\tau_1^n-\tau_2^n|<1/n$ and

$$\|u\|_{L^2_xL^\infty([\tau_1^n,\tau_2^n])}\ge 2\|u_0\|_{L^2}.\tag{2.18}$$

Since (τ_1^n) and (τ_2^n) are bounded sequences, we can suppose that there exist a $\tau\in[0,T]$ such that $\lim_{n\to\infty}\tau_1^n=\lim_{n\to\infty}\tau_2^n=\tau$, using Lemma 2.4 and Lebesgue's Dominated Convergence Theorem, we have that

$$\|u\|_{L^2_xL^\infty([\tau_1^n,\tau_2^n])}\to\|u(\tau)\|_{L^2}=\|u_0\|_{L^2}\quad\text{as } n\to\infty;$$

however, this contradicts the relation (2.18).

Let $0\le t_k\le t$ be a sequence with $t_0=0$, $t_{k+1}-t_k=\delta_0$ and let $n\approx t/\delta_0$ such that $t_n\le t<t_{n+1}$. By Lemma 2.5 and (2.17), it follows that

$$\begin{aligned}\|D^s_x u(t_k)\|_{L^2_x} &\le \|D^s_x u(t_{k-1})\|_{L^2}+c\delta_0^{(1-s)/3}\|u\|^2_{L^2_xL^\infty([t_{k-1},t_k])}\|u_0\|_{L^2}\\ &\quad+\delta_0^{(1-s)/3+1/2}\|u\|^2_{\mathcal{L}^\infty_T\dot H^{1/4}}\|u\|^3_{L^2_xL^\infty([t_{k-1},t_k])}\\ &\le \|D^s_x u(t_{k-1})\|_{L^2}+c\delta_0^{(1-s)/3}\|u_0\|^3_{L^2}(1+\delta_0^{1/2}\|u\|^2_{\mathcal{L}^\infty_T\dot H^{1/4}}),\end{aligned}$$

similarly we have

$$\|D^s_x u(t)\|_{L^2_x}\le\|D^s_x u(t_n)\|_{L^2}+c\delta_0^{(1-s)/3}\|u_0\|^3_{L^2}(1+\delta_0^{1/2}\|u\|^2_{\mathcal{L}^\infty_T\dot H^{1/4}});\tag{2.19}$$

therefore,

$$\begin{aligned}\|D_x^s u(t_n)\|_{L_x^2} - \|D^s u(0)\|_{L_x^2} &= \sum_{k=1}^{n} \big(\|D_x^s u(t_k)\|_{L_x^2} - \|D_x^s u(t_{k-1})\|_{L^2}\big)\\ &\le \sum_{k=1}^{n} c\delta_0^{(1-s)/3}\|u_0\|_{L^2}^3(1+\delta_0^{1/2}\|u\|^2_{\mathcal{L}_T^\infty \dot H^{1/4}})\\ &\le ct\|u_0\|_{L^2}^3 \frac{(1+\delta_0^{1/2}\|u\|^2_{\mathcal{L}_T^\infty \dot H^{1/4}})}{\delta_0^{(2+s)/3}},\end{aligned}$$

so that we conclude

$$\|D_x^s u(t_n)\|_{L_x^2} \le \|D^s u(0)\|_{L^2} + ct\|u_0\|_{L^2}^3 \frac{(1+\delta_0^{1/2}\|u\|^2_{\mathcal{L}_T^\infty \dot H^{1/4}})}{\delta_0^{(2+s)/3}}, \tag{2.20}$$

combining (2.19) and (2.20) we obtain

$$\|D_x^s u(t)\|_{L_x^2} \le \|D^s u(0)\|_{L^2} + \|u_0\|_{L^2}^3 \frac{c(t+\delta_0)}{\delta_0^{(2+s)/3}}(1+\delta_0^{1/2}\|u\|^2_{\mathcal{L}_T^\infty \dot H^{1/4}}).$$

This completes the proof. □

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Xavier Carvajal
Instituto de Matemática - UFRJ Av. Horácio Macedo, Centro de Tecnologia Cidade Universitária, Ilha do Fundão, Caixa Postal 68530 21941-972 Rio de Janeiro, RJ, Brasil
E-mail address: `carvajal@im.ufrj.br`