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MULTIPLE SEMICLASSICAL STATES FOR SINGULAR MAGNETIC NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. By means of a finite-dimensional reduction, we show a multiplicity result of semiclassical solutions $u : \mathbb{R}^N \to \mathbb{C}$ to the singular nonlinear Schrödinger equation

$$\Big(\frac{\varepsilon}{i}\nabla - A(x)\Big)^2 u + u + (V(x) - \gamma(\varepsilon)W(x))u = K(x)|u|^{p-1}u, \quad x \in \mathbb{R}^N,$$

where $N \geq 2$, $1 < p < 2^* - 1$, $A(x), V(x)$ and $K(x)$ are bounded potentials. Such solutions concentrate near (non-degenerate) *local* extrema or a (non-degenerate) *manifold* of stationary points of an auxiliary function Λ related to the unperturbed electric field $V(x)$ and the coefficient $K(x)$ of the nonlinear term.

1. Introduction and main results

In recent years, much attention has been devoted to the search of standing waves solutions of the type $\psi(x,t) = \exp(-i\frac{E}{\hbar}t)u(x)$, $E \in \mathbb{R}$, $u : \mathbb{R}^N \to \mathbb{C}$ to the time-dependent NLS equations (Nonlinear Schrödinger equations) with potentials

$$i\hbar\frac{\partial\psi}{\partial t} = \Big(\frac{\hbar}{i}\nabla - A(x)\Big)^2\psi + U(x)\psi - K(x)|\psi|^{p-1}\psi, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N, \tag{1.1}$$

where i is the imaginary unit and $\hbar$ is the Planck constant. The function $A : \mathbb{R}^N \to \mathbb{R}^N$ denotes a magnetic potential , $U : \mathbb{R}^N \to \mathbb{R}$ represents an electric potential and the nonlinear term grows subcritically, namely for $p > 1$ if $N = 2$ and $1 < p < (N+2)/(N-2)$ if $N \geq 3$.

This leads to solve the complex semilinear elliptic equation

$$\Big(\frac{\varepsilon}{i}\nabla - A(x)\Big)^2 u + (U(x) - E)u = K(x)|u|^{p-1}u, \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $\varepsilon = \hbar$ and $V(x) + 1 = U(x) - E$ is strictly positive on the whole $\mathbb{R}^N$, whose solutions are usually referred as semi-classical ones since their existence is proved by letting $\varepsilon \to 0$ thus performing the transition from Quantum to Classical Mechanics. It has been also investigated the problem of finding a family $\{u_\varepsilon\}$ of such solutions which exhibits a *concentration behavior* around a special point, namely, solutions with a spike shape, a maximum point converging to a point

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located around a prescribed region, while vanishing as $\varepsilon \to 0$ everywhere else in the domain. Such special point has been proved to be a critical point of the potential $V(x)$ and the study of single and multiple spike solutions to (1.2) and related problems has attracted considerable attention in recent years. In the case $A = 0$, different approaches have been carried out in order to study one-bump or multi-bump semi-classical bound states (solutions with finite energy) and different cases have been covered (see [1, 10, 11, 13, 20, 21, 22, 28, 29, 30, 34, 37, 39, 43, 45, 46, 52, 53]). In the case $A \neq 0$, the first existence result is due to Esteban and Lions [31] for $\varepsilon > 0$ fixed by means of concentration-compactness arguments. Later, Kurata [40] has showed, in the semiclassical limit, the existence and the concentration of a least energy solution near global minima of V under suitable assumptions linking the magnetic and the electric potentials in the case $K(x) = 1$. Furthermore, he has proved that the magnetic potential only contributes to the phase factor of the complex solution but it doesn't influence the concentration of its modulus. A first multiplicity result for solutions of (1.2) has been proved by Cingolani in [24], by means of topological arguments that allow to relate the number of the solutions to the richness of the set M of global minima of an auxiliary function Λ defined as

$$\Lambda(x) = \frac{(1+V(x))^{\theta}}{K(x)^{2/(p-1)}}, \quad \theta = \frac{p+1}{p-1} - \frac{N}{2},$$

(see (4.5) in Section 4 for details) on the whole $\mathbb{R}^N$ since $K(x) > 0$ for all $x \in \mathbb{R}^N$, which coincide with global minima of $V(x)$ if $K(x) = 1$. In [23], Cingolani and Secchi have treated the more general case in which Λ has a non-degenerate manifold of stationary points. For bounded electric and magnetic potentials, they have proved a multiplicity result following the new perturbation approach introduced in the paper [2] due to Ambrosetti, Malchiodi and Secchi in the case $A = 0$ (see also [3]). Precisely, by means of a finite-dimensional reduction, the complex valued solutions to (1.2) (after the change of variable $x \to \varepsilon x$) are found *near* least energy solutions of the complex limiting equation

$$\Big(\frac{\nabla}{i} - A(\varepsilon\xi)\Big)^2 u + u + V(\varepsilon\xi)u = K(\varepsilon\xi)|u|^{p-1}u \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

(see Remark 5.2) where $\varepsilon\xi$ is in a neighborhood of M. In such sense, here and in what follows, as $\varepsilon \to 0$, solutions of (1.2) concentrate around stationary points of Λ (see Proof of Theorem 5.1). Furthermore, the boundedness of the electromagnetic potentials assures that the variational setting $H^1(\mathbb{R}^N, \mathbb{C})$ of (1.3) becomes equivalent to the variational framework in which (1.2) is set up. Then, such result has been improved in [25] to degenerate and topologically non-trivial critical points of Λ dropping the boundedness of the magnetic potential. Necessary conditions for a sequence of solutions to (1.2) to concentrate, in different senses, around a given point have been established by Secchi and Squassina in [49]. For multi-peaks, we refer to [8, 14, 26] and for the critical case to [4, 6, 17]. The asymptotic evolution has been recently studied in [50]. Dealing with singular magnetic NLS equations, we cite a recent paper by Barile [7] where the author has obtained a multiplicity result of complex-valued solutions to

$$\Big(\frac{\varepsilon}{i}\nabla - A(x)\Big)^2 u + \left(V(x) - \gamma(\varepsilon)W(x)\right) u = |u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.4}$$

where $2 < p < 2^*$, $\gamma : [0, +\infty) \to [0, +\infty)$ and $W : \mathbb{R}^N \to [0, +\infty)$ is a measurable potential satisfying (W1) like $\frac{1}{|x|}$, $\frac{1}{|x|^2}$ (see [42] in the case $A = 0$). The introduction of singular potentials has important physical interest since they appear in many fields such as Quantum Mechanics and Astrophysics [35, 41], Chemistry [15, 44], Cosmology [9] and Differential Geometry [5] thus being the object of a wide recent mathematical research (e.g. [16, 19, 31, 32, 33, 36, 48, 51]). Furthermore, in such a case, it has a certain relevance from the mathematical point of view since it allows to perturbe the potential $V(x)$ which is supposed to be bounded below so that the resulting potential $V_\varepsilon(x) = V(x) - \gamma(\varepsilon)W(x)$ may be unbounded below and eventually above. Following the variational approach used in [24], it is proved that the number of the solutions to (1.4) can still be related to the topology of the global minima set of the unperturbed potential $V(x)$, provided the perturbation $\gamma(\varepsilon)$ is small with respect to the coefficient ε^2 of the differential term, in the sense that for any $\delta > 0$ there exists $\eta^{**}(\delta) > 0$ such that

$$\limsup_{\varepsilon\to 0} \frac{\gamma(\varepsilon)}{\varepsilon^2} < \eta^{**}(\delta).$$

Thus such result can be seen as a quite natural but important generalization of the one in [24] to the case of unbounded electric potentials and $K(x) = 1$. Our purpose, in this work, is to extend such multiplicity result to

$$\Big(\frac{\varepsilon}{i}\nabla - A(x)\Big)^2 u + u + (V(x) - \gamma(\varepsilon)W(x))\, u = K(x)|u|^{p-1}u, \quad x \in \mathbb{R}^N, \tag{1.5}$$

in the more general case in which the auxiliary function Λ has a manifold M of stationary points, not necessarily global minima and, for bounded magnetic and electric potentials $A(x)$ and $V(x)$, following the perturbation approach used in [23]. Really, we are able to prove that the result in the spirit of [23] holds after the introduction of the new term $-\gamma(\varepsilon)W(x)$ which may be unbounded below, thus generalizing it to the case of electric potentials eventually unbounded.

Without loss of generality we can assume that $V(0) = 0$ and $K(0) = 1$. Performing the change of variable $x \mapsto \varepsilon x$, the problem becomes that of finding some functions $u : \mathbb{R}^N \to \mathbb{C}$ such that

$$\Big(\frac{\nabla}{i} - A(\varepsilon x)\Big)^2 u + u + (V(\varepsilon x) - \gamma(\varepsilon)W(\varepsilon x))\, u = K(\varepsilon x)|u|^{p-1}u \quad \text{in } \mathbb{R}^N. \tag{1.6}$$

Of course, if u is a solution of (1.6), then $u(\cdot/\varepsilon)$ is a solution of (1.5). Since (1.6) is invariant under the multiplicative action of S^1, solutions of (1.6) naturally appear as *orbits* so that we simply speak about solutions. The complex-valued solutions to (1.6) are found *near* least energy solutions of the equation

$$\Big(\frac{\nabla}{i} - A(\varepsilon\xi)\Big)^2 u + u + V(\varepsilon\xi)u = K(\varepsilon\xi)|u|^{p-1}u \quad \text{in } \mathbb{R}^N, \tag{1.7}$$

(see Remark 5.2) where $\varepsilon\xi$ is in a neighborhood of M. The least energy of (1.7) have the form

$$z^{\varepsilon\xi,\sigma} : x \in \mathbb{R}^N \to e^{i\sigma + iA(\varepsilon\xi)\cdot x}\Big(\frac{1 + V(\varepsilon\xi)}{K(\varepsilon\xi)}\Big)^{1/(p-1)} U((1 + V(\varepsilon\xi))^{1/2}(x - \xi)),$$

(see Section 2) where $\varepsilon\xi$ belongs to M and $\sigma \in [0, 2\pi]$. As in [23] (see also [2]), the proof relies on a suitable finite-dimensional reduction and critical points of the Euler functional f_ε associated to problem (1.6) are found *near* critical points of a finite-dimensional functional Φ_ε which is defined on a suitable neighborhood of M (see (4.6) and (4.7)). This allows to use Ljusternik-Schnirelman category in the case M is a set of local maxima or minima of Λ. We remark again that the case of maxima cannot be handled by using direct variational arguments as in [7, 24]. We present a special case of our results.

We will use the following assumptions:

- (K1) $K \in L^\infty(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ is strictly positive and K'' is bounded;
- (V1) $V \in L^\infty(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ satisfies $\inf_{x\in\mathbb{R}^N}(1+V(x)) > 0$, and V'' is bounded;
- (W1) $W : \mathbb{R}^N \to [0,+\infty)$ is a measurable function such that, for some $\alpha_1 > 0$ and $\alpha_2 \geq 0$,
$$\int_{\mathbb{R}^N} W(x)|v|^2 \leq \alpha_1 \|\nabla |v|\|_2^2 + \alpha_2 \|v\|_2^2$$
for any v such that $|v| \in H^1(\mathbb{R}^N, \mathbb{R})$;
- (A1) $A \in L^\infty(\mathbb{R}^N, \mathbb{R}^N) \cap C^1(\mathbb{R}^N, \mathbb{R}^N)$, and the Jacobian J_A of A is globally bounded in $\mathbb{R}^N$;
- (G1) $\gamma : [0,+\infty) \to [0,+\infty)$ is a function which depends on ε such that $G(\varepsilon) := \frac{\gamma(\varepsilon)}{\varepsilon^2} = O(\varepsilon)$.

Theorem 1.1. *Assume* (K1), (V1), (W1), (A1), (G1)*. If the auxiliary function* Λ *has a non-degenerate critical point* $x_0 \in \mathbb{R}^N$*, then for* $\varepsilon > 0$ *small, the problem* (1.6) *has at least a (orbit of) solution concentrating near* x_0*.*

Furthermore, if M is a set of critical points non-degenerate in the sense of Bott (see [12]) we can prove the existence of (at least) cup long of M, denoted by $l(M)$, solutions concentrating near points of M. For the definition of the cup long, refer to Section 5.

Theorem 1.2. *As in Theorem 1.1, assume* (K1), (V1), (W1), (A1), (G1)*. If the auxiliary function* Λ *has a smooth, compact, non-degenerate manifold of critical points* M*, then for* $\varepsilon > 0$ *small, the problem* (1.6) *has at least* $l(M)$ *(orbits of) solutions concentrating near points of* M*.*

We remark that the presence of an external magnetic field produces a phase in the complex wave which depends on the value of A near M, but does not seem to influence the location of the peaks of the modulus of the complex wave.

Notation. 1. The complex conjugate of any number $z \in \mathbb{C}$ will be denoted by $\bar{z}$. 2. The real part of a number $z \in \mathbb{C}$ will be denoted by $\operatorname{Re} z$. 3. The ordinary inner product between two vectors $a, b \in \mathbb{R}^N$ will be denoted by $a \cdot b$. 4. We omit the symbol dx in integrals over $\mathbb{R}^N$ when no confusion can arise. 5. C denotes a generic positive constant, which may vary inside a chain of inequalities. 6. We use the Landau symbols. For example $O(\varepsilon)$ is a generic function such that $\limsup_{\varepsilon\to 0} O(\varepsilon)/\varepsilon < \infty$, and $o(\varepsilon)$ is a function such that $\lim_{\varepsilon\to 0} o(\varepsilon)/\varepsilon = 0$.

2. The variational framework

We work in the real Hilbert space E obtained as the completion of $C_0^\infty(\mathbb{R}^N, \mathbb{C})$ with respect to the norm $\|\cdot\|$ associated to the inner product
$$(u|v) \equiv \operatorname{Re} \int_{\mathbb{R}^N} \nabla u \cdot \overline{\nabla v} + u\bar{v}.$$

Solutions to (1.6) are, under some conditions we are going to point out, critical points of the functional formally defined on E as

$$\begin{aligned} f_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N}\Big(\Big|\Big(\frac{1}{i}\nabla - A(\varepsilon x)\Big)u\Big|^2 + |u|^2 + (V(\varepsilon x) - \gamma(\varepsilon)W(\varepsilon x))|u|^2\Big)\,dx \\ - \frac{1}{p+1}\int_{\mathbb{R}^N} K(\varepsilon x)|u|^{p+1}\,dx. \end{aligned} \tag{2.1}$$

In the following, we shall assume that the functions V, W, K and A satisfy assumptions (V1), (W1), (K1) and (A1). In particular, by the boundedness of the magnetic and electric potentials, the norm $\|\cdot\|^2$ is equivalent to the usual norm

$$\|u\|_\varepsilon^2 \equiv \int_{\mathbb{R}^N}\big(|D^\varepsilon u|^2 + (1 + V(\varepsilon x))|u|^2\big)\,dx < \infty$$

on the real Hilbert space E_ε, defined by the closure of $C_0^\infty(\mathbb{R}^N,\mathbb{C})$ under the scalar product

$$(u|v)_\varepsilon \equiv \operatorname{Re}\int_{\mathbb{R}^N}\big(D^\varepsilon u\overline{D^\varepsilon v} + (1 + V(\varepsilon x))u\overline{v}\big)\,dx,$$

where $D^\varepsilon u = (D_1^\varepsilon u, \dots, D_N^\varepsilon u)$ and $D_j^\varepsilon = i^{-1}\partial_j - A_j(\varepsilon x)$. Indeed,

$$\int_{\mathbb{R}^N}\Big(\Big|\Big(\frac{1}{i}\nabla - A(\varepsilon x)\Big)u\Big|^2\Big)\,dx = \int_{\mathbb{R}^N}\Big(|\nabla u|^2 + |A(\varepsilon x)u|^2 - 2\operatorname{Re}\Big(\frac{\nabla u}{i}\cdot A(\varepsilon x)\overline{u}\Big)\Big)\,dx,$$

and the last integral is finite thanks to the Cauchy-Schwartz inequality and the boundedness of A. The functional spaces E and E_ε are isomorphic so, roughly speaking, we can say that the above variational frameworks become equivalent. This allows us to prove that the integral involving W is finite by assumption (W1) as we need that for all $u \in E$ it results $|u| \in H^1(\mathbb{R}^N,\mathbb{R})$. Since A is real valued, it is easy to deduce that (see, for example, [38, 47]) for any $u \in E_\varepsilon$, the diamagnetic inequality

$$|\nabla|u|(x)| = \Big|\operatorname{Re}\Big(\nabla u\frac{\overline{u}}{|u|}\Big)\Big| = \Big|\operatorname{Re}\Big((\nabla u - iA(\varepsilon x)u)\frac{\overline{u}}{|u|}\Big)\Big| \le |D^\varepsilon u(x)| \tag{2.2}$$

holds a.e. in $\mathbb{R}^N$ and $|u| \in H^1(\mathbb{R}^N,\mathbb{R})$. Furthermore,

$$\int_{\mathbb{R}^N}|\nabla|u\|^2 + |u|^2\,dx \le \int_{\mathbb{R}^N}\big(|D^\varepsilon u|^2 + (1 + V(\varepsilon x))|u|^2\big)\,dx \le c\|u\|^2 \tag{2.3}$$

So, by the change of variable $y = \varepsilon x$, (W1) and (2.3), we have that

$$\begin{aligned} \gamma(\varepsilon)\int_{\mathbb{R}^N} W(\varepsilon x)|u|^2 &\le \frac{\gamma(\varepsilon)}{\varepsilon^N}\Big[\alpha_1\int_{\mathbb{R}^N}\Big|\nabla\Big|u\Big(\frac{y}{\varepsilon}\Big)\Big|\Big|^2 + \alpha_2\int_{\mathbb{R}^N}\Big|u\Big(\frac{y}{\varepsilon}\Big)\Big|^2\Big] \\ &\le \frac{\gamma(\varepsilon)}{\varepsilon^2}\Big[\alpha_1\int_{\mathbb{R}^N}|\nabla|u(x)\|^2 + \alpha_2\varepsilon^2\int_{\mathbb{R}^N}|u(x)|^2\Big] \quad (\text{with } x = \frac{y}{\varepsilon}) \\ &\le \frac{\gamma(\varepsilon)}{\varepsilon^2}\alpha_\varepsilon\Big[\int_{\mathbb{R}^N}|\nabla|u(x)\|^2 + \int_{\mathbb{R}^N}|u(x)|^2\Big] \\ &\le G(\varepsilon)\alpha_\varepsilon c\|u\|^2 \end{aligned} \tag{2.4}$$

is finite for ε small, where $\alpha_\varepsilon := \max\{\alpha_1, \alpha_2\varepsilon^2\} \to \alpha_1$ as $\varepsilon \to 0$. It follows that f_ε is actually well defined on E for ε small enough. In order to find possibly multiple critical points of (2.1), we follow the approach of [2, 23]. Since we need to find complex-valued solutions, some further remarks are due.

Let $\xi \in \mathbb{R}^N$ which will be fixed suitable later on: we look for solutions to (1.6) "close" to a particular solution of the equation

$$\Big(\frac{\nabla}{i} - A(\varepsilon\xi)\Big)^2 u + u + V(\varepsilon\xi)u = K(\varepsilon\xi)|u|^{p-1}u \quad \text{in } \mathbb{R}^N \tag{2.5}$$

(see Remark 5.2). More precisely, we denote by $U_c : \mathbb{R}^N \to \mathbb{C}$ a least-energy solution to the scalar problem

$$-\Delta U_c + U_c + V(\varepsilon\xi)U_c = K(\varepsilon\xi)|U_c|^{p-1}U_c \quad \text{in } \mathbb{R}^N. \tag{2.6}$$

By energy comparison (see [40]), one has that

$$U_c(x) = e^{i\sigma}U^{\xi}(x - y_0)$$

for some choice of $\sigma \in [0, 2\pi]$ and $y_0 \in \mathbb{R}^N$, where $U^{\xi} : \mathbb{R}^N \to \mathbb{R}$ is the unique solution of

$$\begin{cases} -\Delta U^{\xi} + U^{\xi} + V(\varepsilon\xi)U^{\xi} = K(\varepsilon\xi)|U^{\xi}|^{p-1}U^{\xi}, \\ U^{\xi}(0) = \max_{\mathbb{R}^N} U^{\xi}, \quad U^{\xi} > 0. \end{cases} \tag{2.7}$$

If U denotes the unique solution of

$$\begin{cases} -\Delta U + U = U^p \quad \text{in } \mathbb{R}^N, \\ U(0) = \max_{\mathbb{R}^N} U, \quad U > 0, \end{cases} \tag{2.8}$$

then some elementary and direct computations prove that $U^{\xi}(x) = \alpha(\varepsilon\xi)U(\beta(\varepsilon\xi)x)$, where

$$\alpha(\varepsilon\xi) = \Big(\frac{1 + V(\varepsilon\xi)}{K(\varepsilon\xi)}\Big)^{1/(p-1)}, \quad \beta(\varepsilon\xi) = \big(1 + V(\varepsilon\xi)\big)^{1/2},$$

and the function $u(x) = e^{iA(\varepsilon\xi)\cdot x}U_c(x)$ actually solves (2.5).

For $\xi \in \mathbb{R}^N$ and $\sigma \in [0, 2\pi]$, we set

$$z^{\varepsilon\xi,\sigma} : x \in \mathbb{R}^N \to e^{i\sigma + iA(\varepsilon\xi)\cdot x}\alpha(\varepsilon\xi)U(\beta(\varepsilon\xi)(x - \xi)). \tag{2.9}$$

Sometimes, for convenience, we shall identify $[0, 2\pi]$ and $S^1 \subset \mathbb{C}$, through $\eta = e^{i\sigma}$. Introduce the functional $F^{\varepsilon\xi,\sigma} : E \to \mathbb{R}$ defined by

$$\begin{aligned} F^{\varepsilon\xi,\sigma}(u) = \frac{1}{2}\int_{\mathbb{R}^N} \Big(\Big|\Big(\frac{\nabla u}{i} - A(\varepsilon\xi)u\Big)\Big|^2 + |u|^2 + V(\varepsilon\xi)|u|^2\Big)\,dx \\ - \frac{1}{p+1}\int_{\mathbb{R}^N} K(\varepsilon\xi)|u|^{p+1}\,dx, \end{aligned}$$

whose critical points correspond to solutions of (2.5). The set

$$Z^{\varepsilon} = \{z^{\varepsilon\xi,\sigma} | \xi \in \mathbb{R}^N \wedge \sigma \in [0, 2\pi]\} \cong S^1 \times \mathbb{R}^N$$

is a regular manifold of critical points for the functional $F^{\varepsilon\xi,\sigma}$. From elementary differential geometry it follows that

$$T_{z^{\varepsilon\xi,\sigma}}Z^{\varepsilon} = \operatorname{span}_{\mathbb{R}}\Big\{\frac{\partial}{\partial\sigma}z^{\varepsilon\xi,\sigma} = iz^{\varepsilon\xi,\sigma}, \frac{\partial}{\partial\xi_1}z^{\varepsilon\xi,\sigma}, \dots, \frac{\partial}{\partial\xi_N}z^{\varepsilon\xi,\sigma}\Big\}$$

where we mean by the symbol $\operatorname{span}_{\mathbb{R}}$ that all the linear combinations must have real coefficients. We remark that, for $j = 1, \dots, N$,

$$\frac{\partial}{\partial\xi_j}z^{\varepsilon\xi,\sigma} = -\frac{\partial}{\partial x_j}z^{\varepsilon\xi,\sigma} + iz^{\varepsilon\xi,\sigma}A_j(\varepsilon\xi) + O(\varepsilon)\,. \tag{2.10}$$

So that any $\zeta \in T_{z^{\varepsilon\xi,\sigma}} Z^\varepsilon$ can be written as

$$\zeta = i l_1 z^{\varepsilon\xi,\sigma} + \sum_{j=2}^{N+1} l_j \frac{\partial}{\partial x_{j-1}} z^{\varepsilon\xi,\sigma} + O(\varepsilon) \tag{2.11}$$

for some real coefficients $l_1, l_2, \dots, l_{N+1}$.

The next lemma shows that $\nabla f_\varepsilon(z^{\varepsilon\xi,\sigma})$ gets small when $\varepsilon \to 0$, namely $z^{\varepsilon\xi,\sigma}$ is an "almost solution" of (1.6).

Lemma 2.1. *For all $\xi \in \mathbb{R}^N$, all $\eta \in S^1$ and all $\varepsilon > 0$ small, one has that*

$$\begin{aligned} \|\nabla f_\varepsilon(z^{\varepsilon\xi,\sigma})\| \le C\Big(&\varepsilon|\nabla V(\varepsilon\xi)| + \varepsilon|\nabla K(\varepsilon\xi)| + \varepsilon|J_A(\varepsilon\xi)| \\ &+ \varepsilon|\operatorname{div} A(\varepsilon\xi)| + \varepsilon^2 + C(\varepsilon\xi)G(\varepsilon)\Big), \end{aligned}$$

for some constant $C > 0$.

Proof. From

$$\begin{aligned} f_\varepsilon(u) = F^{\varepsilon\xi,\eta}(u) &+ \frac{1}{2}\int_{\mathbb{R}^N} \Big(\Big|\frac{\nabla u}{i} - A(\varepsilon x)u\Big|^2 - \Big|\frac{\nabla u}{i} - A(\varepsilon\xi)u\Big|^2\Big) \\ &+ \frac{1}{2}\int_{\mathbb{R}^N} [V(\varepsilon x) - V(\varepsilon\xi)]\,|u|^2 - \frac{\gamma(\varepsilon)}{2}\int_{\mathbb{R}^N} W(\varepsilon x))|u|^2 \\ &- \frac{1}{p+1}\int_{\mathbb{R}^N} [K(\varepsilon x) - K(\varepsilon\xi)]\,|u|^{p+1} \end{aligned} \tag{2.12}$$

and since $z^{\varepsilon\xi,\eta}$ is a critical point of $F^{\varepsilon\xi,\eta}$, one has (with $z = z^{\varepsilon\xi,\eta}$)

$$\begin{aligned} &\langle \nabla f_\varepsilon(z)|v\rangle \\ &= \varepsilon \operatorname{Re}\int_{\mathbb{R}^N} \frac{1}{i}(\operatorname{div} A(\varepsilon x))z\overline{v} + 2\operatorname{Re}\int_{\mathbb{R}^N} (A(\varepsilon\xi) - A(\varepsilon x))\,z\cdot\overline{\Big(\frac{\nabla}{i} - A(\varepsilon\xi)\Big)v} \\ &\quad + \operatorname{Re}\int_{\mathbb{R}^N} [V(\varepsilon x) - V(\varepsilon\xi)]\,z\overline{v} - \gamma(\varepsilon)\operatorname{Re}\int_{\mathbb{R}^N} W(\varepsilon x)z\overline{v} \\ &\quad - \operatorname{Re}\int_{\mathbb{R}^N} [K(\varepsilon x) - K(\varepsilon\xi)]\,|z|^{p-2}z\overline{v}. \end{aligned}$$

From the assumption that $|D^2V(x)| \le$ const. and direct calculations one infers

$$\int_{\mathbb{R}^N} |V(\varepsilon x) - V(\varepsilon\xi)|^2\,|z^{\varepsilon\xi,\sigma}|^2 \le c_1\varepsilon^2|\nabla V(\varepsilon\xi)|^2 + c_2\varepsilon^4,$$

and similar estimates hold for the terms involving K (see [23]). In particular, after the change of variable $y = \varepsilon x$, by Hölder inequality and (W1) we have

$$\begin{aligned} \gamma(\varepsilon)\int_{\mathbb{R}^N} W(\varepsilon x)|z^{\varepsilon\xi,\sigma}||\overline{v}| &\le \frac{\gamma(\varepsilon)}{\varepsilon^N}\Big[\int_{\mathbb{R}^N} W(y)\Big|z^{\varepsilon\xi,\sigma}\Big(\frac{y}{\varepsilon}\Big)\Big|^2\Big]^{1/2}\Big[\int_{\mathbb{R}^N} W(y)\Big|v\Big(\frac{y}{\varepsilon}\Big)\Big|^2\Big]^{1/2} \\ &\le \frac{\gamma(\varepsilon)}{\varepsilon^N}\Big[\underbrace{\alpha_1\int_{\mathbb{R}^N}\Big|\nabla\Big|z^{\varepsilon\xi,\sigma}\Big(\frac{y}{\varepsilon}\Big)\Big|\Big|^2 + \alpha_2\int_{\mathbb{R}^N}\Big|z^{\varepsilon\xi,\sigma}(\frac{y}{\varepsilon})\Big|^2}_{\tau_1}\Big]^{1/2} \\ &\quad\times\Big[\underbrace{\alpha_1\int_{\mathbb{R}^N}\Big|\nabla\Big|v\Big(\frac{y}{\varepsilon}\Big)\Big|\Big|^2 + \alpha_2\int_{\mathbb{R}^N}\Big|v\Big(\frac{y}{\varepsilon}\Big)\Big|^2}_{\tau_2}\Big]^{1/2} \end{aligned} \tag{2.13}$$

By the change of variable $x = y/\varepsilon$, the definition of z and (2.3) we have

$$\begin{aligned}\tau_1 &= \frac{\varepsilon^N}{\varepsilon^2}\Big[\alpha_1\alpha(\varepsilon\xi)^2\beta(\varepsilon\xi)^{2-N}\int_{\mathbb{R}^N}|\nabla U|^2 + \alpha_2\alpha(\varepsilon\xi)^2\beta(\varepsilon\xi)^{-N}\varepsilon^2\int_{\mathbb{R}^N}U^2\Big]\\ &\le \frac{\varepsilon^N}{\varepsilon^2}\alpha_\varepsilon \underbrace{\underbrace{\alpha(\varepsilon\xi)^2\beta(\varepsilon\xi)^{-N}}_{C^1(\varepsilon\xi)}\underbrace{\max\{1,\beta(\varepsilon\xi)^2\}}_{C^2(\varepsilon\xi)}}_{C^{1,2}(\varepsilon\xi)}\|U\|^2 \end{aligned} \tag{2.14}$$

where $C^1(\varepsilon\xi), C^2(\varepsilon\xi) \to 1$ as $\varepsilon \to 0$ and

$$\tau_2 = \frac{\varepsilon^N}{\varepsilon^2}\Big[\alpha_1\int_{\mathbb{R}^N}|\nabla|\overline{v}\|^2 + \alpha_2\varepsilon^2\int_{\mathbb{R}^N}|\overline{v}|^2\Big] \le \frac{\varepsilon^N}{\varepsilon^2}\alpha_\varepsilon c\|v\|^2$$

so that

$$\gamma(\varepsilon)\int_{\mathbb{R}^N}W(\varepsilon x)|z^{\varepsilon\xi,\sigma}|\,|\overline{v}| \le \frac{\gamma(\varepsilon)}{\varepsilon^2}\alpha_\varepsilon C'(\varepsilon\xi)c'\|U\|\,\|v\| \le G(\varepsilon)\alpha_\varepsilon C'(\varepsilon\xi)c''\|v\|$$

where $C'(\varepsilon\xi) = \big(C^{1,2}(\varepsilon\xi)\big)^{1/2}$. It then follows that

$$\begin{aligned}\|\nabla f_\varepsilon(z^{\varepsilon\xi,\sigma})\| \le C\Big(&\varepsilon|\nabla V(\varepsilon\xi)| + \varepsilon|\nabla K(\varepsilon\xi)| + \varepsilon|J_A(\varepsilon\xi)|\\ &+ \varepsilon|\operatorname{div}A(\varepsilon\xi)| + \varepsilon^2 + C(\varepsilon\xi)G(\varepsilon)\Big),\end{aligned}$$

where $C(\varepsilon\xi) = \alpha_\varepsilon C'(\varepsilon\xi)$. The lemma is proved. $\square$

3. The invertibility of $D^2 f_\varepsilon$ on $(TZ^\varepsilon)^\perp$

To apply the perturbation method, we need to exploit some non-degeneracy properties of the solution $z^{\varepsilon\xi,\sigma}$ as a critical point of $F^{\varepsilon\xi,\sigma}$. Let $L_{\varepsilon,\sigma,\xi} : (T_{z^{\varepsilon\xi,\sigma}}Z^\varepsilon)^\perp \to (T_{z^{\varepsilon\xi,\sigma}}Z^\varepsilon)^\perp$ be the operator defined by

$$\langle L_{\varepsilon,\sigma,\xi}v|w\rangle = D^2 f_\varepsilon(z^{\varepsilon\xi,\sigma})(v,w)$$

for all $v, w \in (T_{z^{\varepsilon\xi,\sigma}}Z^\varepsilon)^\perp$. Recall the following elementary result which will play a fundamental role in the present section.

Lemma 3.1. *Let $M \subset \mathbb{R}^N$ be a bounded set. Then there exists a constant $C > 0$ such that for all $\xi \in M$ one has*

$$\int_{\mathbb{R}^N}\Big|\Big(\frac{\nabla}{i} - A(\xi)\Big)u\Big|^2 + |u|^2 \ge C\int_{\mathbb{R}^N}\big(|\nabla u|^2 + |u|^2\big) \quad \forall u \in E. \tag{3.1}$$

For the proof, we refer to [23]. At this point we shall prove the following result.

Lemma 3.2. *Given $\overline{\xi} > 0$, there exists $C > 0$ such that for $\varepsilon > 0$ small enough one has*

$$|\langle L_{\varepsilon,\sigma,\xi}v|v\rangle| \ge C\|v\|^2, \quad \forall|\xi| \le \overline{\xi},\ \forall\sigma \in [0,2\pi],\ \forall v \in (T_{z^{\varepsilon\xi,\sigma}}Z^\varepsilon)^\perp. \tag{3.2}$$

Proof. We follow the arguments in [23] with some modifications due to the presence of the terms involving W. Recall that

$$T_{z^{\varepsilon\xi,\sigma}}Z^\varepsilon = \operatorname{span}_{\mathbb{R}}\Big\{\frac{\partial}{\partial\xi_1}z^{\varepsilon\xi,\sigma}, \dots, \frac{\partial}{\partial\xi_N}z^{\varepsilon\xi,\sigma}, iz^{\varepsilon\xi,\sigma}\Big\},$$

define
$$\mathcal{N} = \mathrm{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x_1} z^{\varepsilon\xi,\sigma}, \dots, \frac{\partial}{\partial x_N} z^{\varepsilon\xi,\sigma}, z^{\varepsilon\xi,\sigma}, iz^{\varepsilon\xi,\sigma} \right\}.$$
As in [2, 23], it suffices to prove (3.2) for all $v \in \mathrm{span}_{\mathbb{R}}\{z^{\varepsilon\xi,\sigma}, \phi\}$, where $\phi \perp \mathcal{N}$. More precisely, we shall prove that for some constants $C_1 > 0$, $C_2 > 0$, for all ε small enough and all $|\xi| \le \bar{\xi}$ we have
$$\langle L_{\varepsilon,\sigma,\xi} z^{\varepsilon\xi,\sigma} | z^{\varepsilon\xi,\sigma} \rangle \le -C_1 < 0, \tag{3.3}$$
$$\langle L_{\varepsilon,\sigma,\xi} \phi | \phi \rangle \ge C_2 \|\phi\|^2 \quad \forall\, \phi \perp \mathcal{N}. \tag{3.4}$$
From the expression for the second derivative of $F^{\varepsilon\xi,\sigma}$ and the fact that $z^{\varepsilon\xi,\sigma}$, as a solution of (2.5), is a mountain pass critical point of $F^{\varepsilon\xi,\sigma}$, we can find some $c_0 > 0$ such that for all $\varepsilon > 0$ small, all $|\xi| \le \bar{\xi}$ and all $\sigma \in [0, 2\pi]$ it results
$$D^2 F^{\varepsilon\xi,\sigma}(z^{\varepsilon\xi,\sigma})(z^{\varepsilon\xi,\sigma}, z^{\varepsilon\xi,\sigma}) < -c_0 < 0. \tag{3.5}$$
Recalling (2.12), we find
$$\begin{aligned}
\langle L_{\varepsilon,\sigma,\xi} z^{\varepsilon\xi,\sigma} | z^{\varepsilon\xi,\sigma} \rangle &= D^2 F^{\varepsilon\xi,\sigma}(z^{\varepsilon\xi,\sigma})(z^{\varepsilon\xi,\sigma}, z^{\varepsilon\xi,\sigma}) \\
&\quad + \int_{\mathbb{R}^N} \left(\Big| \Big(\frac{\nabla}{i} - A(\varepsilon x) \Big) z^{\varepsilon\xi,\sigma} \Big|^2 - \Big| \Big(\frac{\nabla}{i} - A(\varepsilon \xi) \Big) z^{\varepsilon\xi,\sigma} \Big|^2 \right) \\
&\quad + \int_{\mathbb{R}^N} [V(\varepsilon x) - V(\varepsilon\xi)]\, |z^{\varepsilon\xi,\sigma}|^2 - \gamma(\varepsilon) \int_{\mathbb{R}^N} W(\varepsilon x))|z^{\varepsilon\xi,\sigma}|^2 \\
&\quad - \int_{\mathbb{R}^N} [K(\varepsilon x) - K(\varepsilon\xi)]\, |z^{\varepsilon\xi,\sigma}|^{p+1}.
\end{aligned}$$
Since, following the computations in (2.13) and (2.14),
$$\begin{aligned}
\gamma(\varepsilon) \int_{\mathbb{R}^N} W(\varepsilon x))|z^{\varepsilon\xi,\sigma}|^2 &\le \frac{\gamma(\varepsilon)}{\varepsilon^N} \Big[\underbrace{\alpha_1 \int_{\mathbb{R}^N} \Big| \nabla \Big| z^{\varepsilon\xi,\sigma}\Big(\frac{y}{\varepsilon}\Big) \Big| \Big|^2 + \alpha_2 \int_{\mathbb{R}^N} \Big| z^{\varepsilon\xi,\sigma}\Big(\frac{y}{\varepsilon}\Big) \Big|^2}_{\tau_1} \Big] \\
&\le \frac{\gamma(\varepsilon)}{\varepsilon^2} \alpha_\varepsilon C^{1,2}(\varepsilon\xi) \|U\|^2 \le G(\varepsilon) C' \alpha_\varepsilon C^{1,2}(\varepsilon\xi)
\end{aligned}$$
for ε small enough, we infer that
$$\begin{aligned}
\langle L_{\varepsilon,\sigma,\xi} z^{\varepsilon\xi,\sigma} | z^{\varepsilon\xi,\sigma} \rangle &\le D^2 F^{\varepsilon\xi,\sigma}(z^{\varepsilon\xi,\sigma})(z^{\varepsilon\xi,\sigma}, z^{\varepsilon\xi,\sigma}) + c_1 \varepsilon |\nabla V(\varepsilon\xi)| \\
&\quad + c_2 \varepsilon |\nabla K(\varepsilon\xi)| + c_3 \varepsilon |J_A(\varepsilon\xi)| + c_4 \varepsilon^2 + c_5 C(\varepsilon\xi) G(\varepsilon)
\end{aligned}$$
where $C(\varepsilon\xi) = \alpha_\varepsilon C^{1,2}(\varepsilon\xi)$. Hence (3.3) follows. The proof of (3.4) is more involved. As before, since $z^{\varepsilon\xi,\sigma}$ is a critical point for $F^{\varepsilon\xi,\sigma}$ of mountain-pass type, by standard results (see [18]) there results
$$D^2 F^{\varepsilon\xi,\sigma}(z^{\varepsilon\xi,\sigma})(\phi, \phi) \ge c_1 \|\phi\|^2 \quad \forall \phi \perp \mathcal{N}. \tag{3.6}$$
Let $R >> 1$ and consider a radial smooth function $\chi_1 : \mathbb{R}^N \to \mathbb{R}$ such
$$\chi_1(x) = 1, \quad \text{for } |x| \le R; \quad \chi_1(x) = 0, \quad \text{for } |x| \ge 2R;$$
$$|\nabla \chi_1(x)| \le \frac{2}{R}, \quad \text{for } R \le |x| \le 2R.$$
We also set $\chi_2(x) = 1 - \chi_1(x)$. Given ϕ let us consider the functions
$$\phi_i(x) = \chi_i(x - \xi)\phi(x), \quad i = 1, 2.$$

Due to the definition of χ, straightforward computations yield

$$\|\phi\|^2 = \|\phi_1\|^2 + \|\phi_2\|^2 + 2I_\phi + o_R(1)\|\phi\|^2 \tag{3.7}$$

where $I_\phi = \int_{\mathbb{R}^N} \chi_1\chi_2(\phi^2 + |\nabla\phi|^2)$ and $o_R(1)$ is a function which tends to 0, as $R \to +\infty$.

At this point, let us evaluate the three terms in the equation below:

$$(L_{\varepsilon,\sigma,\xi}\phi|\phi) = \underbrace{(L_{\varepsilon,\sigma,\xi}\phi_1|\phi_1)}_{\alpha_1} + \underbrace{(L_{\varepsilon,\sigma,\xi}\phi_2|\phi_2)}_{\alpha_2} + 2\underbrace{(L_{\varepsilon,\sigma,\xi}\phi_1|\phi_2)}_{\alpha_3}.$$

One has

$$\begin{aligned}
\alpha_1 = \langle L_{\varepsilon,\sigma,\xi}\phi_1|\phi_1\rangle &= D^2F^{\varepsilon\xi,\sigma}(z^{\varepsilon\xi,\sigma})(\phi_1,\phi_1) \\
&\quad + \int_{\mathbb{R}^N} [V(\varepsilon x) - V(\varepsilon\xi)]\,|\phi_1|^2 - \gamma(\varepsilon)\int_{\mathbb{R}^N} W(\varepsilon x))|\phi_1|^2 \\
&\quad - \int_{\mathbb{R}^N} [K(\varepsilon x) - K(\varepsilon\xi)]\,|\phi_1|^{p+1} \\
&\quad + \int_{\mathbb{R}^N} \Big(\Big|\Big(\frac{\nabla}{i} - A(\varepsilon x)\Big)\phi_1\Big|^2 - \Big|\Big(\frac{\nabla}{i} - A(\varepsilon\xi)\Big)\phi_1\Big|^2\Big).
\end{aligned}$$

Using (3.6) (for details, see [23]), we infer

$$D^2F^{\varepsilon\xi}\,[\phi_1,\phi_1] \geq C\|\phi_1\|^2 + o_R(1)\|\phi\|^2. \tag{3.8}$$

Using arguments already carried out before, one has

$$\int_{\mathbb{R}^N} [V(\varepsilon x) - V(\varepsilon\xi)]\,|\phi_1|^2 \leq \varepsilon cR\|\phi\|^2$$

and similarly for the terms containing K. In particular, by the change of variable $y = \varepsilon x$, assumption (W1) and the definition of ϕ_1 we have

$$\begin{aligned}
&\gamma(\varepsilon)\int_{\mathbb{R}^N} W(\varepsilon x)|\phi_1|^2 \leq \frac{\gamma(\varepsilon)}{\varepsilon^2}\Big[\alpha_1\int_{\mathbb{R}^N} |\nabla|\chi_1(x-\xi)\phi|\|^2 + \alpha_2\varepsilon^2\int_{\mathbb{R}^N} |\chi_1(x-\xi)\phi|^2\Big] \\
&= \frac{\gamma(\varepsilon)}{\varepsilon^2}\Big[\alpha_1\Big(\int_{R\leq|y|\leq 2R} |\nabla\chi_1(y)|^2|\phi(y+\xi)|^2 + \int_{|y|\leq 2R} |\chi_1(y)|^2|\nabla\phi(y+\xi)|^2 \\
&\quad + 2\int_{R\leq|y|\leq 2R} \chi_1(y)\phi(y+\xi)\nabla\chi_1(y)\cdot\nabla\phi(y+\xi)\Big) \\
&\quad + \alpha_2\varepsilon^2\int_{|y|\leq 2R} |\chi_1(y)\phi(y+\xi)|^2\Big] \\
&\leq G(\varepsilon)\left[\alpha_\varepsilon\|\phi_1\|^2 + o_R(1)\|\phi\|^2\right]
\end{aligned}$$

It follows that

$$\begin{aligned}
\alpha_1 = (L_{\varepsilon,\sigma,\xi}\phi_1|\phi_1) &\geq c_1\|\phi_1\|^2 - c_2\varepsilon R\|\phi\|^2 + o_R(1)\|\phi\|^2 \\
&\quad - G(\varepsilon)\left[\alpha_\varepsilon\|\phi_1\|^2 + o_R(1)\|\phi\|^2\right]\,.
\end{aligned} \tag{3.9}$$

Let us now estimate α_2. In particular,

$$
\begin{aligned}
&\gamma(\varepsilon)\int_{\mathbb{R}^N} W(\varepsilon x)|\phi_2|^2 \\
&\le \frac{\gamma(\varepsilon)}{\varepsilon^2}\Big[\alpha_1\int_{\mathbb{R}^N}|\nabla|\chi_2(x-\xi)\phi|^2+\alpha_2\varepsilon^2\int_{\mathbb{R}^N}|\chi_2(x-\xi)\phi|^2\Big] \\
&= \frac{\gamma(\varepsilon)}{\varepsilon^2}\Big[\alpha_1\Big(\int_{R\le|y|\le 2R}|\nabla\chi_2(y)|^2|\phi(y+\xi)|^2+\int_{|y|\ge R}|\chi_2(y)|^2|\nabla\phi(y+\xi)|^2 \\
&\quad +2\int_{R\le|y|\le 2R}\chi_2(y)\phi(y+\xi)\nabla\chi_2(y)\cdot\nabla\phi(y+\xi)\Big)+\alpha_2\varepsilon^2\int_{|y|\ge R}|\chi_2(y)\phi(y+\xi)|^2\Big] \\
&\le G(\varepsilon)\left[\alpha_\varepsilon\|\phi_2\|^2+o_R(1)\|\phi\|^2\right].
\end{aligned}
$$

One finds

$$
\alpha_2=(L_{\varepsilon,\sigma,\xi}\phi_2|\phi_2)\ge c_3\|\phi_2\|^2+o_R(1)\|\phi\|^2-G(\varepsilon)\left[\alpha_\varepsilon\|\phi_2\|^2+o_R(1)\|\phi\|^2\right] \tag{3.10}
$$

In a quite similar way one shows that

$$
\begin{aligned}
\alpha_3=(L_{\varepsilon,\sigma,\xi}\phi_1|\phi_2)&\ge c_4 I_\phi+o_R(1)\|\phi\|^2 \\
&\quad -G(\varepsilon)\Big[\left(\alpha_\varepsilon\|\phi_1\|^2+o_R(1)\|\phi\|^2\right)^{1/2}\left(\alpha_\varepsilon\|\phi_2\|^2+o_R(1)\|\phi\|^2\right)^{1/2}\Big]
\end{aligned} \tag{3.11}
$$

Indeed, by the change of variable $y=\varepsilon x$, assumption (W1) and Hölder inequality

$$
\begin{aligned}
&\gamma(\varepsilon)\int_{\mathbb{R}^N}W(\varepsilon x)|\phi_1(x)||\overline{\phi_2(x)}| \\
&\le \frac{\gamma(\varepsilon)}{\varepsilon^N}\Big[\Big(\int_{\mathbb{R}^N}W(y)\Big|\phi_1\Big(\frac{y}{\varepsilon}\Big)\Big|^2\Big)^{1/2}\Big(\int_{\mathbb{R}^N}W(y)\Big|\overline{\phi_2\Big(\frac{y}{\varepsilon}\Big)}\Big|^2\Big)^{1/2}\Big] \\
&\le \frac{\gamma(\varepsilon)}{\varepsilon^2}\Big[\Big(\alpha_1\int_{\mathbb{R}^N}|\nabla|\chi_1(x-\xi)\phi|^2+\alpha_2\varepsilon^2\int_{\mathbb{R}^N}|\chi_1(x-\xi)\phi|^2\Big)^{1/2} \\
&\quad\times\Big(\alpha_1\int_{\mathbb{R}^N}|\nabla|\chi_2(x-\xi)\phi|^2+\alpha_2\varepsilon^2\int_{\mathbb{R}^N}|\chi_2(x-\xi)\phi|^2\Big)^{1/2}\Big] \\
&\le G(\varepsilon)\Big[\left(\alpha_\varepsilon\|\phi_1\|^2+o_R(1)\|\phi\|^2\right)^{1/2}\left(\alpha_\varepsilon\|\phi_2\|^2+o_R(1)\|\phi\|^2\right)^{1/2}\Big]
\end{aligned}
$$

where in the last inequality we have used previous calculations. Finally, (3.9), (3.10), (3.11) and the fact that $I_\phi\ge 0$, yield

$$
\begin{aligned}
(L_{\varepsilon,\sigma,\xi}\phi|\phi)&=\alpha_1+\alpha_2+2\alpha_3 \\
&\ge c_5\left[\|\phi_1\|^2+\|\phi_2\|^2+2I_\phi\right]-c_6R\varepsilon\|\phi\|^2+o_R(1)\|\phi\|^2 \\
&\quad -G(\varepsilon)\alpha_\varepsilon\Big[\|\phi_1\|^2+\|\phi_2\|^2+2\left(\|\phi_1\|^2+o_R(1)\|\phi\|^2\right)^{1/2} \\
&\quad\times\left(\|\phi_2\|^2+o_R(1)\|\phi\|^2\right)^{1/2}+o_R(1)\|\phi\|^2\Big]
\end{aligned}
$$

Recalling (3.7), we infer that

$$
\begin{aligned}
(L_{\varepsilon,\sigma,\xi}\phi|\phi)&\ge c_7\|\phi\|^2-c_8R\varepsilon\|\phi\|^2+o_R(1)\|\phi\|^2 \\
&\quad -G(\varepsilon)\alpha_\varepsilon\Big[\|\phi_1\|^2+\|\phi_2\|^2+2\left(\|\phi_1\|^2+o_R(1)\|\phi\|^2\right)^{1/2} \\
&\quad\times\left(\|\phi_2\|^2+o_R(1)\|\phi\|^2\right)^{1/2}+o_R(1)\|\phi\|^2\Big]
\end{aligned}
$$

Taking $R = \varepsilon^{-1/2}$, and choosing ε small, equation (3.4) follows. This completes the proof. $\square$

4. The finite-dimensional reduction

In this section we will show that the existence of critical points of f_ε can be reduced to the search of critical points of an auxiliary finite-dimensional functional. The proof will be carried out in two subsections dealing, respectively, with a Liapunov-Schmidt reduction, and with the behavior of the auxiliary finite dimensional functional.

4.1. **A Liapunov-Schmidt type reduction.**

The main result of this section is the following lemma.

Lemma 4.1. *For $\varepsilon > 0$ small, $|\xi| \le \bar{\xi}$ and $\sigma \in [0, 2\pi]$, there exists a unique $w = w(\varepsilon, \sigma, \xi) \in (T_{z^{\varepsilon\xi,\sigma}} Z^\varepsilon)^\perp$ such that $\nabla f_\varepsilon(z^{\varepsilon\xi,\sigma} + w) \in T_{z^{\varepsilon\xi,\sigma}} Z^\varepsilon$. Such a $w(\varepsilon, \sigma, \xi)$ is of class C^2, respectively $C^{1,p-1}$, with respect to ξ, provided that $p \ge 2$, respectively $1 < p < 2$. Moreover, the functional $\Phi_\varepsilon(\sigma, \xi) = f_\varepsilon(z^{\varepsilon\xi,\sigma} + w(\varepsilon, \sigma, \xi))$ has the same regularity as w and satisfies:*

$$\nabla \Phi_\varepsilon(\sigma_0, \xi_0) = 0 \Longleftrightarrow \nabla f_\varepsilon(z_{\xi_0} + w(\varepsilon, \sigma_0, \xi_0)) = 0.$$

For the proof of the above lemma, we refer to [23, Lemma 4.1].

Remark 4.2. Since $f_\varepsilon(z^{\varepsilon\xi,\sigma})$ is independent of σ, the implicit function w is constant with respect to that variable. Consequently, there exists a functional $\Psi_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ such that

$$\Phi_\varepsilon(\sigma, \xi) = \Psi_\varepsilon(\xi), \quad \forall \sigma \in [0, 2\pi],\ \forall \xi \in \mathbb{R}^N.$$

For this reason, in the sequel we will omit the dependence of w on σ, even it is defined over $S^1 \times \mathbb{R}^N$.

Remark 4.3. From the proof of Lemma 4.1 (see [23]) and Lemma 2.1, it follows that:

$$\|w\| \le C \left(\varepsilon |\nabla V(\varepsilon\xi)| + \varepsilon |\nabla K(\varepsilon\xi)| + \varepsilon |J_A(\varepsilon\xi)| + \varepsilon^2 + C(\varepsilon\xi) G(\varepsilon) \right), \tag{4.1}$$

where $C > 0$.

For future reference, it is also convenient to estimate the gradient $\nabla_\xi w$.

Lemma 4.4. *It results*

$$\|\nabla_\xi w\| \le c \left(\varepsilon |\nabla V(\varepsilon\xi)| + \varepsilon |\nabla K(\varepsilon\xi)| + \varepsilon |J_A(\varepsilon\xi)| + O(\varepsilon^2) \right)^\gamma, \tag{4.2}$$

where $\gamma = \min\{1, p-1\}$ and $c > 0$ is some constant.

Proof. For the details, we refer to [2, Lemma 4] and [23, Lemma 4.2]. We will denote by $\dot{w}_i$ the components of $\nabla_\xi w$ and $\dot{z}_i = \partial_{\xi_i} z$. Since w satisfies the equation $\langle P \nabla f_\varepsilon(z^{\varepsilon\xi,\sigma} + w), v \rangle = 0$ for all $v \in (T_{z^{\varepsilon\xi,\sigma}} Z^\varepsilon)^\perp$ (with $P =$ the projection onto $(T_{z^{\varepsilon\xi,\sigma}} Z^\varepsilon)^\perp$), we find that $\dot{w}_i$ verifies

$$\partial_{\xi_i} \left(\langle L_{\varepsilon,\sigma,\xi} w, v \rangle + \langle P \nabla f_\varepsilon(z), v \rangle + \langle R(z, w), v \rangle \right) = 0$$

with $R(z, w) = \|o(w)\|$. Taking into account [23, Lemma 4.2], we limit to estimate the ∂_{ξ_i} of $\nabla W_\varepsilon(z)[v]$, namely

$$\partial_{\xi_i} \Big(-\gamma(\varepsilon) \operatorname{Re} \int_{\mathbb{R}^N} W(\varepsilon x) z \bar{v} \Big) = -\gamma(\varepsilon) \operatorname{Re} \int_{\mathbb{R}^N} W(\varepsilon x) \dot{z}_i \bar{v}.$$

As in (2.13), by (W1) and the expression of $\dot z_i$ in (2.10) we get

$$\gamma(\varepsilon)\Big|\int_{\mathbb{R}^N} W(\varepsilon x)\dot z_i \overline{v}\Big| \le \gamma(\varepsilon)\int_{\mathbb{R}^N} W(\varepsilon x)\,|\dot z_i|\,|\overline{v}| \le \widetilde{C}(\varepsilon\xi)\varepsilon G(\varepsilon)\|v\|$$

where $\widetilde{C}(\varepsilon\xi)$ depends on α and β. From [2, Lemma 4], Inequality (4.2) follows without effort. □

4.2. **The finite-dimensional functional.** The purpose of this subsection is to give an explicit form to the finite dimensional functional $\Phi_\varepsilon(\sigma,\xi) = \Psi_\varepsilon(\xi) = f_\varepsilon(z^{\varepsilon\xi,\sigma} + w(\varepsilon,\xi))$. For brevity, we set in the sequel $z = z^{\varepsilon\xi,\sigma}$ and $w = w(\varepsilon,\xi)$. Since z satisfies (2.5) and K'' is bounded we get

$$\begin{aligned}
\Phi_\varepsilon(\sigma,\xi) &= f_\varepsilon(z^{\varepsilon\xi,\sigma} + w(\varepsilon,\sigma,\xi)) \\
&= K(\varepsilon\xi)\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_{\mathbb{R}^N} |z|^{p+1} + \frac{1}{2}\int_{\mathbb{R}^N} |A(\varepsilon\xi) - A(\varepsilon x)|^2 z^2 \\
&\quad + \operatorname{Re}\int_{\mathbb{R}^N} (A(\varepsilon\xi) - A(\varepsilon x))\, z \cdot (A(\varepsilon\xi) - A(\varepsilon x))\,\overline{w} \\
&\quad + \varepsilon\operatorname{Re}\int_{\mathbb{R}^N} \frac{1}{i} z\overline{w}\operatorname{div} A(\varepsilon x) + \frac{1}{2}\int_{\mathbb{R}^N} \Big|\Big(\frac{\nabla}{i} - A(\varepsilon x)\Big) w\Big|^2 \\
&\quad + \operatorname{Re}\int_{\mathbb{R}^N} [V(\varepsilon x) - V(\varepsilon\xi)]\, z\overline{w} + \frac{1}{2}\int_{\mathbb{R}^N} [V(\varepsilon x) - V(\varepsilon\xi)]\,|w|^2 \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} [V(\varepsilon x) - V(\varepsilon\xi)]\,|z|^2 + \frac{1}{2}V(\varepsilon\xi)\int_{\mathbb{R}^N} |w|^2 \\
&\quad - \frac{\gamma(\varepsilon)}{2}\int_{\mathbb{R}^N} W(\varepsilon x)|z|^2 - \gamma(\varepsilon)\operatorname{Re}\int_{\mathbb{R}^N} W(\varepsilon x) z\overline{w} \\
&\quad - \frac{\gamma(\varepsilon)}{2}\int_{\mathbb{R}^N} W(\varepsilon x)|w|^2 \\
&\quad - \frac{1}{p+1}\operatorname{Re}\int_{\mathbb{R}^N} K(\varepsilon x)\big(|z+w|^{p+1} - |z|^{p+1} - (p+1)|z|^{p-1}z\overline{w}\big) \\
&\quad + \operatorname{Re} K(\varepsilon\xi)\int_{\mathbb{R}^N} |z|^{p-1} z\overline{w} + O(\varepsilon^2).
\end{aligned} \tag{4.3}$$

By the definition of $\alpha(\varepsilon\xi)$ and $\beta(\varepsilon\xi)$ we get immediately

$$\int_{\mathbb{R}^N} |z^{\varepsilon\xi,\sigma}|^{p+1} = C_0\Lambda(\varepsilon\xi)\,[K(\varepsilon\xi)]^{-1} \tag{4.4}$$

where we define the auxiliary function

$$\Lambda(x) = \frac{(1+V(x))^\theta}{K(x)^{2/(p-1)}}, \quad \theta = \frac{p+1}{p-1} - \frac{N}{2} \tag{4.5}$$

for all $x \in \mathbb{R}^N$ since, by (K1), K is strictly positive on $\mathbb{R}^N$ and $C_0 = \int_{\mathbb{R}^N} |U|^{p+1}$. Now we can estimate the various terms in (4.3) by means of (4.1) and (4.2) as in

[23]. In particular,

$$\gamma(\varepsilon)\int_{\mathbb{R}^N} W(\varepsilon x)|z|^2 \le \frac{\gamma(\varepsilon)}{\varepsilon^2}\alpha_\varepsilon C^{1,2}(\varepsilon\xi)\|U\|^2 \le G(\varepsilon)C(\varepsilon\xi)C_1,$$

$$\gamma(\varepsilon)\int_{\mathbb{R}^N} W(\varepsilon x)|z||\overline{w}| \le \frac{\gamma(\varepsilon)}{\varepsilon^2}\alpha_\varepsilon C'(\varepsilon\xi)C_2'\|U\|\|w\| \le G(\varepsilon)C''(\varepsilon\xi)C_2\|w\|,$$

$$\gamma(\varepsilon)\int_{\mathbb{R}^N} W(\varepsilon x)|w|^2 \le \frac{\gamma(\varepsilon)}{\varepsilon^2}\alpha_\varepsilon C_3\|w\|^2 \le G(\varepsilon)\alpha_\varepsilon C_3\|w\|^2$$

where

$$C(\varepsilon\xi) = \alpha_\varepsilon C^{1,2}(\varepsilon\xi) = \alpha_\varepsilon \alpha^2(\varepsilon\xi)\beta(\varepsilon\xi)^{-N}\max\{1,\beta^2(\varepsilon\xi)\}$$

and $C''(\varepsilon\xi) = \alpha_\varepsilon C'(\varepsilon\xi)$ with $C'(\varepsilon\xi) = (C^{1,2}(\varepsilon\xi))^{1/2}$. So it results

$$\Phi_\varepsilon(\sigma,\xi) = \Psi_\varepsilon(\xi) = C_1\Lambda(\varepsilon\xi) + O(\varepsilon). \tag{4.6}$$

Similarly,

$$\nabla\Psi_\varepsilon(\xi) = C_1\nabla\Lambda(\varepsilon\xi) + \varepsilon^{1+\gamma}O(1) \tag{4.7}$$

where $C_1 = (\frac{1}{2} - \frac{1}{p+1})C_0$. Indeed, taking account of the result in [23], we limit to consider

$$\begin{aligned}
\nabla_\xi W_\varepsilon(z+w) &= \nabla_\xi\Big(-\frac{\gamma(\varepsilon)}{2}\int_{\mathbb{R}^N} W(\varepsilon x)|z+w|^2\Big) = \langle W_\varepsilon'(z+w), (\nabla_\xi z + \nabla_\xi w)\rangle \\
&= -\gamma(\varepsilon)\,\mathrm{Re}\int_{\mathbb{R}^N} W(\varepsilon x)(z+w)\,\overline{\nabla_\xi z + \nabla_\xi w} \\
&= -\gamma(\varepsilon)\,\mathrm{Re}\int_{\mathbb{R}^N} W(\varepsilon x)z\overline{\nabla_\xi z} - \gamma(\varepsilon)\,\mathrm{Re}\int_{\mathbb{R}^N} W(\varepsilon x)z\overline{\nabla_\xi w} \\
&\quad - \gamma(\varepsilon)\,\mathrm{Re}\int_{\mathbb{R}^N} W(\varepsilon x)w\overline{\nabla_\xi z} - \gamma(\varepsilon)\,\mathrm{Re}\int_{\mathbb{R}^N} W(\varepsilon x)w\overline{\nabla_\xi w}.
\end{aligned}$$

whose last four terms can be estimated as in (2.13) by means of (4.1) and (4.2) again so that (4.7) holds.

5. Statement and proof of the main results

In this section we obtain existence and multiplicity of solutions to (1.5) by means of the finite-dimensional reduction performed in the previous section. Recalling Lemma 4.1, we have to look for critical points of Φ_ε as a function of the variables $(\sigma,\xi) \in [0,2\pi]\times\mathbb{R}^N$ (or, equivalently, $(\eta,\xi)\in S^1\times\mathbb{R}^N$).

We use the following notation: given a set $M\subset\mathbb{R}^N$ and a number $\delta>0$,

$$M_\delta := \{x\in\mathbb{R}^N : \mathrm{dist}(x,\Omega) < \delta\}.$$

If $M\subset N$, $\mathrm{cat}(M,N)$ denotes the Ljusternik-Schnirelman category of M with respect to N, namely the least integer k such that M can be covered by k closed subsets of N, contractible to a point in N. We set $\mathrm{cat}(M) = \mathrm{cat}(M,M)$. We start with the following result, which deals with local extrema.

Theorem 5.1. *Suppose we are in the hypotheses of Theorem 1.1. Assume moreover that there is a compact set $M\subset\mathbb{R}^N$ over which Λ achieves an isolated strict local minimum, resp. maximum, with value a, resp. b, in the sense that for some $\delta>0$,*

$$b := \inf_{x\in\partial M_\delta}\Lambda(x) > a, \quad \textit{resp. } a := \sup_{x\in\partial M_\delta}\Lambda(x) < b.$$

Then there exists $\varepsilon_\delta > 0$ such that (1.6) has at least $\operatorname{cat}(M, M_\delta)$ (orbits of) solutions concentrating near M_δ, for all $0 < \varepsilon < \varepsilon_\delta$.

For the sake of completeness, we rewrite the proof as in [2, 23].

Proof. Recall that $\Phi_\varepsilon(\eta,\xi) = \Psi_\varepsilon(\xi)$ and choose $\bar{\xi} > 0$ such that $M_\delta \subset \{x \in \mathbb{R}^N | \ |x| < \bar{\xi}\}$. Set $N^\varepsilon = \{\xi \in \mathbb{R}^N | \ \varepsilon\xi \in M\}$, $N_\delta^\varepsilon = \{\xi \in \mathbb{R}^N | \ \varepsilon\xi \in M_\delta\}$ and $\Theta^\varepsilon = \{\xi \in \mathbb{R}^N | \ \Psi_\varepsilon(\xi) \le C_1 \frac{a+b}{2}\}$. From (4.6) we get some $\varepsilon_\delta > 0$ such that

$$N^\varepsilon \subset \Theta^\varepsilon \subset N_\delta^\varepsilon, \tag{5.1}$$

for all $0 < \varepsilon < \varepsilon_\delta$. To apply standard category theory, it suffices to prove that Θ_ε cannot touch $\partial N_\delta^\varepsilon$ so that Θ_ε is compact. But if $\varepsilon\xi \in \partial N_\delta^\varepsilon$, one has $\Lambda(\varepsilon\xi) \ge b$ by the definition of δ, and so

$$\Psi_\varepsilon(\xi) \ge C_1\Lambda(\varepsilon\xi) + o_\varepsilon(1) \ge C_1 b + o_\varepsilon(1).$$

On the other hand, for all $\xi \in \Theta^\varepsilon$ one has also $\Psi_\varepsilon(\xi) \le C_1\frac{a+b}{2}$. From (5.1) and elementary properties of the Ljusternik-Schnirelman category we can conclude that Ψ_ε has at least

$$\operatorname{cat}(\Theta^\varepsilon, \Theta^\varepsilon) \ge \operatorname{cat}(N^\varepsilon, N_\delta^\varepsilon) = \operatorname{cat}(N, N_\delta)$$

critical points in Θ^ε, which correspond to at least $\operatorname{cat}(M, M_\delta)$ orbits of solutions to (1.6). Now, let $(\eta^*, \xi^*) \in S^1 \times M_\delta$ a critical point of Φ_ε. By Lemma 4.1, this point localizes a solution $u_{\varepsilon,\eta^*,\xi^*}(x) = z^{\varepsilon\xi^*,\eta^*}(x) + w(\varepsilon,\eta^*,\xi^*)$ of (1.6). By the change of variable which allowed us to pass from (1.5) to (1.6) we find that

$$u_{\varepsilon,\eta^*,\xi^*}(x) \approx z^{\varepsilon\xi^*,\eta^*}\left(\frac{x-\xi^*}{\varepsilon}\right) \tag{5.2}$$

satisfies (1.5) where $\approx$ stands for the concept of "near" or "close" whose sense is explained in the following Remark 5.2. The concentration statement follows as in [2] from standard arguments. The proof of the second part follows with analogous arguments. $\square$

Remark 5.2. By means of a Liapunov-Schmidt type reduction, we have found that a solution of (1.6) has the form $u_{\varepsilon,\eta^*,\xi^*}(x) = z^{\varepsilon\xi^*,\eta^*}(x) + w(\varepsilon,\eta^*,\xi^*)$. From this and the properties of the function $w(\varepsilon,\eta^*,\xi^*)$, it follows that $\|u_{\varepsilon,\eta^*,\xi^*} - z^{\varepsilon\xi^*,\eta^*}\| \to 0$ as $\varepsilon \to 0$. In this sense, we say that the complex solutions $u_{\varepsilon,\eta^*,\xi^*}(x)$ of (1.6) are found "near" or "close" to the least energy solutions $z^{\varepsilon\xi^*,\eta^*}$ of (1.7) and this corresponds, after the change of variable, to (5.2).

Observe that Theorem 1.1 in the Introduction is an immediate corollary of the previous one when x_0 is either a nondegenerate local maximum or minimum for Λ. When Λ has a maximum, the direct variational approach and the arguments in [7] cannot be applied.

To treat the general case, we refer to some topological concepts as the *cup long* of a set $M \subset \mathbb{R}^N$ which is by definition

$$l(M) = 1 + \sup\{k \in \mathbb{N} \mid (\exists \alpha_1, \dots, \alpha_N \in \check{H}^*(M) \setminus \{1\})(\alpha_1 \cup \dots \cup \alpha_k \ne 0)\},$$

where $\check{H}^*(M)$ is the Alexander cohomology of M with real coefficients and $\cup$ denotes the cup product. In some cases as $M = S^{N-1}$, T^N, we have $l(M) = \operatorname{cat}(M)$, but in general $l(M) \le \operatorname{cat}(M)$. Furthermore we recall the following definition which dates back to Bott [12]:

Definition 5.3. We say that M is non-degenerate for a C^2 function $I : \mathbb{R}^N \to \mathbb{R}$ if M consists of Morse theoretically non-degenerate critical points for the restriction $I_{|M^\perp}$.

To prove our existence result, we recall the following result which is an adaptation of [18, Theorem 6.4, Chapter II] and fits into the frame of the Conley theory [27].

Theorem 5.4. *Let $I \in C^1(V)$ and $J \in C^2(V)$ be two functionals defined on the Riemannian manifold V, and let $\Sigma \subset V$ be a smooth, compact, non-degenerate manifold of critical points of J. Denote by $\mathcal{U}$ a neighborhood of Σ. If $\|I - J\|_{C^1(\mathcal{U})}$ is small enough, then the functional I has at least $l(\Sigma)$ critical points contained in $\mathcal{U}$.*

At this point, we can prove an existence and multiplicity result for (1.5).

Theorem 5.5. *Let* (K1), (V1), (W1), (A1), (G1) *hold. If the auxiliary function Λ has a smooth, compact, non-degenerate manifold of critical points M, then for $\varepsilon > 0$ small, the problem* (1.6) *has at least $l(M)$ (orbits) of solutions concentrating near points of M.*

Proof. By Remark 4.2, we have to find critical points of $\Psi_\varepsilon = \Psi_\varepsilon(\xi)$. Since M is compact, we can choose and fix $\overline{\xi} > 0$ so that $|x| < \overline{\xi}$ for all $x \in M$. $\{\eta^*\} \times M$ is obviously a non-degenerate critical manifold. We set $V = \mathbb{R}^N$, $J = \Lambda$, $\Sigma = M$ and $I(\xi) = \Psi_\varepsilon(\eta, \xi/\varepsilon)$. Select $\delta > 0$ so that $M_\delta \subset \{x \in \mathbb{R}^N |\ |x| < \overline{\xi}\}$, and no critical points of Λ are in M_δ, except for those of M. Set $\mathcal{U} = M_\delta$. By the definition of (4.6) and (4.7) and hypotheses (K1) and (V1), it follows that $J \in C^2(\overline{\mathcal{U}})$. As concerns as the regularity of the functional I, we have to prove that the functional

$$\begin{aligned}
\widetilde{W}(\xi) &= \widetilde{W}_\varepsilon(\eta, \xi/\varepsilon) \\
&= -\frac{\gamma(\varepsilon)}{2}\int_{\mathbb{R}^N} W(\varepsilon x)|z^{\xi,\eta}|^2 - \gamma(\varepsilon)\,\mathrm{Re}\int_{\mathbb{R}^N} W(\varepsilon x) z^{\xi,\eta}\overline{w(\varepsilon,\eta,\xi/\varepsilon)} \\
&\quad - \frac{\gamma(\varepsilon)}{2}\int_{\mathbb{R}^N} W(\varepsilon x)|w(\varepsilon,\eta,\xi/\varepsilon)|^2
\end{aligned}$$

is of class $C^1(V)$. Indeed, by its definition, $z^{\xi,\eta}$ depends on the functions $\alpha(\xi)$ and $\beta(\xi)$ so on the potentials $V(\xi)$ and $K(\xi)$ which are both in $C^1(\mathbb{R}^N)$ (with respect to ξ) by hypotheses (K1) and (V1). Furthermore, by Lemma 4.1, w is of class C^2 (if $p \geq 2$) or $C^{1,p-1}$ (if $1 < p < 2$) and the result follows without effort. Again by (4.6) and (4.7), it results that I is close to J in $C^1(\overline{\mathcal{U}})$ when ε is very small. We can apply Theorem 5.4 to find at least $l(M)$ critical points $\{\xi_1, \dots, \xi_{l(M)}\}$ for Ψ_ε, provided ε is small enough. Hence the orbits $S^1 \times \{\xi_1\}, \dots, S^1 \times \{\xi_{l(M)}\}$ consist of critical points for Φ_ε which produce solutions of (1.6). The concentration statement follows as in [2]. □

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