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MULTIPLE POSITIVE SOLUTIONS OF FOURTH-ORDER FOUR-POINT BOUNDARY-VALUE PROBLEMS WITH CHANGING SIGN COEFFICIENT

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Abstract. In this paper, we investigate the existence of multiple positive solutions of the fourth-order four-point boundary-value problems

$$\begin{gathered} y^{(4)}(t) = h(t)g(y(t), y''(t)), \quad 0 < t < 1, \\ y(0) = y(1) = 0, \\ ay''(\xi_1) - by'''(\xi_1) = 0, \quad cy''(\xi_2) + dy'''(\xi_2) = 0, \end{gathered}$$

where $0 < \xi_1 < \xi_2 < 1$. We show the existence of three positive solutions by applying the Avery and Peterson fixed point theorem in a cone, here $h(t)$ may change sign on $[0, 1]$.

1. Introduction

Recently, several authors have studied the existence of positive solutions to boundary-value problems for fourth-order differential equations. For details; see, for example, [3, 4, 5, 6, 7, 8, 9, 10]. Zhong, Chen and Wang [10] investigated the fourth-order nonlinear ordinary differential equation

$$y^{(4)}(t) - f(t, y(t), y''(t)) = 0, \quad 0 \le t \le 1, \tag{1.1}$$

with the four-point boundary conditions

$$\begin{gathered} y(0) = y(1) = 0, \\ ay''(\xi_1) - by'''(\xi_1) = 0, \quad cy''(\xi_2) + dy'''(\xi_2) = 0, \end{gathered} \tag{1.2}$$

where $f \in C([0, 1] \times [0, \infty) \times (-\infty, 0], [0, \infty))$, a, b, c, d are nonnegative constants, and $0 \le \xi_1 < \xi_2 \le 1$. Some results on the existence of at least one positive solution to BVP (1.1)-(1.2) are obtained by using the Krasnoselskii fixed point theorem. Their key result reads as follows.

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Lemma 1.1 ([10, Lemma 2.2]). *If* $\alpha = ad+bc+ac(\xi_2-\xi_1) \neq 0$ *and* $h(t) \in C[\xi_1, \xi_2]$, *then the boundary-value problem*

$$\begin{gathered} u^{(4)}(t) = h(t), \quad 0 < t < 1, \\ u(0) = u(1) = 0, \\ au''(\xi_1) - bu'''(\xi_1) = 0, \quad cu''(\xi_2) + du'''(\xi_2) = 0 \end{gathered}$$

has a unique solution

$$u(t) = \int_0^1 G_1(t,s) \int_{\xi_1}^{\xi_2} G_2(s,\tau)h(\tau)\,d\tau\,ds, \tag{1.3}$$

where

$$\begin{gathered} G_1(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t < s \le 1, \end{cases} \\ G_2(t,s) = \frac{1}{\alpha}\begin{cases} (a(s-\xi_1)+b)(d+c(\xi_2-t)), & s < t \le 1,\ \xi_1 \le s \le \xi_2, \\ (a(t-\xi_1)+b)(d+c(\xi_2-s)), & 0 \le t \le s,\ \xi_1 \le s \le \xi_2. \end{cases} \end{gathered} \tag{1.4}$$

Unfortunately this lemma is wrong. Indeed, by [2, Lemma 2.1], expression (1.3) should be replaced by

$$\begin{aligned} u(t) = &\int_0^1 G_1(t,s) \int_t^{\xi_1} (\tau - s)h(\tau)\,d\tau\,ds \\ &+ \frac{1}{\delta}\int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2} (a(\xi_1 - s) - b)(c(\xi_2-\tau)+d)h(\tau)\,d\tau\,ds, \end{aligned} \tag{1.5}$$

where $\delta = ad+bc+ac(\xi_2-\xi_1) > 0$. So the conclusions in [10] should be reconsidered. If $f(t,y(t),y''(t))$ in (1.1) are replaced by $h(t)g(y(t),y''(t))$, then (1.1) reduces to

$$y^{(4)}(t) - h(t)g(y(t),y''(t)) = 0, \quad 0 \le t \le 1, \tag{1.6}$$

where $h \in C[0,1]$ and $g \in C([0,\infty) \times (-\infty,0], [0,\infty))$.

To the authors' knowledge, no one has studied the existence of positive solutions for problem (1.6), (1.2) using the assumption that $h(t)$ changes sign. Hence, the aim of this paper is to investigate the existence of positive solutions of the BVP (1.6) and (1.2) by using a triple positive fixed-point theorem of Avery and Peterson in [1].

2. Preliminaries

Let $E = \{y \in C^2[0,1] : y(0) = y(1) = 0\}$. Then we have the following lemma.

Lemma 2.1 ([10]). *For* $y \in E$, *we get*

$$\|y\|_\infty \le \|y'\|_\infty \le \|y''\|_\infty,$$

where $\|y\|_\infty = \sup_{t\in[0,1]} |y(t)|$.

By Lemma 2.1, E is a Banach space with the norm $\|y\| = \|y''\|_\infty$. We define the operator $T : E \to E$ by

$$Ty(t) = \int_0^1 G_1(t,s)(Qy)(s)ds, \tag{2.1}$$

where $G_1(t,s)$ as in (1.4), and

$$
\begin{aligned}
(Qy)(s) &= \int_{\xi_1}^{s} (\tau - s)h(\tau)g\big(y(\tau), y''(\tau)\big)d\tau \\
&\quad + \frac{1}{\delta}\int_{\xi_1}^{\xi_2} (b - a(\xi_1 - s))(c(\xi_2 - \tau) + d)h(\tau)g\big(y(\tau), y''(\tau)\big)d\tau.
\end{aligned} \tag{2.2}
$$

Here, $\delta = ad + bc + ac(\xi_2 - \xi_1) > 0$.

From [2, Lemma 2.1], we easily know that $u(t)$ is a solution of the four-point boundary-value problem (1.6), (1.2) if and only if $u(t)$ is a fixed point of the operator T.

It is rather straightforward to show that

$$0 \le G_1(t,s) \le G_1(s,s), \quad 0 \le t, s \le 1, \tag{2.3}$$

and

$$G_1(t,s) \ge \omega G_1(s,s), \quad t \in [\omega, 1-\omega], \quad s \in [0,1], \tag{2.4}$$

where

$$0 < \omega < \min\{\xi_1, 1 - \xi_2\} < \frac{1}{2}. \tag{2.5}$$

For the convenience of the reader, we present some definitions from the cone theory in Banach spaces.

Definition. The map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\alpha : P \to [0,\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \ge t\alpha(x) + (1-t)\alpha(y), \quad \forall x, y \in P, \quad 0 \le t \le 1.$$

Similarly, we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E provided that $\beta : P \to [0,\infty)$ is continuous and

$$\beta(tx + (1-t)y) \le t\beta(x) + (1-t)\beta(y), \quad \forall x, y \in P, \quad 0 \le t \le 1.$$

Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. Then for positive real numbers a, b, c, and d, we define the following convex sets:

$$
\begin{gathered}
P(\gamma, d) = \{x \in P : \gamma(x) < d\}, \\
P(\gamma, \alpha, b, d) = \{x \in P : b \le \alpha(x), \gamma(x) \le d\}, \\
P(\gamma, \theta, \alpha, b, c, d) = \{x \in P : b \le \alpha(x), \theta(x) \le c, \gamma(x) \le d\}, \\
R(\gamma, \psi, a, d) = \{x \in P : a \le \psi(x), \gamma(x) \le d\}.
\end{gathered}
$$

The following fixed-point theorem due to Avery and Peterson is fundamental in the proof of our main result.

Lemma 2.2 ([1]). *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \le \lambda \psi(x)$ for $0 \le \lambda \le 1$, such that for some positive numbers M and d,*

$$\alpha(x) \le \psi(x) \quad \text{and} \quad \|x\| \le M\gamma(x),$$

for all $x \in \overline{P(\gamma, d)}$. Suppose $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers a, b, and c with $a < b$ such that

(i) $\{x \in P(\gamma, \theta, \alpha, b, c, d) : \alpha(x) > b\} \neq$ *and* $\alpha(Tx) > b$ *for* $x \in P(\gamma, \theta, \alpha, b, c, d)$*;*
(ii) $\alpha(Tx) > b$ *for* $x \in P(\gamma, \alpha, b, d)$ *with* $\theta(Tx) > c$*;*
(iii) $0 \notin R(\gamma, \psi, a, d)$ *and* $\psi(Tx) < a$ *for* $x \in R(\gamma, \psi, a, d)$ *with* $\psi(x) = a$*.*

Then T *has at least three fixed points* $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$*, such that*

$$\gamma(x_i) \leq d \textit{ for } i = 1, 2, 3, \quad b < \alpha(x_1), \quad a < \psi(x_2) \text{ with } \alpha(x_2) < b, \quad \psi(x_3) < a.$$

3. Main result

Define the cone $P \subset E = \{y \in C^2[0, 1] : y(0) = y(1) = 0\}$ by

$$P = \{y \in E : y(t) \geq 0,\ y \text{ is concave on } [0, 1]\}.$$

Let the nonnegative, increasing, continuous functionals γ, ψ, θ and α be

$$\gamma(y) = \max_{0 \leq t \leq 1} |y''(t)|, \quad \psi(y) = \theta(y) = \max_{0 \leq t \leq 1} |y(t)|, \quad \alpha(y) = \min_{\omega \leq t \leq 1 - \omega} |y(t)|.$$

We make the following assumptions:

(H1) $g : [0, \infty) \times (-\infty, 0] \to [0, \infty)$ is continuous;
(H2) $h \in C[0, 1]$, $h(t) \leq 0$, $\forall t \in [0, \xi_1]$, $h(t) \geq 0$, $\forall t \in [\xi_1, \xi_2]$, $h(t) \leq 0$, $\forall t \in [\xi_2, 1]$, and $h(t)$ is not identically zero on any subinterval of $[0, 1]$.

Lemma 3.1. *Assume that* (H1)–(H2) *hold. If* $b \geq a\xi_1$ *and* $d \geq c(1 - \xi_2)$*, then* $T : P \to P$ *is completely continuous.*

Proof. For each $t \in [0, 1]$, we consider three cases:
Case 1: $t \in [0, \xi_1]$. For any $y \in P$, we have from (2.2), (H1), (H2) and $b \geq a\xi_1$ that

$$\begin{aligned} (Qy)(t) &= \int_t^{\xi_1} (t - \tau) h(\tau) g\big(y(\tau), y''(\tau)\big) d\tau \\ &\quad + \frac{1}{\delta} \int_{\xi_1}^{\xi_2} (b - a\xi_1 + at)(c(\xi_2 - \tau) + d) h(\tau) g\big(y(\tau), y''(\tau)\big) d\tau \geq 0. \end{aligned} \tag{3.1}$$

Case 2: $t \in [\xi_1, \xi_2]$. For each $y \in P$, we have from (H1), (H2) and (2.2) that

$$\begin{aligned} (Qy)(t) &= \int_{\xi_1}^t (\tau - t) h(\tau) g\big(y(\tau), y''(\tau)\big) d\tau \\ &\quad + \frac{1}{\delta} \int_{\xi_1}^t (b - a\xi_1 + at)(c(\xi_2 - \tau) + d) h(\tau) g\big(y(\tau), y''(\tau)\big) d\tau \\ &\quad + \frac{1}{\delta} \int_t^{\xi_2} (b - a\xi_1 + at)(c(\xi_2 - \tau) + d) h(\tau) g\big(y(\tau), y''(\tau)\big) d\tau \\ &= \frac{1}{\delta} \int_{\xi_1}^t (b + a(\tau - \xi_1))(c(\xi_2 - t) + d) h(\tau) g\big(y(\tau), y''(\tau)\big) d\tau \\ &\quad + \frac{1}{\delta} \int_t^{\xi_2} (b + a(t - \xi_1))(c(\xi_2 - \tau) + d) h(\tau) g\big(y(\tau), y''(\tau)\big) d\tau \geq 0. \end{aligned} \tag{3.2}$$

Case 3: $t \in [\xi_2, 1]$. For any $y \in P$, we have from (H1), (H2), (2.2) and $d \geq (1-\xi_2)c$ that

$$\begin{aligned}(Qy)(t) &= \int_{\xi_1}^{\xi_2} (\tau - t)h(\tau)g\big(y(\tau), y''(\tau)\big)d\tau + \int_{\xi_2}^{t} (\tau - t)h(\tau)g\big(y(\tau), y''(\tau)\big)d\tau \\ &\quad + \frac{1}{\delta}\int_{\xi_1}^{\xi_2} (b - a\xi_1 + at)(c(\xi_2 - \tau) + d)h(\tau)g\big(y(\tau), y''(\tau)\big)d\tau \\ &= \frac{1}{\delta}\int_{\xi_1}^{\xi_2} (b + a(\tau - \xi_1))(d - c(t - \xi_2))h(\tau)g\big(y(\tau), y''(\tau)\big)d\tau \\ &\quad + \int_{\xi_2}^{t} (\tau - t)h(\tau)g\big(y(\tau), y''(\tau)\big)d\tau \geq 0.\end{aligned} \tag{3.3}$$

Thus, from (3.1)-(3.3), we get

$$(Qy)(t) \geq 0, \quad t \in [0, 1]. \tag{3.4}$$

Therefore, by (2.1), $G_1(t, s) \geq 0$ and (3.4), we obtain

$$(Ty)(t) \geq 0, \quad t \in [0, 1]. \tag{3.5}$$

Obviously, we have $(Tu)(0) = (Tu)(1) = 0$, and

$$(Tu)''(t) = -(Qu)(t) \leq 0, \quad t \in [0, 1].$$

Hence, $T : P \to P$. Moreover, it is easy to check by the Arzera-Ascoli theorem that the operator T is completely continuous. □

Remark 3.2. By $\delta = ad + bc + ac(\xi_2 - \xi_1) > 0$, $b \geq a\xi_1$ and $d \geq c(1 - \xi_2)$, we have $b > 0$ and $d > 0$.

For convenience of notation, we set

$$M = \int_0^{\xi_1} -\tau h(\tau)d\tau + \int_{\xi_2}^{1} -(1 - \tau)h(\tau)d\tau + \big(\xi_2 - \xi_1 + \frac{bd}{\delta}\big)\int_{\xi_1}^{\xi_2} h(\tau)d\tau, \tag{3.6}$$

$$m = \min\{m_1, m_2\}, \tag{3.7}$$

where

$$\begin{gathered} m_1 = \frac{bd}{\delta}\int_{\xi_1}^{\xi_2} h(\tau)d\tau \int_{\xi_1}^{\xi_2} G_1(\omega, s)ds, \\ m_2 = \frac{bd}{\delta}\int_{\xi_1}^{\xi_2} h(\tau)d\tau \int_{\xi_1}^{\xi_2} G_1(1 - \omega, s)ds. \end{gathered} \tag{3.8}$$

We are now in a position to present and prove our main results.

Theorem 3.3. *Let $b \geq a\xi_1$ and $d \geq c(1 - \xi_2)$. Assume* (H1)–(H2) *hold. Suppose there exist constants $0 < p < q < \min\{\omega, \frac{1}{8}\}r$ such that*

- (H3) *$g(u, v) \leq r/M$, for $(u, v) \in [0, r] \times [-r, 0]$,*
- (H4) *$g(u, v) > q/m$, for $(u, v) \in [q, q/\omega] \times [-r, 0]$,*
- (H5) *$g(u, v) < 8p/M$, for $(u, v) \in [0, p] \times [-r, 0]$,*

where M, m are as in (3.6)-(3.7)*, then* (1.6), (1.2) *has at least three positive solutions y_1, y_2, and y_3 such that*

$$\max_{0 \leq t \leq 1} |y_i''(t)| \leq r, \quad \text{for } i = 1, 2, 3;$$

$$\min_{\omega\le t\le 1-\omega}|y_1(t)|>q;\quad p<\max_{0\le t\le 1}|y_2(t)|;$$
$$\min_{\omega\le t\le 1-\omega}|y_2(t)|<q;\quad \max_{0\le t\le 1}|y_3(t)|<p.$$

Proof. From Lemma 3.1, $T:P\to P$ is completely continuous. We now show that all the conditions of Lemma 2.2 are satisfied.

If $y\in\overline{P(\gamma,r)}$, then $\gamma(y)=\max_{0\le t\le 1}|y''(t)|\le r$. By Lemma 2.1, we have $\max_{0\le t\le 1}|y(t)|\le r$, then assumption (H3) implies $g(y(t),y''(t))\le r/M$. On the other hand, from (3.1)-(3.3), we have

$$\begin{aligned}\max_{0\le t\le \xi_1}(Qy)(t)&\le\int_0^{\xi_1}-\tau h(\tau)g\big(y(\tau),y''(\tau)\big)d\tau\\&\quad+\frac{1}{\delta}\int_{\xi_1}^{\xi_2}b(c(\xi_2-\tau)+d)h(\tau)g\big(y(\tau),y''(\tau)\big)d\tau\\&\le\int_0^{\xi_1}-\tau h(\tau)g\big(y(\tau),y''(\tau)\big)d\tau\\&\quad+\frac{1}{\delta}b(c(\xi_2-\xi_1)+d)\int_{\xi_1}^{\xi_2}h(\tau)g\big(y(\tau),y''(\tau)\big)d\tau,\end{aligned}\tag{3.9}$$

$$\begin{aligned}\max_{\xi_1\le t\le \xi_2}(Qy)(t)&\le\frac{1}{\delta}\int_{\xi_1}^{t}(b+a(t-\xi_1))(c(\xi_2-\tau)+d)h(\tau)g\big(y(\tau),y''(\tau)\big)d\tau\\&\quad+\frac{1}{\delta}\int_{t}^{\xi_2}(b+a(t-\xi_1))(c(\xi_2-\tau)+d)h(\tau)g\big(y(\tau),y''(\tau)\big)d\tau\\&=\frac{1}{\delta}\int_{\xi_1}^{\xi_2}(b+a(t-\xi_1))(c(\xi_2-\tau)+d)h(\tau)g\big(y(\tau),y''(\tau)\big)d\tau\\&\le\frac{1}{\delta}(b+a(\xi_2-\xi_1))(c(\xi_2-\xi_1)+d)\int_{\xi_1}^{\xi_2}h(\tau)g\big(y(\tau),y''(\tau)\big)d\tau,\end{aligned}\tag{3.10}$$

and

$$\begin{aligned}\max_{\xi_2\le t\le 1}(Qy)(t)&\le\frac{1}{\delta}\int_{\xi_1}^{\xi_2}d(b+a(\tau-\xi_1))h(\tau)g\big(y(\tau),y''(\tau)\big)d\tau\\&\quad+\int_{\xi_2}^{1}-(1-\tau)h(\tau)g\big(y(\tau),y''(\tau)\big)d\tau\\&\le\frac{1}{\delta}d(b+a(\xi_2-\xi_1))\int_{\xi_1}^{\xi_2}h(\tau)g\big(y(\tau),y''(\tau)\big)d\tau\\&\quad+\int_{\xi_2}^{1}-(1-\tau)h(\tau)g\big(y(\tau),y''(\tau)\big)d\tau.\end{aligned}\tag{3.11}$$

By (3.9)-(3.11), we get

$$\begin{aligned}\gamma(Ty)&=\max_{t\in[0,1]}|(Ty)''(t)|=\max_{t\in[0,1]}|(Qy)(t)|\\&=\max\Big\{\max_{0\le t\le\xi_1}|(Qy)(t)|,\max_{\xi_1\le t\le\xi_2}|(Qy)(t)|,\max_{\xi_2\le t\le 1}|(Qy)(t)|\Big\}\\&\le\int_0^{\xi_1}-\tau h(\tau)g\big(y(\tau),y''(\tau)\big)d\tau+\int_{\xi_2}^{1}-(1-\tau)h(\tau)g\big(y(\tau),y''(\tau)\big)d\tau\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\delta}(b+a(\xi_2-\xi_1))(c(\xi_2-\xi_1)+d)\int_{\xi_1}^{\xi_2} h(\tau)g\big(y(\tau),y''(\tau)\big)d\tau \qquad (3.12)\\
&\le \frac{r}{M}\Big(\int_0^{\xi_1} -\tau h(\tau)d\tau+\int_{\xi_2}^1 -(1-\tau)h(\tau)d\tau+\big(\xi_2-\xi_1+\frac{bd}{\sigma}\big)\int_{\xi_1}^{\xi_2} h(\tau)d\tau\Big)\\
&= \frac{r}{M}M=r.
\end{aligned}
$$

Hence, $T:\overline{P(\gamma,r)}\to\overline{P(\gamma,r)}$.

To check condition (i) of Lemma 2.2, we choose $y(t)=q/\omega$, $0\le t\le 1$. It is easy to see that $y(t)=q/\omega\in P(\gamma,\theta,\alpha,q,q/\omega,r)$ and $\alpha(y)=q/\omega>q$, and so $\{y\in P(\gamma,\theta,\alpha,q,q/\omega,r):\alpha(y)>q\}\neq\emptyset$. Hence, if $y\in P(\gamma,\theta,\alpha,q,q/\omega,r)$, then $q\le y(t)\le q/\omega$, $-r\le y''(t)\le 0$ for $\omega\le t\le 1-\omega$. From assumption (H4), we have $g(y(t),y''(t))>b/m$ for $\omega\le t\le 1-\omega$, and by the definitions of α and the cone P, we distinguish two cases as follows:

Case (1): $\alpha(Ty)=(Ty)(\omega)$. By (3.4) and (3.2), we have

$$
\begin{aligned}
&\alpha(Ty)\\
&=(Ty)(\omega)=\int_0^1 G_1(\omega,s)(Qy)(s)ds\\
&>\int_{\xi_1}^{\xi_2} G_1(\omega,s)(Qy)(s)ds\\
&\ge\frac{1}{\delta}\int_{\xi_1}^{\xi_2} G_1(\omega,s)ds\Big[\int_{\xi_1}^{s} bdh(\tau)g\big(y(\tau),y''(\tau)\big)d\tau+\int_{s}^{\xi_2} bdh(\tau)g\big(y(\tau),y''(\tau)\big)d\tau\Big]\\
&=\frac{bd}{\delta}\int_{\xi_1}^{\xi_2} G_1(\omega,s)ds\int_{\xi_1}^{\xi_2} h(\tau)g\big(y(\tau),y''(\tau)\big)d\tau\\
&\ge\frac{bd}{\delta}\frac{q}{m}\int_{\xi_1}^{\xi_2} G_1(\omega,s)ds\int_{\xi_1}^{\xi_2} h(\tau)d\tau\\
&=\frac{q}{m}\cdot m_1\ge q.
\end{aligned}
$$

Case (2): $\alpha(Ty)=(Ty)(1-\omega)$. Similarly, we obtain

$$
\begin{aligned}
\alpha(Ty)=(Ty)(1-\omega)&>\int_{\xi_1}^{\xi_2} G_1(1-\omega,s)(Qy)(s)ds\\
&\ge\frac{bd}{\delta}\frac{q}{m}\int_{\xi_1}^{\xi_2} G_1(1-\omega,s)ds\int_{\xi_1}^{\xi_2} h(\tau)d\tau\\
&=\frac{q}{m}\cdot m_2\ge q.
\end{aligned}
$$

i.e.,

$$
\alpha(Ty)>q,\quad \forall y\in P(\gamma,\theta,\alpha,q,\frac{q}{\omega},r).
$$

This show that condition (i) of Lemma 2.2 is satisfied. Secondly, we have

$$
\alpha(Ty)=\min_{\omega\le t\le 1-\omega}|(Ty)(t)|\ge\omega\|Ty\|_\infty=\omega\theta(Ty)>\omega\frac{q}{\omega}=q,
$$

for all $y\in P(\gamma,\alpha,q,r)$ with $\theta(Ty)>q/\omega$. Thus, condition (ii) of Lemma 2.2 is satisfied.

We finally show that (iii) of Lemma 2.2 also holds. Clearly, as $\psi(0) = 0 < p$, there holds that $0 \notin R(\gamma, \psi, p, r)$. Suppose that $y \in R(\gamma, \psi, p, r)$ with $\psi(y) = p$. Then, by (H5) and (3.12), we get

$$
\begin{aligned}
\psi(Ty) &= \max_{0\le t\le 1} |(Ty(t)| = \max_{0\le t\le 1} \int_0^1 G_1(t,s)(Qy)(s)ds \\
&= \max_{0\le t\le 1} \Big| \int_0^{\xi_1} G_1(t,s)(Qy)(s)ds + \int_{\xi_1}^{\xi_2} G_1(t,s)(Qy)(s)ds \\
&\quad + \int_{\xi_2}^1 G_1(t,s)(Qy)(s)ds \Big| \\
&\le \max_{0\le t\le 1} \Big[\max_{0\le s\le \xi_1} (Qy)(s) \int_0^{\xi_1} G_1(t,s)ds + \max_{\xi_1\le s\le \xi_2} (Qy)(s) \int_{\xi_1}^{\xi_2} G_1(t,s)ds \\
&\quad + \max_{\xi_2\le s\le 1} (Qy)(s) \int_{\xi_2}^1 G_1(t,s)ds \Big] \\
&\le \max\big\{ \max_{0\le s\le \xi_1} (Qy)(s),\ \max_{\xi_1\le s\le \xi_2} (Qy)(s),\ \max_{\xi_2\le s\le 1} (Qy)(s) \big\} \max_{0\le t\le 1} \int_0^1 G_1(t,s)ds \\
&\le \max_{0\le t\le 1} \int_0^1 G_1(t,s)ds \Big[\int_0^{\xi_1} -\tau h(\tau) g\big(y(\tau), y''(\tau)\big)d\tau \\
&\quad + \int_{\xi_2}^1 -(1-\tau)h(\tau) g\big(y(\tau), y''(\tau)\big)d\tau \\
&\quad + \frac{1}{\delta}(b + a(\xi_2 - \xi_1))(c(\xi_2 - \xi_1) + d) \int_{\xi_1}^{\xi_2} h(\tau) g\big(y(\tau), y''(\tau)\big)d\tau \Big] \\
&\le \max_{0\le t\le 1} \int_0^1 G_1(t,s)ds \cdot \frac{8p}{M} \Big[\int_0^{\xi_1} -\tau h(\tau)d\tau + \int_{\xi_2}^1 -(1-\tau)h(\tau)d\tau \\
&\quad + \big(\xi_2 - \xi_1 + \frac{bd}{\sigma}\big) \int_{\xi_1}^{\xi_2} h(\tau)d\tau \Big] \\
&= \frac{1}{8} \cdot \frac{8p}{M} \cdot M = p.
\end{aligned}
$$

So, condition (iii) of Lemma 2.2 is satisfied. Therefore, an application of Lemma 2.2 imply the boundary-value problem (1.6), (1.2) has at least three positive solutions y_1, y_2, and y_3 such that

$$
\begin{gathered}
\max_{0\le t\le 1} |y_i''(t)| \le r, \quad \text{for } i = 1,2,3; \quad \min_{\omega\le t\le 1-\omega} |y_1(t)| > q; \\
p < \max_{0\le t\le 1} |y_2(t)|, \quad \min_{\omega\le t\le 1-\omega} |y_2(t)| < q; \quad \max_{0\le t\le 1} |y_3(t)| < p.
\end{gathered}
$$

The proof is complete. □

Now, we give an example to demonstrate our result. Consider the fourth-order four-point boundary-value problem

$$
y^{(4)}(t) = h(t) g(y(t), y''(t)), \quad 0 < t < 1, \tag{3.13}
$$

$$
\begin{gathered}
y(0) = y(1) = 0, \\
y''(\frac{1}{3}) - y'''(\frac{1}{3}) = 0, \quad y''(\frac{2}{3}) + y'''(\frac{2}{3}) = 0,
\end{gathered} \tag{3.14}
$$

where $\xi_1 = \frac{1}{3}$, $\xi_2 = \frac{2}{3}$, $h(t) = 9\pi \sin(3t-1)\pi$, and

$$g(u,v) = \begin{cases} \frac{u^2}{2} - (\frac{v}{150})^3, & 0 \le u \le 1,\ v \le 0, \\ 11\sqrt[4]{u-1} - (\frac{v}{150})^3 + \frac{1}{2}, & 1 < u \le 9,\ v \le 0, \\ 11\sqrt[4]{8} + \frac{1}{2} - (\frac{v}{150})^3, & u > 9,\ v \le 0. \end{cases}$$

It is easy to check that the functions h and g satisfy (H1) and (H2). Set $\omega = 1/3$. It follows from a direct calculation that

$$\begin{aligned} M &= 9\pi \Big[\int_0^{1/3} -\tau \sin(3\tau-1)\pi d\tau + \int_{2/3}^1 -(1-\tau)\sin(3\tau-1)\pi d\tau \\ &\quad + \frac{16}{21}\int_{1/3}^{2/3} \sin(3\tau-1)\pi d\tau \Big] \\ &= \frac{46}{7}, \end{aligned}$$

and

$$m = 9\pi \cdot \frac{3}{7}\int_{1/3}^{2/3} \sin(3\tau-1)\pi d\tau \cdot \min\Big\{ \int_{1/3}^{2/3} G(\frac{1}{3},s)ds, \int_{1/3}^{2/3} G(\frac{2}{3},s)ds \Big\} = \frac{2}{7}.$$

Choose $p = 1$, $q = 3$ and $r = 130$, then we have

$$\begin{gathered} g(u,v) \le 1.151 < 1.21 = \frac{8p}{M}, \quad \text{for } 0 \le u \le 1,\ -130 \le v \le 0; \\ g(u,v) \ge 14.232 > 10.5 = \frac{q}{m}, \quad \text{for } 3 \le u \le 9,\ -130 \le v \le 0; \\ g(u,v) \le 19.651 < 19.78 = \frac{r}{M}, \quad \text{for } 0 \le u \le 130,\ -130 \le v \le 0. \end{gathered}$$

Noticing that $b > \xi_1 a$ and $d > (1-\xi_2)c$ hold, then all conditions of Theorem 3.3 hold. Hence, by Theorem 3.3, BVP (3.13), (3.14) has at least three positive solutions y_1, y_2 and y_3 such that

$$\begin{gathered} \max_{0\le t\le 1} |y_i''(t)| \le 130, \quad \text{for } i = 1,2,3; \quad \min_{\frac{1}{3}\le t\le \frac{2}{3}} |y_1(t)| > 3; \\ 1 < \max_{0\le t\le 1} |y_2(t)|, \quad \min_{\frac{1}{3}\le t\le \frac{2}{3}} |y_2(t)| < 3 \quad \max_{0\le t\le 1} |y_3(t)| < 1. \end{gathered}$$

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