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ASYMPTOTIC CONDITIONS FOR SOLVING NON-SYMMETRIC PROBLEMS OF THIRD ORDER NONLINEAR DIFFERENTIAL SYSTEMS

AWAR SIMON UKPERA

Abstract. This article presents asymptotic conditions, under various sharp and nonuniform nonresonance assumptions, for solving the forced third order nonlinear system $X'''+AX''+BX'+sH(t,X)=P(t)$, in which A, B and other associated matrices are not necessarily symmetric. This work generalizes some results in the literature which are hinged basically on symmetry considerations.

1. Introduction

We shall establish new solvability criteria for the forced third-order nonlinear system

$$X''' + AX'' + BX' + sH(t, X) = P(t) \tag{1.1}$$

subject to the periodic boundary conditions

$$X(0) - X(T) = X'(0) - X'(T) = X''(0) - X''(T) = 0 \tag{1.2}$$

on an interval $[0, T]$ with $T > 0$.

Here, $X \equiv (x_i)_{1\le i\le n} : [0, T] \to \mathbb{R}^n$, $\mathbb{R}^n$ is the n-dimensional Euclidean space, equipped with the usual norm $\|\cdot\|$ and scalar product $\langle X, Y\rangle = \sum_{i=1}^n x_i y_i$, so that $\langle X, X\rangle = \|X\|^2$ is the usual Euclidean norm in $\mathbb{R}^n$. A and B are constant real $n\times n$ matrices (not necessarily symmetric), $H \equiv (h_i(t, X))_{1\le i\le n} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $P \equiv (p_i)_{1\le i\le n} : [0, T] \to \mathbb{R}^n$ are n-vectors, which are T-periodic in t. Furthermore, H satisfies the Caratheodory conditions, and $s \in \{-1, 1\}$.

The hypotheses for the present problem will be formulated as a transition from some known results, in previous studies involving symmetric matrices, to the recent studies carried out in our joint papers with Afuwape [3, 12], and also in [11].

The classical spaces of k times continuously differentiable functions shall be denoted by $C^k([0, T], \mathbb{R}^n), k \ge 0$ an integer, with norm $\|X\|_{C^k}$; $L^p = L^p([0, T])$, $1 \le p \le \infty$, will denote the usual Lebesgue spaces, with their respective norms $\|X\|_{L^p}$; while $W_T^{k,p}([0, T], \mathbb{R}^n)$, will denote the Sobolev space of T-periodic functions

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of order k, defined by

$$\begin{aligned} W_T^{k,p} =&\{X : [0,T] \to \mathbb{R}^n : X, X', \dots, X^{(k-1)} \text{ are absolutely continuous on } [0,T], \\ &X^{(k)} \in L^p(0,T) \text{ and } X^{(i)}(0) - X^{(i)}(T) = 0, i = 0,1,2,\dots,k-1,\ k \in \mathbb{R}\} \end{aligned}$$

with corresponding norm $\|X\|_{W_T^{k,p}}$.

It is standard result that if D is a real $n \times n$ symmetric matrix, then for any $X \in \mathbb{R}^n$,

$$\delta_d \|X\|^2 \le \langle DX, X\rangle \le \Delta_d \|X\|^2, \tag{1.3}$$

where δ_d and Δ_d are real constants which represent respectively, the least and greatest eigenvalues of D. In general, $\lambda_i(D)$, $i = 1,\dots,n$, shall denote the eigenvalues of any matrix D; and $\|D\|$ denote the norm of D thought as a linear operator in $\mathbb{R}^n$ (that is, spectral norm of D with respect to the inner product $\langle \cdot,\cdot\rangle$).

Furthermore, the following algebraic inequalities also hold: Let D_1 and D_2 be any two real $n \times n$ commuting symmetric matrices. Then

(i) the eigenvalues $\lambda_i(D_1 + D_2)$, $i = 1,\dots,n$, of the sum of matrices D_1 and D_2 are all real and satisfy

$$\begin{aligned} \min_{1\le j\le n} \lambda_j(D_1) + \min_{1\le k\le n} \lambda_k(D_2) &\le \lambda_i(D_1 + D_2) \\ &\le \max_{1\le j\le n} \lambda_j(D_1) + \max_{1\le k\le n} \lambda_k(D_2) \end{aligned} \tag{1.4}$$

(ii) the eigenvalues $\lambda_i(D_1D_2)$, $i = 1,\dots,n$, of the product of matrices D_1 and D_2 are all real and satisfy

$$\min_{1\le j,k\le n} \{\lambda_j(D_1)\lambda_k(D_2)\} \le \lambda_i(D_1D_2) \le \max_{1\le j,k\le n} \{\lambda_j(D_1)\lambda_k(D_2)\} \tag{1.5}$$

(See [1, 6] for proofs). However, note that if D_1 and D_2 are two commuting symmetric matrices, their product D_1D_2 is also symmetric so that its eigenvalues are real.

Relying significantly on the analysis of an underlying linear differential operator, various results at both the resonance and nonresonance situations were proposed in [3] without much rigorous verification. Given under symmetry assumptions, our main result was built around the sharp nonresonance relation

$$k^2\omega^2 < \Delta_a^{-1}\delta_h \le \frac{\langle sA^{-1}H(t,X), X\rangle}{\|X\|^2} \le \delta_a^{-1}\Delta_h < (k+1)^2\omega^2 \tag{1.6}$$

holding for some $k \in \mathbb{N}$ and $\omega = \frac{2\pi}{T}$, where δ_h, Δ_h are constants which satisfy $0 < \delta_h \le \lambda_i(J_h(t,X)) \le \Delta_h$ uniformly in X for a.e. $t \in [0,T]$, $J_h(t,X)$ being the Jacobian matrix of $H(t,X)$, for the existence of T-periodic solutions of (1.1).

We have obtained a further generalisation of [3], still subject to symmetry considerations, by weakening the sharp nonresonance relation $(\mathcal{H}_1)$, and thereby permitting some interaction between the ratio $\langle sA^{-1}H(t,X), X\rangle / \|X\|^2$ and the spectrum $k^2\omega^2$ of the associated eigenvalue problem. This coincided with the problem of finding vector analogues to Ezeilo and Nkashama's [5] nonuniform nonresonance assumptions

$$k^2 \le \frac{\gamma^-(t)}{a} \le \liminf_{|x|\to\infty} \frac{h(t,x)}{ax} \le \limsup_{|x|\to\infty} \frac{h(t,x)}{ax} \le \frac{\gamma^+(t)}{a} \le (k+1)^2 \tag{1.7}$$

holding uniformly in $x \in \mathbb{R}$ for a.e. $t \in [0, 2\pi]$, where $\gamma^{\pm} \in L^1(0, 2\pi)$ such that strict inequalities hold on subsets of $[0, 2\pi]$ of positive measure; for the existence of 2π-periodic solutions of a scalar prototype of (1.1). This was achieved by employing two intermediate symmetric matrices $M(t)$ and $N(t)$ whose eigenvalues interact with the spectrum for many values of $t \in [0, T]$. Thus, as has been demonstrated in [11, 12], we require that the inequalities

$$\begin{aligned} k^2\omega^2 &\le \frac{\langle A^{-1}M(t)X, X\rangle}{\|X\|^2} \\ &\le \liminf_{\|X\|\to\infty} \frac{\langle sA^{-1}H(t,X), X\rangle}{\|X\|^2} \\ &\le \limsup_{\|X\|\to\infty} \frac{\langle sA^{-1}H(t,X), X\rangle}{\|X\|^2} \\ &\le \frac{\langle A^{-1}N(t)X, X\rangle}{\|X\|^2} \\ &\le (k+1)^2\omega^2 \end{aligned} \tag{1.8}$$

hold uniformly in $X \in \mathbb{R}^n$ for a.e. $t \in [0, T]$, $k \in \mathbb{N}$, with $M, N \in L^1([0, T], \mathbb{R}^{n^2})$ symmetric matrices such that $k^2\omega^2\|X\|^2 < \langle A^{-1}M(t)X, X\rangle$ and $\langle A^{-1}N(t)X, X\rangle < (k + 1)^2\omega^2\|X\|^2$ on subsets of $[0, T]$ of positive measure; with the corresponding expression for uniqueness given by

$$\begin{aligned} k^2\omega^2 &\le \frac{\langle A^{-1}M(t)(X_1 - X_2), X_1 - X_2\rangle}{\|X_1 - X_2\|^2} \\ &\le \frac{\langle sA^{-1}\big(H(t, X_1) - H(t, X_2)\big), X_1 - X_2\rangle}{\|X_1 - X_2\|^2} \\ &\le \frac{\langle A^{-1}N(t)(X_1 - X_2), X_1 - X_2\rangle}{\|X_1 - X_2\|^2} \\ &\le (k+1)^2\omega^2 \end{aligned} \tag{1.9}$$

for $X_1, X_2 \in \mathbb{R}^n$ with $X_1 \ne X_2$ where $k \in \mathbb{N}$ and M, N are as in (1.8).

The relation (1.9) is in harmony with some abstract results given in Amann [4] for the unique solvability of semi-linear operator equations of the form $\mathbf{L}U = F(U)$ in a real Hilbert space $\mathcal{H}$, for $\mathbf{L}$ self-adjoint, where it is required that there exist symmetric operators $B^{\pm} \in \mathcal{L}(\mathcal{H})$ such that

$$B^- \le F'(U) \le B^+ \forall U \in \mathcal{H}$$
$$\langle(\mathbf{L} - B^-)U, U\rangle \le -\gamma\|U\|^2, \quad \langle(\mathbf{L} - B^+)U, U\rangle \ge \gamma\|U\|^2, \quad \gamma > 0$$

Clearly, (1.9) is an application of this result to systems of type (1.1), even though our linearity here is not self-adjoint, with the intermediate operators B^- and B^+ replaced by the generalised intermediate matrices M and N in $L^1([0, T], \mathbb{R}^{n^2})$ respectively.

The basic challenge of this paper is to investigate the changes to these solvability conditions arising from the transition to non-symmetry assumptions. Specifically, we shall find analogous repesentations to some of the above relations when symmetry considerations on A, B and other associated matrices are dropped.

2. Preliminary estimates and results

Consider the eigenvalue problem

$$X''' + AX'' + BX' = -A\lambda X \tag{2.1}$$

together with (1.2), with A, B nonsingular and not neccessarily symmetric, and λ a real parameter. The following results are vector analogues and adaptations of their corresponding scalar based counterparts found in Ezeilo-Nkashama [5] and Minhós [9] respectively:

Proposition 2.1. *The following statements hold for* (2.1)*:*

(i) $\lambda = 0$ *is an eigenvalue;*
(ii) $\lambda = k^2\omega^2$*, for some* $k = 1, 2, \dots, \omega = \frac{2\pi}{T}$*, is an eigenvalue if and only if* $|\lambda_i(B)| = k^2\omega^2$*;*
(iii) *any* $\lambda \neq k^2\omega^2$*, for each* $k = 1, 2, \dots, \omega = \frac{2\pi}{T}$ *is not an eigenvalue*

Furthermore, let $\mathcal{E}_k$ *be the eigenspace corresponding to the unique eigenvalue* $k^2\omega^2$*, when it exists. Then for every* $X \in W_T^{3,2}(0, 2\pi)$*, we have*

$$\int_0^T \langle X''' + AX'' + BX' + k^2\omega^2 AX, X''' + AX'' + BX' + (k+1)^2\omega^2 AX\rangle\, dt \geq 0,$$

and the equality holds if and only if $X = 0$ *or either* $k^2\omega^2$ *or* $(k+1)^2\omega^2$ *is an eigenvalue of* (2.1) *and* $X \in \mathcal{E}_k$ *or* $X \in \mathcal{E}_{k+1}$*, respectively.*

Proof. Let the solution X of (2.1)-(1.2) have the Fourier series expansion

$$X(t) \sim \sum_{i=1}^{n}\sum_{k=0}^{\infty}(c_{k,i}\cos k\omega t + d_{k,i}\sin k\omega t),$$

$d_{0,i} = 0$, $\omega = \frac{2\pi}{T}$, $k \in \mathbb{N}$, $i = 1, \dots, n$. Then substituting into (2.1) yields

$$\begin{aligned}
0 &= I\sum_{i=1}^{n}\sum_{k=0}^{\infty} -(k\omega)^3\left[c_{k,i}\sin(k\omega t) + d_{k,i}\cos(k\omega t)\right]\\
&\quad + A\sum_{i=1}^{n}\sum_{k=0}^{\infty}(k\omega)^2\left[c_{k,i}\cos(k\omega t) + d_{k,i}\sin(k\omega t)\right]\\
&\quad + B\sum_{i=1}^{n}\sum_{k=0}^{\infty}k\omega\left[c_{k,i}\sin(k\omega t) - d_{k,i}\cos(k\omega t)\right]\\
&\quad - \lambda A\sum_{i=1}^{n}\sum_{k=0}^{\infty}\left[c_{k,i}\cos(k\omega t) + d_{k,i}\sin(k\omega t)\right]\\
&= -\lambda A\sum_{i=1}^{n}c_{0,i} - \sum_{i=1}^{n}\sum_{k=1}^{\infty}\left[A(\lambda - k^2\omega^2)c_{k,i} + k\omega(B - k^2\omega^2 I)d_{k,i}\right]\cos(k\omega t)\\
&\quad + \sum_{i=1}^{n}\sum_{k=1}^{\infty}\left[k\omega(B - k^2\omega^2 I)c_{k,i} - A(\lambda - k^2\omega^2)d_{k,i}\right]\sin(k\omega t)
\end{aligned}$$

where I is the identity matrix. Thus, for nontrivial solution to exist, we conclude that statements (i)–(iii) must hold.

Also, noting from the last equality that

$$\begin{aligned}
&X''' + AX'' + BX' + k^2\omega^2 AX \\
&= k^2\omega^2 A\sum_{i=1}^{n} c_{0,i} + \sum_{i=1}^{n}\sum_{r=1}^{\infty}\big[A\omega^2(k^2-r^2)c_{r,i} + r\omega(B - r^2\omega^2 I)d_{r,i}\big]\cos(r\omega t) \\
&\quad - \sum_{i=1}^{n}\sum_{r=1}^{\infty}\big[r\omega(B-r^2\omega^2 I)c_{r,i} - A\omega^2(k^2-r^2)d_{r,i}\big]\sin(r\omega t)
\end{aligned}$$

we obtain

$$\begin{aligned}
&\int_0^T \langle X''' + AX'' + BX' + k^2\omega^2 AX, X''' + AX'' + BX' + (k+1)^2\omega^2 AX\rangle\, dt \\
&\geq \frac{T}{2}\sum_{i=1}^{n}\sum_{r=1}^{\infty}\big[\|A\|\omega^2(k^2-r^2)c_{r,i} + r\omega(|\lambda_i(B)| - r^2\omega^2)d_{r,i}\big] \\
&\quad\times\big[\|A\|\omega^2((k+1)^2-r^2)c_{r,i} + r\omega(|\lambda_i(B)| - r^2\omega^2)d_{r,i}\big] \\
&\quad + \frac{T}{2}\sum_{i=1}^{n}\sum_{r=1}^{\infty}\big[r\omega(|\lambda_i(B)| - r^2\omega^2)c_{r,i} - \|A\|\omega^2(k^2-r^2)d_{r,i}\big] \\
&\quad\times\big[r\omega(|\lambda_i(B)| - r^2\omega^2)c_{r,i} - \|A\|\omega^2((k+1)^2-r^2)d_{r,i}\big] \\
&\geq \frac{T}{2}\sum_{i=1}^{n}\sum_{r=1}^{\infty}\big[\omega^4\|A\|^2(k^2-r^2)((k+1)^2-r^2) + r^2\omega^2(|\lambda_i(B)| - r^2\omega^2)\big] \\
&\quad\times\big[c_{r,i}^2 + d_{r,i}^2\big] \geq 0
\end{aligned}$$

It follows that the equality holds if and only if $c_{r,i} = d_{r,i} = 0$ unless $r^2 = k^2$ or $r^2 = (k+1)^2$ and $|\lambda_i(B)| = k^2\omega^2 I$. That means, if and only if $X = 0$ or either $k^2\omega^2$ or $(k+1)^2\omega^2$ is an eigenvalue of (2.1) and $X \in \mathcal{E}_k$ or $X \in \mathcal{E}_{k+1}$, respectively. $\square$

Each of the statements (i)–(iii) above has direct implication on the solvability of the corresponding non-autonomous system

$$X''' + AX'' + BX' + \lambda AX = P(t) \equiv P(t+T), P \in L^1 \tag{2.2}$$

It is clear for instance, from (iii) and the Fredholm alternative, that a solution for (2.2)-(1.2) exists, provided that some control is put on the closeness of λ to $k^2\omega^2$ and $(k+1)^2\omega^2$. Furthermore, if $\lambda = 0, k^2\omega^2$, then (2.2)-(1.2) does not, in general, have a T-periodic solution. This observation underscores the importance of any existence result for (1.1) in which B is arbitrary, to be such that $\lim_{\|X\|\to\infty}\frac{\langle sA^{-1}H(t,X),X\rangle}{\langle X,X\rangle}$ is allowed to 'touch' $k^2\omega^2$ for some values of t.

For any given $n\times n$ matrix D (not necessarily symmetric), we set $D_s = (D + D^T)/2$ and $D_d = (D - D^T)/2$, where D^T is the transpose of D. Moreover we define $\gamma_D = |\min\{0,\sigma_{\min}\}|$, $\Gamma_D = |\max\{0,\sigma_{\max}\}|$, where $\sigma_{\min}$ and $\sigma_{\max}$ represent respectively the minimum and the maximum eigenvalues of D_s (the symmetric part of D). Then for any $X \in \mathbb{R}^n$ (see Omari-Zanolin [10])

$$-\gamma_D\|X\|^2 \leq \sigma_{\min}\|X\|^2 \leq \langle D_s X, X\rangle = \langle DX, X\rangle \leq \sigma_{\max}\|X\|^2 \leq \Gamma_D\|X\|^2. \tag{2.3}$$

Moreover,

$$\max\{\|D_s\|, \|D_d\|\} \leq \|D\|, \|D_s\| = \max\{\gamma_D, \Gamma_D\}.$$

Let $\rho(t)$ be defined by

$$\rho(t) = \frac{1}{2}(\min \lambda_i(A^{-1}C(t)) + \max \lambda_i(A^{-1}C(t))) = \frac{1}{2}(\Gamma_A^{-1}\gamma_C(t) + \gamma_A^{-1}\Gamma_C(t))$$

so that

$$k^2\omega^2 < \Gamma_A^{-1}\gamma_C(t) \le \rho(t) \le \gamma_A^{-1}\Gamma_C(t) < (k+1)^2\omega^2$$

Thus setting

$$\alpha = \frac{1}{2}\max(\gamma_A^{-1}\Gamma_C(t) - \Gamma_A^{-1}\gamma_C(t)) \text{ and } \beta = \min\{(k+1)^2\omega^2 - \rho(t), \rho(t) - k^2\omega^2\},$$

we easily deduce that

$$0 \le \alpha \le \beta \tag{2.4}$$

with the strict inequality holding on subsets of $[0,T]$ of positive measure.

Let us now consider the linear system

$$X'''(t) + AX''(t) + BX'(t) + C(t)X(t) = 0 \tag{2.5}$$

together with (1.2), where A and B are constant, not necessarily symmetric matrices, and $C(t) \equiv (c_{ij}(t))$, with $c_{ij} \in L^1(0,T)$, is an $n \times n$ arbitrary matrix function. The following result holds for system (2.5)-(1.2).

Lemma 2.2. *Let A be a nonsingular matrix which commutes with $C(t)$, and suppose that*

$$k^2\omega^2\|X\|^2 \le \Gamma_A^{-1}\gamma_C(t)\|X\|^2 \le \langle A^{-1}C(t)X, X\rangle \le \gamma_A^{-1}\Gamma_C(t)\|X\|^2 \le (k+1)^2\omega^2\|X\|^2 \tag{2.6}$$

uniformly for $X \in \mathbb{R}^n$ and a.e. $t \in [0,T]$, with the strict inequality holding on subsets of $[0,T]$ of positive measure, where $\gamma_A, \Gamma_A, \gamma_C, \Gamma_C$ are associated to A and $C(t)$ as defined in (2.3). Then, for any arbitrary matrix B, (2.5)-(1.2) admits in $W^{3,1}([0,T],\mathbb{R}^n)$ only the trivial solution.

Moreover, there exist positive constants β and δ_0, with $\beta = \min\{(k+1)^2\omega^2 - \rho(t), \rho(t) - k^2\omega^2\}$, for $\rho(t) = \frac{1}{2}(\Gamma_A^{-1}\gamma_C(t) + \gamma_A^{-1}\Gamma_C(t))$ for a.e $t \in [0,T]$, such that

$$\beta^2 \int_0^T \|X\|^2\,dt \le \int_0^T \|X'' + A^{-1}C(t)X\|^2\,dt\,, \tag{2.7}$$

$$\int_0^T \|X''\|^2\,dt \le \delta_0 \int_0^T \|X'' + A^{-1}C(t)X\|^2\,dt\,. \tag{2.8}$$

Proof. Let the solution be $X(t) = \overline{X}(t) + \widetilde{X}(t)$, where

$$\overline{X} = \sum_{i=1}^n \Big(c_{0,i} + \sum_{k=1}^N (c_{k,i}\cos k\omega t + d_{k,i}\sin k\omega t)\Big),$$

$$\widetilde{X} = \sum_{i=1}^n \sum_{k=N+1}^\infty \big(c_{k,i}\cos k\omega t + d_{k,i}\sin k\omega t\big).$$

Then multiplying (2.5) scalarly by $\overline{X}(t)-\widetilde{X}(t)$ and integrating over $[0,T]$, setting $D(t)=A^{-1}C(t)$, we obtain (see [11])

$$\begin{aligned} 0 &= \int_0^T \big(\|\widetilde{X}'(t)\|^2 - \langle D(t)\widetilde{X}(t),\widetilde{X}(t)\rangle\big)\,dt - \int_0^T \big(\|\overline{X}'(t)\|^2 - \langle D(t)\overline{X}(t),\overline{X}(t)\rangle\big)\,dt \\ &\geq \int_0^T (\|\widetilde{X}'(t)\|^2 - \rho(t)\|\widetilde{X}(t)\|^2)\,dt - \int_0^T (\|\overline{X}'(t)\|^2 - \rho(t)\|\overline{X}(t)\|^2)\,dt \end{aligned} \tag{2.9}$$

where we fix $\rho(t)=\frac{1}{2}(\Gamma_A^{-1}\gamma_C(t)+\gamma_A^{-1}\Gamma_C(t))$, for a.e $t\in[0,T]$, so that

$$k^2\omega^2 \leq \rho(t) \leq (k+1)^2\omega^2, \quad \text{for a.e } t\in[0,T],$$
$$k^2\omega^2 < \rho(t) < (k+1)^2\omega^2, \quad \text{on subsets of } [0,T] \text{ of positive measure.}$$

By Parseval's identity given by

$$\int_0^T \|X\|^2\,dt = \sum_{i=1}^n \Big(c_{0,i}^2 T + \frac{T}{2}\sum_{k=1}^\infty (c_{k,i}^2 + d_{k,i}^2)\Big),$$

Equation (2.9) becomes

$$\frac{T}{2}\sum_{i=1}^n \Big[\sum_{k=N+1}^\infty (k^2\omega^2-\rho(t))(c_{k,i}^2+d_{k,i}^2) + 2\rho c_{0,i}^2 T + \sum_{k=1}^N (\rho(t)-k^2\omega^2)(c_{k,i}^2+d_{k,i}^2)\Big] \leq 0$$

It follows that $c_{k,i}=0 (k=0,1,2,\dots)$ and $d_{k,i}=0 (k=1,2,\dots)$, for all $i=1,\dots,n$. Thus, $X\equiv 0$.

Furthermore, using the Fourier expansion of $X(t)$ given above

$$\begin{aligned} \int_0^T \|X''+A^{-1}C(t)X\|^2\,dt &\geq \rho(t)\sum_{i=1}^n ({a_{0i}}^2 T + \frac{T}{2}\sum_{k=1}^\infty (k^2\omega^2-\rho(t))^2({a_{k,i}}^2+{b_{k,i}}^2)) \\ &\geq \beta^2 \sum_{i=1}^n ({a_{0i}}^2 T + \frac{T}{2}\sum_{k_i=1}^\infty ({a_{k,i}}^2+{b_{k,i}}^2)) \end{aligned}$$

by the definition of $\rho(t)$ and β given above. Thus, (2.7) now follows by Parseval's identity. Again

$$\begin{aligned} \int_0^T \|X''+A^{-1}C(t)X\|^2\,dt &\geq \rho(t)\sum_{i=1}^n ({a_{0i}}^2 T + \frac{T}{2}\sum_{k=1}^\infty (k^2\omega^2-\rho(t))^2({a_{k,i}}^2+{b_{k,i}}^2)) \\ &\geq \frac{T}{2}\sum_{i=1}^n\sum_{k=1}^\infty (k^2\omega^2-\rho(t))^2({a_{k,i}}^2+{b_{k,i}}^2) \end{aligned} \tag{2.10}$$

Now, fix an integer $N_0=N_0(\nu)$ such that $(k^2\omega^2-\rho(t))^2 \geq \frac{1}{4}k^4\omega^4$, for all $k\geq N_0$. Then, we have

$$\frac{T}{2}\sum_{i=1}^n\sum_{k=N_0+1}^\infty (k^2\omega^2-\rho(t))^2({a_{k,i}}^2+{b_{k,i}}^2) \geq \frac{1}{4}\sum_{i=1}^n\sum_{k=N_0+1}^\infty k^4\omega^4({a_{k,i}}^2+{b_{k,i}}^2). \tag{2.11}$$

But then, we can fix a constant δ_1 such that

$$\delta_1\sum_{i=1}^n\sum_{k=1}^{N_0}(k^2\omega^2-\rho(t))^2({a_{k,i}}^2+{b_{k,i}}^2) \geq \sum_{i=1}^n\sum_{k=1}^{N_0} k^4\omega^4({a_{k,i}}^2+{b_{k,i}}^2). \tag{2.12}$$

Combining (2.11) and (2.12) now gives

$$\int_0^T \|X''\|^2\,dt = \sum_{i=1}^n\sum_{k=1}^\infty k^4\omega^4({a_{k,i}}^2+{b_{k,i}}^2) \le \delta_2\frac{T}{2}\sum_{i=1}^n\sum_{k=1}^\infty (k^2\omega^2-\rho(t))^2({a_{k,i}}^2+{b_{k,i}}^2)$$

where $\delta_2=\max\{4,\delta_1\}$, and the inequality (2.8) now follows from (2.10). $\square$

Lemma 2.3. *Let A and $C(t)$ be as in Lemma 2.2, and suppose that $M,N\in L^1([0,T],\mathbb{R}^{n^2})$ are nonsingular matrices which commute with A, and satisfy the following condition*

$$\Lambda_1(t)\|X\|^2 \le \langle A^{-1}M(t)X,X\rangle \le \langle A^{-1}N(t)X,X\rangle \le \Lambda_2(t)\|X\|^2 \tag{2.13}$$

uniformly in $X\in\mathbb{R}^n$, for a.e. $t\in[0,T]$, where $\Lambda_1(t)=\min\lambda_i(A^{-1}M(t))$, $\Lambda_2(t)=\max\lambda_i(A^{-1}N(t))$, are such that $k^2\omega^2\le|\Lambda_1(t)|$ and $|\Lambda_2(t)|\le(k+1)^2\omega^2$, $k\in\mathbb{N}$, $\omega=\frac{2\pi}{T}$, with the strict inequality holding on subsets of $[0,T]$ of positive measure.

Then, there exist constants $\epsilon=\epsilon(M,N,C)>0$ and $\delta=\delta(M,N,C)>0$ uniformly a.e. on $[0,T]$, such that for all $D(t)\equiv A^{-1}C(t)\in L^1([0,T],\mathbb{R}^{n^2})$ satisfying

$$\langle (A^{-1}M(t)-\epsilon I)X,X\rangle \le \langle D(t)X,X\rangle \le \langle (A^{-1}M(t)+\epsilon I)X,X\rangle \tag{2.14}$$

uniformly in $X\in\mathbb{R}^n$, a.e. on $[0,T]$, and all $X\in W_T^{3,1}([0,T],\mathbb{R}^n)$, one has

$$\int_0^T \|A^{-1}X'''+X''+A^{-1}BX'+A^{-1}C(t)X\|\,dt \ge \delta\|X\|_{W_T^{3,1}} \tag{2.15}$$

Proof. We shall prove by contradiction. Hence, let us assume that the conclusion of the Lemma does not hold, that is, ϵ and δ do not exist. Then there exists a sequence $(X_n)\in W^{3,1}([0,T],\mathbb{R}^n)$ with $\|X_n\|_{W^{3,1}}=1$, and a sequence $(C_n)\in L^1([0,T],\mathbb{R}^{n^2})$ of nonsingular matrices with

$$\langle (A^{-1}M(t)-\frac{1}{n}I)X,X\rangle \le \langle D_n(t)X,X\rangle \le \langle (A^{-1}N(t)+\frac{1}{n}I)X,X\rangle, \quad n\in\mathbb{N}, \tag{2.16}$$

uniformly in $X\in\mathbb{R}^n$, for a.e. $t\in[0,T]$, where $D_n(t)\equiv A^{-1}C_n(t)$, such that for all $X\in W^{3,1}$, one has

$$\int_0^T \|A^{-1}X_n'''(t)+X_n''(t)+A^{-1}BX_n'(t)+A^{-1}C_n(t)X_n\|\,dt < \frac{1}{n} \tag{2.17}$$

Then, by (2.16), there exists some $\kappa\in L^1([0,T],\mathbb{R})$ such that $\|D_n(t)\|\le\kappa(t)$, $n=1,2,\dots$ for a.e. $t\in[0,T], n\in\mathbb{N}$. For example, one can take

$$\kappa(t)\equiv\frac{1}{\|X\|^2}\Big[\|\langle(A^{-1}M(t)-I)X,X\rangle\|+\|\langle(A^{-1}N(t)+I)X,X\rangle\|\Big].$$

Now, by the compact embedding of $W^{3,1}([0,T],\mathbb{R}^n)$ into $W^{2,1}([0,T],\mathbb{R}^n)$ and the continuous embedding of $W^{2,1}([0,T],\mathbb{R}^n)$ into $C^1([0,T],\mathbb{R}^n)$ we infer that by going to subsequences if necessary, we can assume that

$$X_n\to X \text{ in } C^1([0,T],\mathbb{R}^n), X_n'\to X', X_n''\to X'' \text{ in } L^1([0,T],\mathbb{R}^n) \tag{2.18}$$

Moreover, $D_n\rightharpoonup D$ in $L^1([0,T],\mathbb{R}^{n^2})$, so that by (2.16),

$$\langle A^{-1}M(t)X,X\rangle \le \langle D(t)X,X\rangle \equiv \langle A^{-1}C(t)X,X\rangle \le \langle A^{-1}N(t)X,X\rangle \tag{2.19}$$

for a.e. $t \in [0,T]$. Thus for every $\Phi \in L^\infty([0,T],\mathbb{R}^n)$, we have by the Schwartz inequality

$$
\begin{aligned}
&\| \int_0^T \big\langle D_n(t)X_n(t) - D(t)X(t), \Phi(t)\big\rangle \,dt\| \\
&\le \| \int_0^T \big\langle D_n(t)\big(X_n(t) - X(t)\big), \Phi(t)\big\rangle \,dt\| + \| \int_0^T \big\langle \big(D_n(t) - D(t)\big)X(t), \Phi(t)\big\rangle \,dt\| \\
&\le \|\Phi\|_\infty \|\kappa\|_{L^1} \|X_n - X\|_\infty + \| \int_0^T \big\langle \big(D_n(t) - D(t)\big)X(t), \Phi(t)\big\rangle \,dt\|
\end{aligned}
$$

The right hand side of the above inequality clearly tends to zero, and we deduce that $D_nX_n \rightharpoonup DX$ in $L^1([0,T],\mathbb{R}^n)$ It follows that

$$
\begin{aligned}
X_n''' &= -AX_n'' - BX_n' - C_n(\cdot)X_n \\
&\rightharpoonup -AX'' - BX' - C(\cdot)X \quad \text{in } L^1([0,T],\mathbb{R}^n)
\end{aligned}
\tag{2.20}
$$

Since the operator $\frac{d^3}{dt^3} : W^{3,1}([0,T],\mathbb{R}^n) \subset L^1([0,T],\mathbb{R}^n) \to L^1([0,T],\mathbb{R}^n)$ is weakly closed, this implies that $X \in W_T^{3,1}([0,T],\mathbb{R}^n)$, and $X''' = -AX'' - BX' - C(\cdot)X$, that is,

$$
X'''(t) + AX''(t) + BX'(t) + C(t)X(t) = 0, \tag{2.21}
$$

for a.e. $t \in [0,T]$ and $X \in W^{3,1}([0,T],\mathbb{R}^n)$. It follows from (2.13), (2.19) and Lemma 2.2 that $X \equiv 0$, that is, $X_n \to 0$ in $W^{3,1}([0,T],\mathbb{R}^n)$ as $n \to \infty$. But this clearly contradicts the initial assumption that $\|X_n\|_{W^{3,1}} = 1$ for all n, and the proof is complete. ◻

3. Some sharp and nonuniform nonresonance results

We shall prove the following sharp nonresonance result.

Theorem 3.1. *Let A and $C(t)$ be real $n \times n$ (not necessarily symmetric) matrices which commute, with A nonsingular, and satisfying the relation* (2.6). *Suppose that the Jacobian matrix $J_H(t,X) \equiv (\frac{\partial h_i}{\partial x_j})$ of $H(t,X)$ (not necessarily symmetric) exists, and has eigenvalues $\lambda_i(J_H(t,X))$ which satisfy $\delta_h(t) \le \lambda_i(J_H(t,X)) \le \Delta_h(t)$, $i = 1,\dots,n$, uniformly in $X \in \mathbb{R}^n$ for a.e $t \in [0,T]$, where δ_h and Δ_h are respectively the least and greatest eigenvalues of $J_H(t,X)$; and further the relation*

$$
\begin{aligned}
k^2\omega^2\|X\|^2 &< \Gamma_A^{-1}\gamma_H(t)\|X\|^2 \\
&\le \langle sA^{-1}H(t,X), X\rangle \\
&\le \gamma_A^{-1}\Gamma_H(t)\|X\|^2 \\
&< (k+1)^2\omega^2\|X\|^2
\end{aligned}
\tag{3.1}
$$

holds for a.e $t \in [0,T]$ and $X \in \mathbb{R}^n$ with $\|X\| \ge R_0 > 0$, with $k \in \mathbb{N}$, $\omega = \frac{2\pi}{T}$, $s \in \{-1,1\}$, where $\gamma_H(t) = |\min\{0,\delta_h(t)\}|$, $\Gamma_H(t) = |\max\{0,\Delta_h(t)\}|$, and γ_A, Γ_A are as defined in Lemma 2.2.

Then for each $s \in \{-1,1\}$ and all arbitrary matrix B, the system (1.1)-(1.2) *has at least one solution for every $P \in L^1([0,T],\mathbb{R}^n)$.*

Proof. Let us embed (1.1) in the λ parameter-dependent system

$$
X''' + AX'' + BX' + (1-\lambda)C(t)X + \lambda sH(t,X) = \lambda P(t) \tag{3.2}
$$

with $\lambda \in [0,1]$, or equivalently

$$A^{-1}X''' + X'' + A^{-1}BX' + (1-\lambda)A^{-1}C(t)X + \lambda s A^{-1}H(t,X) = \lambda A^{-1}P(t) \tag{3.3}$$

Then proceeding as in [3], we multiply both sides of (3.3) scalarly by $X''+A^{-1}C(t)X$ and integrate over $[0,T]$, yielding after simplification

$$\begin{aligned} &\int_0^T \big(\langle X'', X''\rangle + \langle X'', A^{-1}C(t)X\rangle + \langle H_\lambda(t,X), X''\rangle \\ &+ \langle H_\lambda(t,X), A^{-1}C(t)X\rangle\big)\,dt \\ &= \int_0^T \langle \lambda A^{-1}P(t), X'' + A^{-1}C(t)X\rangle\,dt \end{aligned} \tag{3.4}$$

where we have set $H_\lambda(t,X) = (1-\lambda)A^{-1}C(t)X + \lambda s A^{-1}H(t,X)$. One can easily verify that the integrand on the left-hand-side of (3.4) is identically equal to

$$\frac{1}{2}\{\|X'' + A^{-1}C(t)X\|^2 + \|X'' + H_\lambda(t,X)\|^2 - \|H_\lambda(t,X) - A^{-1}C(t)X\|^2\}$$

Thus, since $\|H_\lambda(t,X) - A^{-1}C(t)X\|^2 = \lambda^2\|sA^{-1}H(t,X) - A^{-1}C(t)X\|^2$, it follows from (3.4) that

$$\begin{aligned} \int_0^T \|X'' + A^{-1}C(t)X\|^2\,dt &\le \int_0^T \|sA^{-1}H(t,X) - A^{-1}C(t)X\|^2\,dt \\ &\quad + \int_0^T 2\|A^{-1}\|\|P(t)\|\|X'' + A^{-1}C(t)X\|\,dt \end{aligned} \tag{3.5}$$

Now, if $\|X\| \ge R_0$, then by (2.10) and (3.1),

$$\begin{aligned} \|sA^{-1}H(t,X) - A^{-1}C(t)X\| &= \Big(\frac{\langle sA^{-1}H(t,X), X\rangle - \langle A^{-1}C(t)X, X\rangle}{\langle X, X\rangle}\Big)\|X\| \\ &\le (\gamma_A^{-1}\Gamma_H(t) - \rho(t))\|X\| \le \alpha\|X\| \end{aligned} \tag{3.6}$$

uniformly for a.e. $t \in [0,T]$, where $\rho(t) = \frac{1}{2}(\Gamma_A^{-1}\gamma_C(t) + \gamma_A^{-1}\Gamma_C(t))$ and $\alpha = \frac{1}{2}\max(\gamma_A^{-1}\Gamma_C(t) - \Gamma_A^{-1}\gamma_C(t))$. On the other hand, if $\|X\| < R_0$, then there exists a constant $\delta_3 > 0$ such that $\|sA^{-1}H(t,X) - A^{-1}C(t)X\| \le \delta_3$, uniformly for a.e. $t \in [0,T]$. Hence,

$$\|sA^{-1}H(t,X) - A^{-1}C(t)X\| \le \alpha\|X\| + \delta_3$$

which implies that for all $X \in \mathbb{R}^n$ and a.e. $t \in [0,T]$, there exist constants $\delta_4 > 0$, $\delta_5 > 0$ such that

$$\|sA^{-1}H(t,X) - A^{-1}C(t)X\|^2 \le \alpha^2\|X\|^2 + \delta_4\|X\| + \delta_5 \tag{3.7}$$

Next, noting that $2\|A^{-1}\|\|P(t)\|\|X'' + A^{-1}C(t)X\| \le \delta_6(\|X''\| + \|X\|)$, for some $\delta_6 > 0$, (3.5) becomes

$$\int_0^T \|X'' + A^{-1}C(t)X\|^2\,dt \le \delta_7 + \alpha^2\int_0^T \|X\|^2\,dt + \delta_8\int_0^T (\|X''\| + \|X\|)dt, \tag{3.8}$$

where $\delta_7 = \delta_5 T$ and $\delta_8 = \delta_4 + \delta_6$. Thus, from the identity

$$\int_0^T \|X'' + A^{-1}C(t)X\|^2\,dt = \Big(\frac{\beta^2 - \alpha^2}{\beta^2} + \frac{\alpha^2}{\beta^2}\Big)\int_0^T \|X'' + A^{-1}C(t)X\|^2\,dt$$

and using relation (2.7), (3.8) now becomes

$$(\beta^2-\alpha^2)\int_0^T \|X''+A^{-1}C(t)X\|^2\,dt \le \delta_9+\delta_{10}\int_0^T (\|X''\|+\|X\|)dt, \tag{3.9}$$

where $\delta_9=\beta^2\delta_7$, $\delta_{10}=\beta^2\delta_8$, and $\beta^2-\alpha^2>0$ a.e on $[0,T]$ by relation (2.4). That is,

$$\begin{aligned}\int_0^T \|X''+A^{-1}C(t)X\|^2\,dt &\le \delta_{11}+\delta_{12}\int_0^T (\|X''\|+\|X\|)dt\\ &\le \delta_{11}+\delta_{13}\Big(\int_0^T \|X''+A^{-1}C(t)X\|^2\,dt\Big)^{\frac12}\end{aligned} \tag{3.10}$$

by the Cauchy-Schwartz inequality and the relations (2.7) and (2.8). We therefore deduce that

$$\int_0^T \|X''+A^{-1}C(t)X\|^2\,dt \le \delta_{14} \tag{3.11}$$

for some $\delta_{14}>0$. Thus, by (2.7), (2.8) and Wirtinger's inequality, there exists a constant $\delta_{15}>0$ such that

$$\int_0^T \|X\|^2\,dt \le \delta_{15},\ \int_0^T \|X'\|^2\,dt \le \delta_{15},\ \int_0^T \|X''\|^2\,dt \le \delta_{15} \tag{3.12}$$

Finally, multiply (3.3) scalarly by $X'''(t)$ and integrate over $[0,T]$ with respect to t, and using the estimates in (3.6), it is easily verified that

$$\int_0^T \|X'''\|^2\,dt \le \delta_{16} \tag{3.13}$$

for some constant $\delta_{16}>0$. The estimates (3.6) and (3.7) show clearly that there exist a constant $K>0$, independent of $\lambda\in(0,1)$ such that any T-periodic solution $X(t)$ of (3.3) satisfies $\|X\|_{C^2}\le K$, and the proof is complete. □

The following result is a nonuniform counterpart of Theorem 3.1.

Theorem 3.2. *Let A,C,M and N be the matrices defined in Lemma 2.3, and suppose that H is an L^1-Carathéodory function which satisfies*

$$\begin{aligned}\langle A^{-1}M(t)X,X\rangle &\le \liminf_{\|X\|\to\infty}\langle sA^{-1}H(t,X),X\rangle\\ &\le \limsup_{\|X\|\to\infty}\langle sA^{-1}H(t,X),X\rangle\\ &\le \langle A^{-1}N(t)X,X\rangle\end{aligned} \tag{3.14}$$

uniformly in $X\in\mathbb{R}^n$ for a.e. $t\in[0,T]$, such that $\Lambda_1(t)\|X\|^2\le\langle A^{-1}M(t)X,X\rangle$, $\langle A^{-1}N(t)X,X\rangle\le\Lambda_2(t)\|X\|^2$ a.e. on $[0,T]$, where Λ_1, Λ_2 are as defined in (2.13).

Then there exists $\epsilon=\epsilon(M,N,C)>0$ satisfying (2.14) *such that for each $s\in\{-1,1\}$ and all arbitrary matrix B, the system* (1.1)-(1.2) *has at least one solution for every $P\in L^1([0,T],\mathbb{R}^n)$.*

Proof. Let $\epsilon>0$ be associated to M,N by Lemma 2.3. Then, by $(\mathcal{H}_5)$, there exists a constant vector $\zeta=\zeta(\epsilon)$ with each $\zeta_i>0$ such that

$$\langle (A^{-1}M(t)-\epsilon I)X,X\rangle \le \langle sA^{-1}H(t,X),X\rangle \le \langle (A^{-1}N(t)+\epsilon I)X,X\rangle \tag{3.15}$$

for a.e. $t\in[0,T]$ and all $X\in\mathbb{R}^n$ with $|x_i|\ge\zeta_i$.

Let us now define $\mu(t,X) \equiv \big(\mu_i(t,X)\big)_{1\le i\le n} : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ by

$$\mu_i(t,X) = \begin{cases} x_i^{-1}h_i(t,X), & \text{if } |x_i| \ge \zeta_i;\\ x_i\zeta_i^{-2}h_i(t,x_1,\dots,x_{i-1},\zeta_i,x_{i+1},\dots,x_n) & \\ +(1-\frac{x_i}{\zeta_i})\kappa(t), & \text{if } 0\le x_i<\zeta_i;\\ x_i\zeta_i^{-2}h_i(t,x_1,\dots,x_{i-1},-\zeta_i,x_{i+1},\dots,x_n) & \\ +(1+\frac{x_i}{\zeta_i})\kappa(t), & \text{if } -\zeta_i\le x_i<0. \end{cases}$$

for a.e. $t\in[0,T]$, where

$$\kappa(t) \equiv \frac{1}{\|X\|^2}\Big[\|\langle (A^{-1}M(t)-I)X,X\rangle\| + \|\langle (A^{-1}N(t)+I)X,X\rangle\|\Big] \tag{3.16}$$

so that by construction and (3.15), we deduce that for a.e. $t\in[0,T]$ and $X\in\mathbb{R}^n$

$$\langle A^{-1}M(t)-\epsilon I)X,X\rangle \le \langle sA^{-1}\mu(t,X),X\rangle \le \langle A^{-1}N(t)+\epsilon I)X,X\rangle \tag{3.17}$$

The function $\tilde H \equiv (\tilde h_i(t,X))_{1\le i\le n} : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ defined by $\tilde h_i(t,X) = \mu_i(t,X)x_i$ satisfies the Carathéodory conditions, by construction. Hence, setting $\Theta(t,X) = H(t,X) - \tilde H(t,X)$, then $\Theta(t,X)$ is also L^1-Carathéodory with

$$\|\Theta(t,X)\| \le \sup_{\|X\|\le\zeta}\|H(t,X)-\tilde H(t,X)\| \le \varphi(t) \tag{3.18}$$

for a.e. $t\in[0,T]$ and $X\in\mathbb{R}^n$, for some $\varphi\in L^1([0,T],\mathbb{R})$ depending only on M,N and γ_r, associated with H by the Carathéodory assumption $\|H(t,X)\|\le\gamma_r$.

Finally, (1.1) is equivalent to

$$X'''(t) + AX''(t) + BX'(t) + s\tilde H(t,X(t)) + s\Theta(t,X(t)) = P(t) \tag{3.19}$$

By the Leray-Schauder technique (see Mawhin [7]), the proof of the theorem now follows by showing that there is a constant $K>0$, independent of $\lambda\in[0,1]$, such that $\|X\|_{C^2}<K$, for all possible solutions X of the homotopy

$$X''' + AX'' + BX' + (1-\lambda)N(t)X + s\lambda\tilde H(t,X) + s\lambda\Theta(t,X) = \lambda P(t) \tag{3.20}$$

with $\lambda\in[0,1]$. Let $\mathbf{X} := C^2([0,T],\mathbb{R}^n)$, $\mathbf{Z} := L^1([0,T],\mathbb{R}^n)$ and $\operatorname{dom}L \subset W^{3,1}(0,T)$ and define the mappings

$$\begin{gathered} L:\operatorname{dom}L\subset\mathbf{X}\to\mathbf{Z}, X\mapsto X'''(\cdot)+AX''(\cdot)+BX'(\cdot),\\ Q:\mathbf{X}\to\mathbf{Z}, X\mapsto s\widetilde H(\cdot,X(\cdot)) = s\langle\mu(\cdot,X(\cdot)),X(\cdot)\rangle,\\ R:\mathbf{X}\to\mathbf{Z}, X\mapsto s\Theta(\cdot,X(\cdot)) - P(\cdot),\\ \widetilde A:\mathbf{X}\to\mathbf{Z}, X\mapsto N(\cdot)X(\cdot) = \langle\mu(\cdot,0),X(\cdot)\rangle \end{gathered}$$

Accordingly, the existence of a solution will follow from [7, Theorem 4.5] if we can show that the possible solutions of the equivalent homotopy

$$LX + (1-\lambda)\widetilde AX + \lambda(QX+RX) = 0 \tag{3.21}$$

in $\operatorname{dom}L$ are *a-priori* bounded independently of $\lambda\in(0,1)$ and of the solutions. Let us re-write (3.20) as

$$\begin{aligned} &A^{-1}X''' + X'' + A^{-1}BX' + (1-\lambda)A^{-1}N(t)X\\ &+ s\lambda A^{-1}\tilde H(t,X) + s\lambda A^{-1}\Theta(t,X)\\ &= A^{-1}\lambda P(t) \end{aligned} \tag{3.22}$$

Then, from (3.17), for all $X \in \operatorname{dom} L$ we have

$$\begin{aligned}\langle A^{-1}M(t)X, X\rangle - \epsilon\|X\|^2 &\le \langle (1-\lambda)A^{-1}N(t)X + s\lambda A^{-1}\tilde{H}(t,X), X\rangle \\ &\le \langle A^{-1}N(t)X, X\rangle + \epsilon\|X\|^2\end{aligned} \tag{3.23}$$

for a.e. $t \in [0,T]$ and $\lambda \in [0,1]$. Thus, we may set

$$(1-\lambda)A^{-1}N(t)X + s\lambda A^{-1}\tilde{H}(t,X) \equiv A^{-1}C(t)X, \tag{3.24}$$

for a.e. $t \in [0,T], X \in domL$ and $\lambda \in [0,1]$, where, by (3.23), $C(t)$ is such that

$$\langle (A^{-1}M(t) - \epsilon I)X, X\rangle \le \langle A^{-1}C(t)X, X\rangle \le \langle (A^{-1}N(t) + \epsilon I)X, X\rangle \tag{3.25}$$

for a.e. $t \in [0,T], X \in domL$ and $\lambda \in [0,1]$. Thus, if $X \in \operatorname{dom} L$ is a solution of (3.22), then

$$\begin{aligned}0 = \int_0^T &\mid A^{-1}X'''(t) + X''(t) + A^{-1}BX'(t) + (1-\lambda)A^{-1}N(t)X(t) \\ &+ s\lambda A^{-1}\tilde{H}(t,X) + \lambda A^{-1}\big(s\Theta(t,X(t)) - P(t)\big) \mid dt\end{aligned} \tag{3.26}$$

It follows from (3.24) that

$$\begin{aligned}0 \ge \int_0^T &\|A^{-1}X''' + A^{-1}X'' + A^{-1}BX'(t) + A^{-1}C(t)X\|\,dt \\ &- \gamma_A^{-1}\big(\int_0^T \|\Theta(t,X)\|\,dt - \int_0^T \|P(t)\|\,dt\big)\end{aligned} \tag{3.27}$$

where $\gamma_A > 0$ is as defined in the previous section. That is, by Lemma 2.3 and (3.18), we obtain

$$0 \ge \delta\|X\|_{W^{3,1}} - \gamma_A^{-1}\big(\|\varphi\|_{L^1} - \|P\|_{L^1}\big), \tag{3.28}$$

which finally yields a constant $K_0 > 0$ such that $\|X\|_{W^{3,1}} \le K_0$. Hence, we obtain the required constant $K > 0$ such that $\|X\|_{C^2} < K$, following a standard procedure just as in [5], completing the proof. $\square$

Lastly, we give some uniqueness results for (1.1)-(1.2).

Theorem 3.3. *Let A be nonsingular and suppose that H is such that either*

$$k^2\omega^2 < \Gamma_A^{-1}\gamma_H(t) \le \frac{\|sA^{-1}(H(t,X_1) - H(t,X_2))\|}{\|X_1 - X_2\|} \le \gamma_A^{-1}\Gamma_H(t) < (k+1)^2\omega^2 \tag{3.29}$$

or

$$\begin{aligned}\Lambda_1(t) &\le \frac{\langle A^{-1}M(t)(X_1 - X_2), X_1 - X_2\rangle}{\|X_1 - X_2\|^2} \\ &\le \frac{\langle sA^{-1}\big(H(t,X_1) - H(t,X_2)\big), X_1 - X_2\rangle}{\|X_1 - X_2\|^2} \\ &\le \frac{\langle A^{-1}N(t)(X_1 - X_2), X_1 - X_2\rangle}{\|X_1 - X_2\|^2} \\ &\le \Lambda_2(t)\end{aligned} \tag{3.30}$$

uniformly for $X_1, X_2 \in \mathbb{R}^n$ with $X_1 \ne X_2$, and a.e. $t \in [0,T]$, $k \in \mathbb{N}$, $\omega = \frac{2\pi}{T}$, where γ_A, Γ_A, γ_H, Γ_H, and Λ_1, Λ_2, M, N are as appearing in Theorems 3.1 and 3.2 respectively.

Then, for each $s \in \{-1,1\}$ and all arbitrary $n \times n$ matrix B, the system (1.1)-(1.2) *has a unique T-periodic solution for every $P \in L^1([0,T],\mathbb{R}^n)$.*

Proof. **Case (i)** (H subject to (3.29): Suppose that $X_1(t)$ and $X_2(t)$ are two T-periodic solutions of (1.1)-(1.2) and set $X(t) = X_1(t) - X_2(t)$. Then $X(t)$ satisfies

$$X'''(t) + AX''(t) + BX'(t) + \widetilde{H}(t, X(t)) = 0 \tag{3.31}$$

where $\widetilde{H}(t,X) = sH(t,X_1) - sH(t,X_2)$. Rewrite (3.31) in the form (3.3), multiply scalarly by $X'' + A^{-1}C(t)X$ and integrate over $[0,T]$ yields

$$\int_0^T \langle X'' + sA^{-1}\widetilde{H}(t,X), X'' + A^{-1}C(t)X\rangle\, dt = 0 \tag{3.32}$$

We observe as before that the integral in (3.32) can be reset in the form

$$\frac{1}{2}\int_0^T \{\|X'' + A^{-1}C(t)X\|^2 + \|X'' + \widetilde{H}(t,X)\|^2 - \|\widetilde{H}(t,X) - A^{-1}C(t)X\|^2\}\, dt = 0$$

so that

$$\int_0^T \|X'' + A^{-1}C(t)X\|^2\, dt \le \int_0^T \|A^{-1}\widetilde{H}(t,X) - A^{-1}C(t)X\|^2\, dt \tag{3.33}$$

Now if $X(t) \neq 0$, then by (3.29),

$$\begin{aligned}
&\|A^{-1}\widetilde{H}(t,X) - A^{-1}C(t)X\| \\
&= \Big|\frac{\|sA^{-1}H(t,X) - sA^{-1}H(t,Y)\| - \langle A^{-1}C(t)X, X\rangle^{\frac{1}{2}}}{\|X-Y\|}\Big|\|X\| \\
&\le (\gamma_A^{-1}\Gamma_H(t) - \rho(t))\|X\| \\
&\le \alpha\|X\|,
\end{aligned} \tag{3.34}$$

Hence (3.33) becomes

$$\int_0^T \|X'' + A^{-1}C(t)X\|^2\, dt \le \alpha^2 \int_0^T \|X\|^2\, dt$$

Hence, by relation (2.7) of Lemma 2.2,

$$\beta^2 \int_0^T \|X\|^2\, dt \le \alpha^2 \int_0^T \|X\|^2\, dt \tag{3.35}$$

But $\alpha^2 < \beta^2$ a.e on $[0,T]$ by relation (2.4), so that (3.35) holds only when $\int_0^T \|X\|^2\, dt = 0$. Thus $X \equiv 0$; that is, $X_1 = X_2$, and are done.
Case (ii) (H subject to (3.30): Again setting $X = X_1 - X_2$, then here X is a solution of the problem

$$X'''(t) + AX''(t) + BX'(t) + \widetilde{C}(t, X(t))X(t) = 0, \tag{3.36}$$

satisfying (1.2), where the matrix $\widetilde{C}(\cdot, X(\cdot))$ is defined by

$$\widetilde{C}(t,X(t))X(t) = \begin{cases} sH(t, X+X_2) - sH(t,X_2) & \text{if } X \neq 0 \\ M(t), & \text{if } X = 0 \end{cases}$$

and by (3.30),

$$\Lambda_1(t)\|X\|^2 \le \langle A^{-1}M(t)X, X\rangle \le \langle A^{-1}\widetilde{C}(t)X, X\rangle \le \langle A^{-1}N(t)X, X\rangle \le \Lambda_2(t)\|X\|^2$$

uniformly for $X \in \mathbb{R}^n$ and a.e. $t \in [0,T]$, where $k^2\omega^2 \le |\Lambda_1(t)|$ and $|\Lambda_2(t)| \le (k+1)^2\omega^2$, with the strict inequality holding on subsets of $[0,T]$ of positive measure.

By Lemma 2.2, we deduce that $X = 0$, and the uniqueness result is thereby established. $\square$

References

[1] A. U. Afuwape; On the convergence of solutions of certain systems of nonlinear third order differential equations, *Simon Steven*, **57** (4), (1983), 255 - 271.

[2] A. U. Afuwape, P. Omari, F. Zanolin; Nonlinear perturbations of differential operators with nontrivial kernel and applications to third-order periodic boundary value problems, *J. Math. Anal. Appl.*, **143** (1), (1989), 35 - 56.

[3] A. U. Afuwape, A. S. Ukpera; Existence of solutions of periodic boundary value problems for some vector third order differential equations, *J. Nig. Math. Soc.*, **20**, (2001), 1 - 17.

[4] H. Amann; On the unique solvability of semi-linear operator equations in Hilbert spaces, *J. Math. Pures Appl.*, **61**, (1982), 149 - 175.

[5] J. O. C. Ezeilo, M. N. Nkashama; Resonant and nonresonant oscillations for third-order nonlinear ordinary differential equations, *Nonlinear Analysis, T.M.A.*, **12** (10), (1988), 1029 - 1046.

[6] J. O. C. Ezeilo; H. O. Tejumola; Boundedness and periodicity of solutions of a certain system of third-order nonlinear differential equations, *Ann. Math. Pura Appl.*, **74** (4), (1966), 283 - 316.

[7] J. Mawhin; *Topological Degree Methods in Nonlinear Boundary Value Problems*, CBMS Regional Conference Series in Mathematics **40**, American Math. Soc., Providence, R.I. (1979).

[8] J. Mawhin, J. R. Ward; Nonuniform nonresonance conditions at the two first eigenvalues for periodic solutions of forced Liénard and Duffing equations, *Rocky Mount. J. Math.*, **12** (4), (1982), 643 - 653.

[9] F. Minhós; Periodic solutions for a third order differential equation under conditions on the potential, *Portugaliae Math.*, **55** (4) (1998), 475 - 484.

[10] P. Omari, F. Zanolin; On forced nonlinear oscillations in n-th order differential systems with geometric conditions, *Nonlinear Analysis T.M.A.* **8** (7), (1984), 723 - 748.

[11] A. S. Ukpera; Nonuniform conditions for certain forced third order nonlinear periodic systems, *Internat.J. Diff. Equ. Appl* **9** (4), (2004) , 289 - 307.

[12] A. S. Ukpera, a. U. Afuwape; Resonant oscillations for forced third order nonlinear periodic systems, *Nonlinear Studies*, **12** (3), (2005), 215 - 231.

Awar Simon Ukpera
Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria
E-mail address: aukpera@oauife.edu.ng