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A NOTE ON RADIAL NONLINEAR SCHRÖDINGER SYSTEMS WITH NONLINEARITY SPATIALLY MODULATED

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Abstract. First, we prove that for Schrödinger radial systems the polar angular coordinate must satisfy $\theta' = 0$. Then using radial symmetry, we transform the system into a generalized Ermakov-Pinney equation and prove the existence of positive periodic solutions.

1. Introduction

This note concerns the existence of solutions for the nonlinear Schrödinger systems with nonlinearity spatially modulated and radial symmetry in $1D$

$$u_1''(x) + a(x)u_1(x) = b(x)f(u_1^2 + u_2^2)u_1 \tag{1.1a}$$

$$u_2''(x) + a(x)u_2(x) = b(x)f(u_1^2 + u_2^2)u_2 \tag{1.1b}$$

where $f(u_1^2 + u_2^2)$ is a positive continuous function with radial symmetry, and a and b are positive, continuous and L-periodic functions; i.e.,

$$a(x) = a(x + L), \quad b(x) = b(x + L). \tag{1.2}$$

Such solutions satisfy the boundary conditions

$$\lim_{|x|\to\infty} u_1(x) = \lim_{|x|\to\infty} u_2(x) = 0, \tag{1.3a}$$

$$\lim_{|x|\to\infty} u_1'(x) = \lim_{|x|\to\infty} u_2'(x) = 0 \tag{1.3b}$$

The study of the existence of positive solutions for systems like (1.1a), (1.1b) with one coupled lineal term has gained the interest of many mathematicians in recent years. We refer to the surveys [1, 2, 3]. In these papers, the authors show the existence of positive solutions for different systems, using critical point theory or a variational approach. Another different approximation to this kind of problems can be found in Ref. [4].

From of physical point of view, this kind of systems has gained a lot of interest in the last years, in particular in the context of systems for the mean field dynamics of Bose-Einstein condensates [12] and in applications to fields as nonlinear and fibers optics [13].

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On the other hand, the existence of positive solutions for the nonlinear Schrödinger equation

$$u'' + a(x)u = b(x)f(u(x)) \tag{1.4}$$

was proved in Ref. [10]. Thus, the existence of semitrivial solutions $(u_1, 0)$ and $(0, u_2)$ of the system (1.1a) is guaranteed by the Ref. [10].

We can transform the system (1.1a) in a equation, doing $y = (u_1, u_2)$

$$y'' + a(x)y = b(x)f(I)y \tag{1.5}$$

with $I = u_1^2 + u_2^2$.

With the change of variable

$$u_1 = \rho\cos\theta, \quad u_2 = \rho\sin\theta. \tag{1.6}$$

equation (1.5) becomes

$$\left[\rho'' - \rho(\theta')^2 + a(x)\rho\right]\cos\theta - [2\rho'\theta' + \rho\theta'']\sin\theta = b(x)f(\rho^2)\rho\cos\theta \tag{1.7}$$

The aim of this paper is to show that for Schrödinger radial systems, as (1.1a), (1.1b) with conditions (1.3a), (1.3b), can only exist solutions with $\theta' = 0$, specifically, we are thinking in the semitrivial solutions $(u_1, 0)$ and $(0, u_2)$. On the other hand, for $\theta' \neq 0$, there not exist solutions of the system (1.1a), (1.1b) with conditions (1.3a), (1.3b).

Moreover, we can transform the system, by using the radial symmetry, to a generalized Ermakov-Pinney equation and study positive periodic solutions for this equation.

The rest of the papers is organized as follows. In section 2 we prove that the only solutions of system (1.1a), (1.1b) with conditions (1.3a), (1.3b), if they exist, are given by solutions which verify $\theta' = 0$. In section 3, we prove the existence of positive periodic solutions of the system (1.1a), (1.1b), with periodic conditions.

In this note, $\|\cdot\|$ denotes the supremum norm.

2. Nonexistence of solutions for $\theta' \neq 0$ and existence for $\theta' = 0$

Physically, when a physical system possesses a symmetry, it means that a physical quantity is conserved. As the system (1.1a), (1.1b) has radial symmetry, the conserved quantity is the angular momentum. In polar coordinates, the conservation of the angular momentum is given by

$$\rho^2\theta' = \mu, \tag{2.1}$$

where μ is a constant. Using this fact, (1.7) becomes

$$\rho'' + a(x)\rho = b(x)f(\rho^2)\rho + \frac{\mu^2}{\rho^3}, \tag{2.2}$$

which can be taken as a generalized Ermakov-Pinney [5, 9].

Now, it is easy to prove that, if there exist solutions of the system (1.1a), (1.1b), with the boundary conditions (1.3a), (1.3b), they must satisfy the condition $\theta' = 0$: for these solutions, θ is constant and these solutions can be solutions of (1.5). In fact, we can find two examples of solutions for this case: the semitrivial solutions $(u_1, 0)$ and $(0, u_2)$ are solutions of the system (1.1a), (1.1b), with conditions (1.3a), (1.3b) (see Ref. [10]).

On the other hand, for one solution (u_1, u_2) with $\theta' \neq 0$, one has $\mu \neq 0$. Thus, if would exist a solution (u_1, u_2) of the system (1.1a), (1.1b) with the boundary conditions (1.3a), (1.3b) it would exist a solution ρ that would verify $\rho \to 0$ as $|x| \to \infty$. But it is impossible, by the singularity of (2.2).

Thus, we are in disposition to formulate the following theorem.

Theorem 2.1. *Let system* (1.1) *be with conditions* (1.3) *where* $a(x)$ *and* $b(x)$ *are positive, continuous and* L*-periodic functions. Then, if there exist solutions of the system* (1.1)*, with the conditions* (1.3)*, different of the trivial solution, they must satisfy the condition* $\theta' = 0$*, where* θ *is the polar angular coordinate in* (1.6)*.*

Remark 2.2. Specifically, for $\theta = k\pi$ or $\theta = \frac{k}{2}\pi$, for any $k \in \mathbb{Z}$, we obtain the semitrivial solutions. These solutions are called bright solitons in the physical literature. The dark solitons are also solutions of the system (1.1a), (1.1b) but with different boundary conditions [7]. It is straightforward to prove that, for this case, the only solutions are the former with $\theta' = 0$, provided that $a(x)$ is different to $b(x)$.

Remark 2.3. We can use another approximation, where one can see the universality of the method exposed here. Thus, let the nonlinear Schrödinger equation be

$$iu_t + u_{xx} + b(x)f(|u|^2)u + V(x)u = 0 \tag{2.3}$$

with $V(x)$ a L-periodic function. If we have the change of variable $u(t,x) = (v(x) + iw(x))\,e^{i\lambda t}$ and if we separate in real and imaginary part, we obtain

$$v'' + (V(x) - \lambda)\,v + b(x)f(v^2 + w^2)v = 0$$
$$w'' + (V(x) - \lambda)\,w + b(x)f(v^2 + w^2)w = 0$$

which is similar to the system (1.1a), (1.1b) for $a(x) = V(x) - \lambda$.

3. Periodic Solutions

As we showed in the previous section, system (1.1a) (1.1b), or equation (2.3), can be reduced to (2.2). Thus, we can describe the behaviour of solutions of (1.1a)–(1.1b) (or (2.3)) using (2.2)

Then, the aim of this section is to provide some existence result for the periodic boundary-value problem

$$\rho'' + a(x)\rho = b(x)f(\rho^2)\rho + \frac{\mu^2}{\rho^3}, \tag{3.1}$$

with $\rho(0) = \rho(L)$, $\rho'(0) = \rho'(L)$, where $a(x)$ and $b(x)$ are positive, continuous and L-periodic functions. To do it, we will use the following fixed-point theorem for a completely continuous operator in a Banach space, due to Krasnoselskii [8].

Theorem 3.1. *Let* X *be a Banach space, and let* $P \subset X$ *be a cone in* X*. Assume* Ω_1, Ω_2 *are open subsets of* X *with* $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$ *and let* $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ *be a completely continuous operator such that one of the following conditions is satisfied*

1. $\|Tu\| \le \|u\|$*, if* $u \in P \cap \partial\Omega_1$*, and* $\|Tu\| \ge \|u\|$*, if* $u \in P \cap \partial\Omega_2$*.*
2. $\|Tu\| \ge \|u\|$*, if* $u \in P \cap \partial\Omega_1$*, and* $\|Tu\| \le \|u\|$*, if* $u \in P \cap \partial\Omega_2$*.*

Then, T *has at least one fixed point in* $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$*.*

From the physical explanation, (3.1) has a repulsive singularity at $x = 0$. In order to apply Theorem 3.1, we need some information about the properties of the Green's function. Thus, let us consider the linear equation

$$\rho'' + a(x)\rho = 0, \tag{3.2}$$

with periodic conditions

$$\rho(0) = \rho(L), \quad \rho'(0) = \rho'(L) \tag{3.3}$$

In this section, we assume conditions under which the only solution of problem (3.2)-(3.3) is the trivial one. As a consequence of Fredholm's alternative, the nonhomogeneous equation

$$\rho'' + a(x)\rho = h(x), \tag{3.4}$$

admits a unique T-periodic solution which can be written as

$$\rho(x) = \int_0^L G(x,s)h(s)ds, \tag{3.5}$$

where $G(x,s)$ is the Green's function of problem (3.2)-(3.3). Following [6], we assume that problem (3.2) satisfies that the Green function, $G(x,s)$, associated with problem (3.4), is positive for all $(x,s) \in [0,L] \times [0,L]$. Moreover, following [11], we denote

$$M = \max_{x,s\in[0,L]} G(x,s), \quad m = \min_{x,s\in[0,L]} G(x,s) \tag{3.6}$$

where $M > m > 0$.

Theorem 3.2. *Let us assume the following hypotheses*

(i) *$a(x)$ and $b(x)$ are continuous and L-periodic functions with $a > 0, b > 0$.*
(ii) *$f(s) \geq 0$ for every $s \geq 0$.*
(iii) *There exists $r > 0$ such that*
$$A_r \max_{x\in[0,L]} \int_0^L G(x,s)b(s)ds + B_r \max_{x\in[0,L]} \int_0^L G(x,s)ds \leq r$$
for $A_r = \max_{s\in[0,r]} f(s^2)s$ and $B_r = \max_{s\in[0,r]} \mu^2/s^3$.
(iv) *There exist $R > r > 0$ such that*
$$A_R \min_{x\in[0,L]} \int_0^L G(x,s)b(s)ds + B_R \min_{x\in[0,L]} \int_0^L G(x,s)ds \geq \frac{M}{m}R$$
for $A_R = \min_{s\in[R,(M/m)R]} f(s^2)s$ and $B_R = \min_{s\in[R,(M/m)R]} \mu^2/s^3$.

Then, (3.1) has a positive periodic solution ρ with $\frac{m}{M}r \leq \rho(x) \leq \frac{M}{m}R$.

Proof. Let $X = C[0,L]$ with the supremum norm $\|\cdot\|$. We define the open sets

$$\Omega_1 = \{\rho \in X : \|\rho\| < r\}$$
$$\Omega_2 = \{\rho \in X : \|\rho\| < \frac{M}{m}R\}$$

Define the cone

$$P = \{\rho \in X : \rho \geq 0 \min_{x\in[0,L]} \rho \geq \frac{m}{M}\|\rho\|\}.$$

It is easy to prove that if $\rho \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$, then

$$\frac{m}{M}r \leq \rho(x) \leq \frac{M}{m}R, \quad \forall x$$

Let us define the operator

$$T\rho = \int_0^L G(x,s)\big[b(s)f(\rho^2(s))\rho(s) + \frac{\mu^2}{\rho^3(s)}\big] \tag{3.7}$$

We note that such operator is completely continuous. Clearly, a solution of problem (3.1) is just a fixed point of this operator.

If $\rho \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$, then

$$T\rho \geq \frac{m}{M} \int_0^L \max_{x\in[0,L]} G(x,s)\big[b(s)f(\rho^2(s))\rho(s) + \frac{\mu^2}{\rho^3(s)}\big] ds = \frac{m}{M}\|T\rho\|$$

that is, $T\left(P \cap (\overline{\Omega}_2 \backslash \Omega_1)\right) \subset P$.

Now, if $\rho \in \partial\Omega_1 \cap P$, then $\|\rho\| = r$ and $(m/M)r \leq \rho(x) \leq r$ for all x. Therefore, using (iii),

$$\|T\rho\| = \max_{x\in[0,L]} T\rho(x) \leq A_r \max_{x\in[0,L]} \int_0^L G(x,s)b(s)ds + B_r \max_{x\in[0,L]} \int_0^L G(x,s)ds \leq r$$

Similarly, if $x \in \partial\Omega_2 \cap P$, then $\|\rho\| = (M/m)R$ and $R \leq \rho(x) \leq (M/m)R$, for all x. Then, using the hypotheses (iv),

$$\begin{aligned}
\|T\rho\| &= \max_{x\in[0,L]} T\rho(x) \\
&= \max_{x\in[0,L]} \int_0^L G(x,s)\big[b(s)f(\rho^2(s))\rho(s) + \frac{\mu^2}{\rho^3(s)}\big] ds \\
&\geq A_R \min_{x\in[0,L]} \int_0^L G(x,s)b(s)ds + B_R \min_{x\in[0,L]} \int_0^L G(x,s)ds \geq \frac{M}{m}R
\end{aligned}$$

Now, from Theorem 3.1 there exists $\rho \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$ which is a solution of problem (3.1). Therefore,

$$\frac{m}{M} r \leq \rho(x) \leq \frac{M}{m} R$$

□

Corollary 3.3. *Under the conditions of Theorem 2, system* (1.1a)–(1.1b) *and equation* (2.3)*, with periodic conditions, have positive periodic solutions.*

In the framework of Bose-Einstein condensates [12] or nonlinear optics [13], such positive periodic solutions are called periodic matter waves.

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