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FLUCTUATIONS IN A MIXED IS-LM BUSINESS CYCLE MODEL

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Abstract. In the present paper, we extend a delayed IS-LM business cycle model by introducing an additional advance (anticipated capital stock) in the investment function. The resulting model is represented in terms of mixed differential equations. For the deviating argument τ (advance and delay) being a bifurcation parameter we investigate the local stability and the local Hopf bifurcation. Also some numerical simulations are given to support the theoretical analysis.

1. Introduction

Differential equations with delayed and advanced argument (also called mixed differential equations) occur in many problems of economy, biology and physics (see for example [8, 12, 10, 1, 6]), because mixed differential equations are much more suitable than delay differential equations for an adequate treatment of dynamic phenomena. The concept of delay is related to a memory of system, the past events are importance for the present current behavior, and the concept of advance is related to a potential future events which can be known at the current present time which could useful for decision making. The study of various problems for mixed differential equations can be found in many works, we cite for example [20, 19, 18, 11, 7, 2].

In the present paper, we extend a delayed IS-LM business cycle model (see [13]), by introducing an additional advance (anticipated capital stock) in the investment function as follows:

$$
\begin{gathered}
\frac{dY}{dt} = \alpha[I(Y(t), K(t+\tau), R(t)) - S(Y(t), R(t))], \\
\frac{dK}{dt} = I(Y(t-\tau), K(t-\tau), R(t-\tau)) - \delta K(t), \\
\frac{dR}{dt} = \beta[L(Y(t), R(t)) - \widetilde{M}],
\end{gathered} \tag{1.1}
$$

where Y is the gross product, K is the capital stock, R is the interest rate , $\widetilde{M}$ is the constant money supply, α is the adjustment coefficient in the goods market, β

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is the adjustment coefficient in money market, δ is the depreciation rate of capital stock, $I(Y,K,R)$ is the investment function, $S(Y,R)$ is the saving function, $L(Y,R)$ is the demand for money and τ is the Kalecki's time delay (1935, [15]); i.e. there is a time lag needed for new capital to be installed.

The need of incorporation of an anticipated capital stock $K(t+\tau)$ in business cycle model is of great interest for government to know by anticipation the capital stock at future time (see [6], 2004).

The first dynamic IS-LM model is proposed by ordinary differential equations by Torre in (1977, [22]) as follows:

$$\begin{aligned}\frac{dY}{dt} &= \alpha[I(Y(t),R(t)) - S(Y(t),R(t))],\\ \frac{dR}{dt} &= \beta[L(Y(t),R(t)) - \widetilde{M}],\end{aligned} \tag{1.2}$$

In (1989, [9]), Gabisch and Lorenz considered the following augmented IS-LM business cycle model:

$$\begin{aligned}\frac{dY}{dt} &= \alpha[I(Y(t),K(t),R(t)) - S(Y(t),R(t))],\\ \frac{dK}{dt} &= I(Y(t),K(t),R(t)) - \delta K(t),\\ \frac{dR}{dt} &= \beta[L(Y(t),R(t)) - \widetilde{M}],\end{aligned} \tag{1.3}$$

Recently, there have been many works devoted on the introduction of Kalecki's time delay into dynamic of investment processus (see [17, 21, 3, 23, 14]).

In (2008, [13]), we proposed a delayed IS-LM model by introducing time delay into capital stock, interest rate and gross product in capital accumulation equation as follows:

$$\begin{aligned}\frac{dY}{dt} &= \alpha[I(Y(t),K(t),R(t)) - S(Y(t),R(t))],\\ \frac{dK}{dt} &= I(Y(t-\tau),K(t-\tau),R(t-\tau)) - \delta K(t),\\ \frac{dR}{dt} &= \beta[L(Y(t),R(t)) - \widetilde{M}],\end{aligned} \tag{1.4}$$

Clearly, this reformulation of Gabisch and Lorenz model is more reasonable, because the change in the capital stock is due to the past investment decisions.

In this work, the dynamics of the system (1.1) are studied in terms of local stability and of the description of the Hopf bifurcation, that is proven to exist as the deviating argument τ (advance and delay) cross some critical value. A numerical illustrations are given to support the theoretical analysis.

2. Steady state and local stability analysis

As in Cai (2005, [3]), we assume that the investment function I, the saving function S, and the demand for money L are given by

$$\begin{aligned}I(Y,K,R) &= \eta Y - \delta_1 K - \beta_1 R,\\ S(Y,R) &= l_1 Y + \beta_2 R,\\ L(Y,R) &= l_2 Y - \beta_3 R,\end{aligned}$$

with $\delta_1, l_1, l_2, \beta_1, \beta_2, \beta_3$ are positive constants. Then system (1.1) becomes

$$\begin{gathered}\frac{dY}{dt} = \alpha[(\eta - l_1)Y(t) - \delta_1 K(t+\tau) - (\beta_1+\beta_2)R(t))],\\ \frac{dK}{dt} = \eta Y(t-\tau) - \delta_1 K(t-\tau) - \delta K(t) - \beta_1 R(t-\tau),\\ \frac{dR}{dt} = \beta[l_2 Y(t) - \beta_3 R(t) - \widetilde{M}].\end{gathered} \tag{2.1}$$

In the following proposition, we give a sufficient conditions for the existence and uniqueness of positive equilibrium E^* of the system (2.1).

Theorem 2.1 ([13]). *Define*

$$\Theta = \delta(\beta_3\eta - \beta_1 l_2) - (\delta+\delta_1)(\beta_2 l_2 + \beta_3 l_1),$$

and suppose that

(H1) *:* $\Theta < 0$*;*

(H2) *:* $(\delta+\delta_1)l_1 - \delta\eta \le 0$.

Then there exists a unique positive equilibrium $E^* = (Y^*, K^*, R^*)$ *of system* (2.1)*, where* Y^*, K^*, R^* *are given by*

$$Y^* = \frac{-((\beta_1+\beta_2)\delta + \beta_2\delta_1)\widetilde{M}}{\Theta}, \tag{2.2}$$

$$K^* = \frac{-(\beta_1 l_1 + \beta_2\eta)\widetilde{M}}{\Theta}, \tag{2.3}$$

$$R^* = \frac{((\delta+\delta_1)l_1 - \delta\eta)\widetilde{M}}{\Theta}. \tag{2.4}$$

In the next, we will study the stability of the positive equilibrium E^* with respect to the time parameter τ. Introducing the variable change $Ka(t) = K(t+\tau)$, the system (2.1) leads:

$$\begin{gathered}\frac{dY}{dt} = \alpha[(\eta - l_1)Y(t) - \delta_1 Ka(t) - (\beta_1+\beta_2)R(t))],\\ \frac{dKa}{dt} = \eta Y(t) - \delta_1 Ka(t-\tau) - \delta Ka(t) - \beta_1 R(t),\\ \frac{dR}{dt} = \beta[l_2 Y(t) - \beta_3 R(t) - \widetilde{M}].\end{gathered} \tag{2.5}$$

The characteristic equation associated to system (2.5) takes the general form

$$\lambda^3 + A\lambda^2 + B\lambda + C + (D\lambda^2 + E\lambda + F)exp(-\lambda\tau) = 0, \tag{2.6}$$

where

$$\begin{gathered}A = \delta + \beta\beta_3 - \alpha(\eta - l_1),\\ B = \alpha\delta_1\eta - \alpha\beta\beta_3(\eta - l_1) + \alpha\beta(\beta_1+\beta_2)l_2 - \alpha(\eta - l_1)\delta + \beta\beta_3\delta,\\ C = \alpha\beta\{\delta[(\beta_1+\beta_2)l_2 - \beta_3(\eta - l_1)] + \delta_1[\beta_3\eta - \beta_1 l_2]\},\\ D = \delta_1, \quad E = \delta_1(\beta\beta_3 - \alpha(\eta - l_1)),\\ F = \alpha\beta\delta_1[(\beta_1+\beta_2)l_2 - \beta_3(\eta - l_1)].\end{gathered}$$

We begin by considering the case $\tau = 0$. This case is of importance, because if the positive equilibrium of (2.1) is stable when $\tau = 0$, we seek conditions on the model parameters to obtain the local stability for all nonnegative values of τ, or to find a critical values τ_0 of the delay which could destabilize the equilibrium.

When $\tau = 0$ the characteristic equation (2.6) reads as

$$\lambda^3 + (A+D)\lambda^2 + (B+E)\lambda + (C+F) = 0. \tag{2.7}$$

From (H1) in proposition 2.1, we have $C + F > 0$. Hence, according to the Routh-Hurwitz criterion, we have the following result.

Theorem 2.2 ([13]). *For $\tau = 0$, the equilibrium E^* is locally asymptotically stable if and only if*

(H3) $A + D > 0$;
(H4) $(A+D)(B+E) - (C+F) > 0$;

where A, B, C, D, E, F are defined in (2.6).

We assume in the sequel, that hypotheses (H1), (H2), (H3) and (H4) hold, and we return to the study of (2.6) with $\tau > 0$. Clearly, $\lambda(\tau) = u(\tau) + iv(\tau)$ is a root of equation (2.6) if and only if

$$\begin{aligned} &u^3 - 3uv^2 + Au^2 - Av^2 + Bu + C \\ &= -\exp(-u\tau)\{Du^2\cos(v\tau) - Dv^2\cos(v\tau) + Eu\cos(v\tau) \\ &\quad + F\cos(v\tau) + 2Duv\sin(v\tau) + Ev\sin(v\tau)\}, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &3u^2v - v^3 + 2Auv + Bv \\ &= -\exp(-u\tau)\{2Duv\cos(v\tau) + Ev\cos(v\tau) \\ &\quad - Du^2\sin(v\tau) + Dv^2\sin(v\tau) - Eu\sin(v\tau) - F\sin(v\tau)\}, \end{aligned} \tag{2.9}$$

We set $u = 0$ into the two equation (2.8) and (2.9) to get

$$-Av^2 + C = (Dv^2 - F)\cos(v\tau) - Ev\sin(v\tau), \tag{2.10}$$

$$v^3 - Bv = Ev\cos(v\tau) + (Dv^2 - F)\sin(v\tau). \tag{2.11}$$

Squaring and adding the squares together, we obtain

$$v^6 + av^4 + bv^2 + c = 0, \tag{2.12}$$

with $a = A^2 - D^2 - 2B$, $b = B^2 - 2AC - E^2 + 2DF$, $c = C^2 - F^2$. Letting $z = v^2$, equation (2.12) becomes the cubic equation

$$h(z) := z^3 + az^2 + bz + c = 0, \tag{2.13}$$

Using the results from [4], we have the following two lemmas.

Lemma 2.3. *Suppose that* (H1)-(H4) *hold, then the following taxonomy holds:*

(i) *If* (2.13) *has no positive solutions, then no stability switches exists.*
(ii) *If* (2.13) *has one or two positive solutions, then there exists only one stability switch.*
(iii) *If* (2.13) *has three positive solutions, then there exists at least a stability switch.*

Lemma 2.4. *If τ^* is a stability switch and v^* corresponding to τ^* is a simple root of equation* (2.13)*, then a Hopf bifurcation occurs at τ^*.*

Now, define

$$\Delta = a^2 - 3b, \tag{2.14}$$

$$\overline{z_1} := \frac{1}{3}(-a + \sqrt{\Delta}), \tag{2.15}$$

$$\overline{z_2} := \frac{1}{3}(-a - \sqrt{\Delta}). \tag{2.16}$$

Lemma 2.5. *Suppose that $c < 0$.*
(i) If one of the following two conditions

(S1) $\Delta < 0$
(S2) $\Delta > 0$, $\overline{z_1} < 0$ *or* $\overline{z_1} > 0$ *and* $\overline{z_2} < 0$; *or* $\overline{z_2} > 0$ *and* $h(\overline{z_1})h(\overline{z_2}) > 0$;

is satisfied, then (2.13) *has unique simple positive root.*
(ii) If

(S3) $\Delta > 0$, $\overline{z_2} > 0$ *and* $h(\overline{z_1})h(\overline{z_2}) < 0$,

then (2.13) *has three simple positive roots.*

Proof. By differentiating $h(z)$, we have

$$\frac{dh(z)}{dz} = 3z^2 + 2az + b\,.$$

Set

$$3z^2 + 2az + b = 0. \tag{2.17}$$

If $\Delta < 0$, then equation (2.17) does not have real roots, so the function h is monotone increasing in z. It follows from $h(0) = c < 0$ that equation (2.13) has unique simple positive root. If $\Delta > 0$, then the equation (2.17) has two roots $\overline{z_1}$ and $\overline{z_2}$, where $\overline{z_1} > \overline{z_2}$, are defined by (2.15) and (2.16).

Clearly, $\overline{z_1}$ is the local minimum of $h(z)$. Thus, if $\overline{z_1} < 0$ or $\overline{z_1} > 0$ and $\overline{z_2} < 0$, or $\overline{z_2} > 0$ and $h(\overline{z_1})h(\overline{z_2}) > 0$, then equation (2.13) has unique simple positive root. (ii) If $\Delta > 0$, $\overline{z_1} > 0$, $\overline{z_2} > 0$ and $h(\overline{z_1})h(\overline{z_2}) < 0$, then equation (2.13) has three simple positive roots. □

By similar arguments the following lemma can be proved.

Lemma 2.6. *Suppose that $c > 0$. If*

(S4) $\Delta > 0$, $\overline{z_1} > 0$ *and* $h(\overline{z_1}) < 0$,

then (2.13) *has two simple positive roots.*

Suppose that equation (2.13) has simple positive roots. Without loss of generality, we assume that it has three positive roots, denoted by z_1, z_2 and z_3, respectively. Then equation (2.6) has three positive roots, say $v_1 = \sqrt{z_1}$; $v_2 = \sqrt{z_2}$; $v_3 = \sqrt{z_3}$. Let

$$\tau_l = \frac{1}{v_l}[\arccos(\frac{(Av_l^2 - C)(F - Dv_l^2) + (v_l^3 - Bv_l)Ev_l}{(Dv_l - F)^2 + E^2 v_l^2})], \quad l = 1, 2, 3.$$

Then $\pm iv_l$ is a pair of purely imaginary roots of equation (2.6) corresponding to $\tau = \tau_l$, $l = 1, 2, 3$. Define

$$\tau_0 = \tau_{l_0} = \min_{l=1,2,3}(\tau_l), \quad v_0 = v_{l_0}, \quad z_0 = v_0^2. \tag{2.18}$$

From lemmas 2.3, 2.4, 2.5 and 2.6, we have the following result.

Theorem 2.7. *Suppose that* (H1)-(H4) *hold. If one of the conditions* (S1), (S2), (S3) *or* (S4) *holds, then there exists a critical positive deviating argument τ_0 such that, when $\tau \in [0, \tau_0)$ the steady state E^* is locally asymptotically stable, and a Hopf bifurcation occurs as τ passes through τ_0, where τ_0 is given by* (2.18)*. Moreover,*

$$\frac{d \operatorname{Re} \lambda(\tau_0)}{d\tau} > 0.$$

Proof. We need to prove only

$$\frac{d \operatorname{Re} \lambda(\tau_0)}{d\tau} > 0.$$

Let $\lambda(\tau) = u(\tau) + iv(\tau)$ be the root of (2.6) satisfying $u(\tau_0) = 0$, and $v(\tau_0) = v_0$.

By differentiating (2.8) and (2.9) with respect to τ and then setting $\tau = \tau_0$, we obtain

$$G_1 \frac{du(\tau_0)}{d\tau} + G_2 \frac{dv(\tau_0)}{d\tau} = H_1, \tag{2.19}$$

$$-G_2 \frac{du(\tau_0)}{d\tau} + G_1 \frac{dv(\tau_0)}{d\tau} = H_2, \tag{2.20}$$

where

$$\begin{gathered}
G_1 = -3v_0^2 + B + (E + Dv_0^2\tau_0 - F\tau_0)\cos(v_0\tau_0) + (2Dv_0 - Ev_0\tau_0)\sin(v_0\tau_0), \\
G_2 = -2Av_0 + (-2Dv_0 + Ev_0\tau_0)\cos(v_0\tau_0) + (E + Dv_0^2\tau_0 - F\tau_0)\sin(v_0\tau_0), \\
H_1 = (-Dv_0^3 + Fv_0)\sin(v_0\tau_0) - Ev_0^2\cos(v_0\tau_0), \\
H_2 = (-Dv_0^3 + Fv_0)\cos(v_0\tau_0) + Ev_0^2\sin(v_0\tau_0).
\end{gathered}$$

Solving for $\frac{du(\tau_0)}{d\tau}$ we get

$$\frac{du(\tau_0)}{d\tau} = \frac{G_1H_1 - G_2H_2}{G_1^2 + G_2^2}. \tag{2.21}$$

Therefore, we have

$$\frac{du(\tau_0)}{d\tau} = \frac{v_0^2 h'(z_0)}{G_1^2 + G_2^2}. \tag{2.22}$$

Thus, we have the transversally condition

$$\frac{du(\tau_0)}{d\tau} \neq 0.$$

If $\frac{du(\tau_0)}{d\tau} < 0$, for $\tau < \tau_0$ and sufficiently close to τ_0, then equation (2.6) has a root $\lambda(\tau) = u(\tau) + iv(\tau)$ satisfying $u(\tau) > 0$, which contradicts the fact that E^* is locally asymptotically stable for all $\tau \in [0, \tau_0)$. This completes the proof. $\square$

3. Hopf bifurcation

From theorem 2.7, we have the following result.

Theorem 3.1 ([5])**.** *Suppose that* (H1)-(H4) *hold. If one of the conditions* (S1), (S2), (S3), (S4) *holds, then there exists $\varepsilon_0 > 0$ such that for each $0 \le \varepsilon < \varepsilon_0$, system* (2.1) *has a family of periodic solutions $p = p(\varepsilon)$ with period $T = T(\varepsilon)$, for the parameter values $\tau = \tau(\varepsilon)$ such that $p(0) = 0$, $T(0) = \frac{2\pi}{v_0}$ and $\tau(0) = \tau_0$.*

4. Numerical Application

In this section, we give a numerical simulation supporting the theoretical analysis given in section 2 and 3. Consider the following parameters:

$$\alpha = 1.5, \quad \beta = 2, \quad \delta = 0.2, \quad \delta_1 = 0.5, \quad \widetilde{M} = 0.05,$$
$$l_1 = 0.1, \quad l_2 = 0.2, \quad \beta_1 = \beta_2 = \beta_3 = 0.2, \quad \eta = 0.4.$$

System (2.1) has the unique positive equilibrium $E^* = (0.2647, 0.1470, 0.0147)$. It follows from 2, that the critical positive deviating argument $\tau_0 = 2.030488132$. Thus from theorem 2.7 we know that when $0 \le \tau < \tau_0$, E^* is asymptotically stable (see Fig.1). When τ passes through the critical value τ_0, E^* loses its stability and a family of periodic solutions with period $T(0) = 7.627527841$ bifurcating from E^* occurs (see Fig.2 and Fig.3).

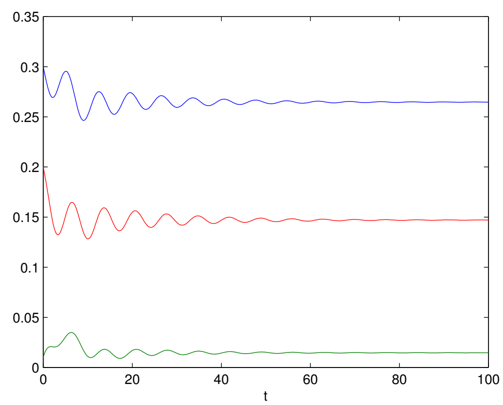

Figure 1. For $\tau = 1.8$, the solutions $(Y(t)\ , K(t)\ , R(t))$ of (1.2) are asymptotically stable and converge to the equilibrium E^*.

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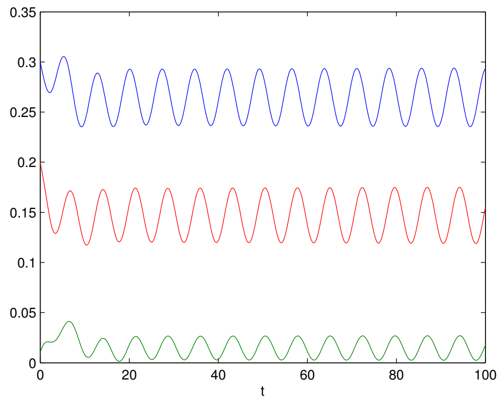

FIGURE 2. When $\tau = 2.03048832$, a Hopf bifurcation occurs and periodic solutions appear, with same period $T(0) = 7.627527841$.

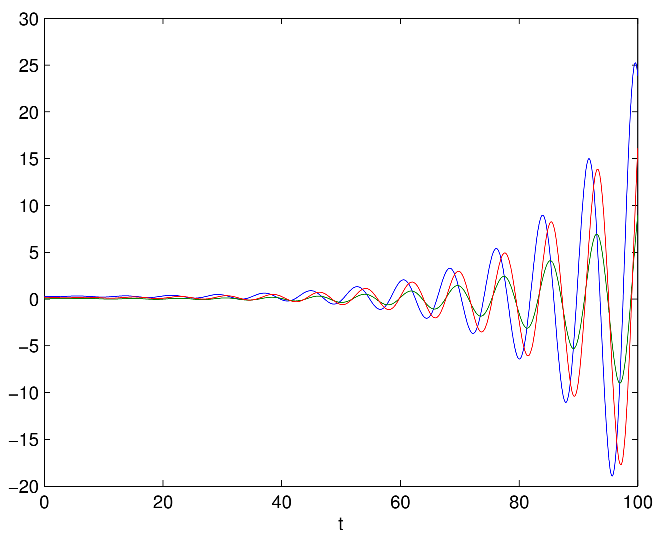

FIGURE 3. For $\tau = 2.5$, the equilibrium E^* of (1.2) is unstable.

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