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EXISTENCE OF ALMOST AUTOMORPHIC SOLUTIONS TO SOME NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. In this paper we obtain the existence of almost automorphic solutions for some neutral first-order functional differential equations with S^p-almost automorphic coefficients.

1. Introduction

The impetus of this paper comes from two main sources. The first source is a paper by N'Guérékata and Pankov [34], in which the concept of Stepanov-like almost automorphy (or $\mathbf{S}^p$-almost automorphy) was introduced. Such a notion was, subsequently, utilized to study the existence of weak almost automorphic solutions to some parabolic evolution equations. The second source is a paper by Diagana and N'Guérékata [4] in which, the concept of Stepanov-like almost automorphy was extensively utilized to obtain the existence and uniqueness of almost automorphic solutions to the semilinear differential equations

$$u'(t) = Au(t) + F(t, u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is a densely defined closed linear operator on a Banach space $\mathbb{X}$, which also is the infinitesimal generator of an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathbb{X}$ and $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ is $\mathbf{S}^p$-almost automorphic for $p > 1$ and jointly continuous.

In this paper we study more general differential equations than (1.1), that is, we investigate the existence and uniqueness of an almost automorphic solution to the neutral first-order functional differential equation

$$\frac{d}{dt}\left[u(t) + f(t, u_t)\right] = Au(t) + g(t, u_t), \quad \forall t \in \mathbb{R}, \tag{1.2}$$

where $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is a densely defined closed linear operator for $t \in \mathbb{R}$, the history $u_t : (-\infty, 0] \mapsto \mathbb{X}$ defined by $u_t(\tau) = u(t + \tau)$ belongs to some abstract phase space $\mathcal{B}$, which is defined axiomatically, and the coefficients f, g are $\mathbf{S}^p$-almost automorphic for $p > 1$ and jointly continuous. It is worth mentioning that since the space $AS^p(\mathbb{X})$ of Stepanov-like almost automorphic functions contains the space

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$AA(\mathbb{X})$ of almost automoprhic functions, it turns out that the results of this paper generalize in particular the existence results established in Diagana et al. [7].

As an application, our main result will be utilized to study the existence of almost automorphic solutions to a slightly modified integrodifferential equation which was considered in Diagana et al. [11] in the pseudo almost periodic case.

The existence of almost automorphic, almost periodic, asymptotically almost periodic, and pseudo almost periodic solutions is certainly one of the most attractive topics in qualitative theory of differential equations due to their significance and various applications. The concept of almost automorphy, which is the central issue in this paper was first introduced in the literature by Bochner in the earlier sixties [1] and is a natural generalization of the notion of almost periodicity. Since then, such a topic has generated several developments and extensions. For the most recent developments, we refer the reader to the book by N'Guérékata [32].

Existence results related to almost periodic and asymptotically almost periodic solutions to ordinary neutral differential equations and abstract partial neutral differential equations have recently been established in [29, 38, 20], respectively. To the best of our knowledge, there are few papers devoted to the existence of almost automorphic solutions to functional-differential equations with delay in the literature, among them are for instance [23, 12, 13, 7]. However, the existence of almost automoprhic solutions to neutral functional differential equations of the form (1.2) in the case when the forcing terms f, g are $\mathbf{S}^p$-almost automorphic is an untreated topic and constitutes the main motivation of the present paper. One should point out that neutral differential equations arise in many areas of applied mathematics. For this reason, those equations have been of a great interest for several mathematicians during the past few decades. The literature relative to ordinary neutral differential equations is quite extensive and so for more on this topic and related issues we refer the reader to [16, 36, 37, 17, 18, 19] and the references therein.

2. Preliminaries

In what follows we recall some definitions and notations needed in the sequel. Most of these definitions and notations come from [7].

Let $(\mathbb{Z}, \|\cdot\|_{\mathbb{Z}})$, $(\mathbb{W}, \|\cdot\|_{\mathbb{W}})$ be Banach spaces. The notation $L(\mathbb{Z}, \mathbb{W})$ stands for the Banach space of bounded linear operators from $\mathbb{Z}$ into $\mathbb{W}$ equipped with its natural topology; in particular, this is simply denoted $L(\mathbb{Z})$ when $\mathbb{Z} = \mathbb{W}$. The spaces $C(\mathbb{R}, \mathbb{Z})$ and $BC(\mathbb{R}, \mathbb{Z})$ stand respectively for the collection of all continuous functions from $\mathbb{R}$ into $\mathbb{Z}$ and the Banach space of all bounded continuous functions from $\mathbb{R}$ into $\mathbb{Z}$ equipped with the sup norm defined by

$$\|f\|_\infty := \sup_{t\in\mathbb{R}} \|f(t)\|.$$

We have similar definitions as above for both $C(\mathbb{R}\times\mathbb{Z}, \mathbb{W})$ and $BC(\mathbb{R}\times\mathbb{Z}, \mathbb{W})$.

In this paper, $(\mathbb{X}, \|\cdot\|)$ stands for a Banach space and the linear operator A is the infinitesimal generator of a C_0-semigroup $(T(s))_{s\geq 0}$, which is asymptotically stable. Namely, there exist some constants $M, \delta > 0$ such that

$$\|T(t)\| \leq Me^{-\delta t}$$

for every $t \geq 0$.

3. $\mathbf{S}^p$-Almost Automorphy

Definition 3.1 (Bochner)**.** A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$.

Remark 3.2. The function g in Definition 3.1 is measurable, but not necessarily continuous. Moreover, if g is continuous, then f is uniformly continuous [33, Theorem 2.6]. If the convergence above is uniform in $t \in \mathbb{R}$, then f is almost periodic. Denote by $AA(\mathbb{X})$ the collection of all almost automorphic functions $\mathbb{R} \to \mathbb{X}$. Note that $AA(\mathbb{X})$ equipped with the sup norm, $\|\cdot\|_\infty$, turns out to be a Banach space. Among other things, almost automorphic functions satisfy the following properties.

Theorem 3.3 ([2], [30, Theorem 2.1.3])**.** *If $f, f_1, f_2 \in AA(\mathbb{X})$, then*

- (i) $f_1 + f_2 \in AA(\mathbb{X})$,
- (ii) $\lambda f \in AA(\mathbb{X})$ *for any scalar* λ,
- (iii) $f_\alpha \in AA(\mathbb{X})$ *where* $f_\alpha : \mathbb{R} \to \mathbb{X}$ *is defined by* $f_\alpha(\cdot) = f(\cdot + \alpha)$,
- (iv) *the range* $\mathcal{R}_f := \{f(t) : t \in \mathbb{R}\}$ *is relatively compact in* $\mathbb{X}$, *thus* f *is bounded in norm,*
- (v) *if* $f_n \to f$ *uniformly on* $\mathbb{R}$ *where each* $f_n \in AA(\mathbb{X})$, *then* $f \in AA(\mathbb{X})$ *too.*
- (vi) *if* $g \in L^1(\mathbb{R})$, *then* $f * g \in AA(\mathbb{R})$, *where* $f * g$ *is the convolution of* f *with* g *on* $\mathbb{R}$.

For more on almost automorphic functions and related issues we refer the reader to the following books by N'Guérékata [30, 32].

We will denote by $AA_u(\mathbb{X})$ the closed subspace of all functions $f \in AA(\mathbb{X})$ with $g \in C(\mathbb{R}, \mathbb{X})$. Equivalently, $f \in AA_u(\mathbb{X})$ if and only if f is almost automorphic and the convergence in Definition 3.1 are uniform on compact intervals, i.e. in the Fréchet space $C(\mathbb{R}, \mathbb{X})$. Indeed, if f is almost automorphic, then, by Theorem 2.1.3(iv) [30], its range is relatively compact.

Obviously, the following inclusions hold:

$$AP(\mathbb{X}) \subset AA_u(\mathbb{X}) \subset AA(\mathbb{X}) \subset BC(\mathbb{X})\,,$$

where $AP(\mathbb{X})$ stands for the collection of all $\mathbb{X}$-valued almost periodic functions.

Definition 3.4. The Bochner transform $f^b(t, s)$, $t \in \mathbb{R}$, $s \in [0, 1]$, of a function $f : \mathbb{R} \mapsto \mathbb{X}$, is defined by

$$f^b(t, s) := f(t + s).$$

Remark 3.5. Note that a function $\varphi(t, s)$, $t \in \mathbb{R}$, $s \in [0, 1]$, is the Bochner transform of a certain function $f(t)$,

$$\varphi(t, s) = f^b(t, s)\,,$$

if and only if $\varphi(t + \tau, s - \tau) = \varphi(s, t)$ for all $t \in \mathbb{R}$, $s \in [0, 1]$ and $\tau \in [s - 1, s]$.

Definition 3.6 ([35]). Let $p \in [1,\infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with values in $\mathbb{X}$ such that $f^b \in L^\infty\big(\mathbb{R}, L^p(0,1;\mathbb{X})\big)$. This is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(\tau)\|^p \, d\tau \Big)^{1/p}.$$

Definition 3.7 ([34]). The space $AS^p(\mathbb{X})$ of Stepanov-like almost automorphic functions (or $\mathbf{S}^p$-almost automorphic) consists of all $f \in BS^p(\mathbb{X})$ such that $f^b \in AA\big(L^p(0,1;\mathbb{X})\big)$.

In other words, a function $f \in L^p_{loc}(\mathbb{R};\mathbb{X})$ is said to be $\mathbf{S}^p$-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p(0,1;\mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $g \in L^p_{loc}(\mathbb{R};\mathbb{X})$ such that

$$\Big[\int_t^{t+1} \|f(s_n+s) - g(s)\|^p ds \Big]^{1/p} \to 0,$$
$$\Big[\int_t^{t+1} \|g(s-s_n) - f(s)\|^p ds \Big]^{1/p} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}$.

Remark 3.8. It is clear that if $1 \le p < q < \infty$ and $f \in L^q_{loc}(\mathbb{R};\mathbb{X})$ is $\mathbf{S}^q$-almost automorphic, then f is $\mathbf{S}^p$-almost automorphic. Also if $f \in AA(\mathbb{X})$, then f is $\mathbf{S}^p$-almost automorphic for any $1 \le p < \infty$.

It is also clear that $f \in AA_u(\mathbb{X})$ if and only if $f^b \in AA(L^\infty(0,1;\mathbb{X}))$. Thus, $AA_u(\mathbb{X})$ can be considered as $AS^\infty(\mathbb{X})$.

Example 3.9 ([34]). Let $x = (x_n)_{n\in\mathbb{Z}} \in l^\infty(\mathbb{X})$ be an almost automorphic sequence and let $\varepsilon_0 \in (0, \frac{1}{2})$. Let $f(t) = x_n$ if $t \in (n-\varepsilon_0, n+\varepsilon_0)$ and $f(t) = 0$, otherwise. Then $f \in AS^p(\mathbb{X})$ for $p \ge 1$ but $f \notin AA(\mathbb{X})$, as f is discontinuous.

Theorem 3.10 ([34]). *The following statements are equivalent:*

(i) $f \in AS^p(\mathbb{X})$;
(ii) $f^b \in AA_u(L^p(0,1;\mathbb{X}))$;
(iii) *for every sequence (s'_n) of real numbers there exists a subsequence (s_n) such that*
$$g(t) := \lim_{n\to\infty} f(t+s_n) \tag{3.1}$$
exists in the space $L^p_{loc}(\mathbb{R};\mathbb{X})$ and
$$f(t) = \lim_{n\to\infty} g(t-s_n) \tag{3.2}$$
in the sense of $L^p_{loc}(\mathbb{R};\mathbb{X})$.

In view of the above, the following inclusions hold:

$$AP(\mathbb{X}) \subset AA_u(\mathbb{X}) \subset AA(\mathbb{X}) \subset AS^p(\mathbb{X}) \subset BS^p(\mathbb{X}).$$

Definition 3.11. A function $F : \mathbb{R}\times\mathbb{X} \mapsto \mathbb{X}, (t,u) \mapsto F(t,u)$ with $F(\cdot,u) \in L^p_{loc}(\mathbb{R};\mathbb{X})$ for each $u \in \mathbb{X}$, is said to be $\mathbf{S}^p$-almost automorphic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{X}$ if $t \mapsto F(t,u)$ is $\mathbf{S}^p$-almost automorphic for each $u \in \mathbb{X}$, that is, for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $G(\cdot,u) \in L^p_{loc}(\mathbb{R};\mathbb{X})$ such that

$$\Big[\int_t^{t+1} \|F(s_n+s,u)-G(s,u)\|^p ds\Big]^{1/p} \to 0,$$
$$\Big[\int_t^{t+1} \|G(s-s_n,u)-F(s,u)\|^p ds\Big]^{1/p} \to 0$$

as $n\to\infty$ pointwise on $\mathbb{R}$ for each $u \in \mathbb{X}$.

The collection of those $\mathbf{S}^p$-almost automorphic functions $F : \mathbb{R}\times\mathbb{X} \mapsto \mathbb{X}$ will be denoted by $AS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$.

The next composition theorem is a slight generalization of [4, Theorem 2.15].

Theorem 3.12. *Let $F : \mathbb{R}\times\mathbb{Z} \mapsto \mathbb{W}$ be a $\mathbf{S}^p$-almost automoprhic. Suppose that there exists a continuous function $L_F : \mathbb{R} \mapsto (0,\infty)$ satisfying $L_F := \sup_{t\in\mathbb{R}} L_F(t) < \infty$ and such that*

$$\|F(t,u)-F(t,v)\|_{\mathbb{W}} \le L_F(t)\,.\,\|u-v\|_{\mathbb{Z}} \tag{3.3}$$

for all $t\in\mathbb{R}, (u,v) \in \mathbb{Z}\times\mathbb{Z}$.

If $\varphi \in AS^p(\mathbb{Z})$, then $\Gamma : \mathbb{R} \to \mathbb{W}$ defined by $\Gamma(\cdot) := F(\cdot,\varphi(\cdot))$ belongs to $AS^p(\mathbb{W})$.

4. The Phase Space $\mathcal{B}$

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$, which is similar to the one utilized in [25]. More precisely, $\mathcal{B}$ is a vector space of functions mapping $(-\infty,0]$ into $\mathbb{X}$ endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ such that the next assumptions hold.

- (A) If $u : (-\infty,\sigma+a) \mapsto \mathbb{X}$, $a > 0$, $\sigma\in\mathbb{R}$, is continuous on $[\sigma,\sigma+a)$ and $u_\sigma \in \mathcal{B}$, then for every $t \in [\sigma,\sigma+a)$ the following hold:
  - (i) u_t is in $\mathcal{B}$;
  - (ii) $\|u(t)\| \le H\|u_t\|_{\mathcal{B}}$;
  - (iii) $\|u_t\|_{\mathcal{B}} \le K(t-\sigma)\sup\{\|u(s)\| : \sigma \le s \le t\} + M(t-\sigma)\|u_\sigma\|_{\mathcal{B}}$,

  where $H > 0$ is a constant; $K, M : [0,\infty) \mapsto [1,\infty)$, K is continuous, M is locally bounded and H, K, M are independent of $u(\cdot)$.
- (A1) For the function $u(\cdot)$ appearing in (A), its corresponding history $t \to u_t$ is continuous from $[\sigma,\sigma+a)$ into $\mathcal{B}$.
- (B) The space $\mathcal{B}$ is complete.
- (C2) If $(v_n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence in $C((-\infty,0],\mathbb{X})$ given by functions with compact support and $v_n \to \varphi$ in the compact-open topology, then $v \in \mathcal{B}$ and $\|v_n - v\|_{\mathcal{B}} \to 0$ as $n\to\infty$.

In what follows, we let $\mathcal{B}_0 = \{v \in \mathcal{B} : v(0) = 0\}$.

Definition 4.1. Let $S(t) : \mathcal{B} \to \mathcal{B}$ be the C_0-semigroup defined by $S(t)v(\theta) = v(0)$ on $[-t,0]$ and $S(t)v(\theta) = v(t+\theta)$ on $(-\infty,-t]$. The phase space $\mathcal{B}$ is called a fading memory if $\|S(t)v\|_{\mathcal{B}} \to 0$ as $t\to\infty$ for every $v \in \mathcal{B}_0$. Now, $\mathcal{B}$ is called uniform fading memory whenever $\|S(t)\|_{L(\mathcal{B}_0)} \to 0$ as $t\to\infty$.

Remark 4.2. In this paper we suppose $Q > 0$ is such that $\|v\|_{\mathcal{B}} \le Q\sup_{\theta\le 0}\|v(\theta)\|$ for each $v \in \mathcal{B}$ bounded continuous (see [25, Proposition 7.1.1]). Moreover, if $\mathcal{B}$ is a fading memory, we assume that $\max\{K(t), M(t)\} \le Q'$ for $t \ge 0$, (see [25, Proposition 7.1.5]).

Remark 4.3. It is worth mentioning that in [25, p. 190] it is shown that the phase $\mathcal{B}$ is a uniform fading memory space if and only if axiom (**C2**) holds, the function $K(\cdot)$ is then bounded and $\lim_{t\to\infty} M(t) = 0$.

Example 4.4 (The phase space $\mathbf{C_r} \times \mathbf{L^p}(\rho, \mathbb{X})$)**.** Let $r \geq 0$, $1 \leq p < \infty$ and let $\rho : (-\infty, -r] \mapsto \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions [25, (g-5)-(g-6)]. Basically, this means that ρ is locally integrable and there exists a nonnegative locally bounded function γ on $(-\infty, 0]$ such that $\rho(\xi+\theta) \leq \gamma(\xi)\rho(\theta)$ for all $\xi \leq 0$ with $\theta \in (-\infty, -r)\setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a subset whose Lebesgue measure is zero. The space $\mathcal{B} = C_r \times L^p(\rho, \mathbb{X})$ consists of all classes of functions $\varphi : (-\infty, 0] \mapsto \mathbb{X}$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable, and $\rho\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(\rho, \mathbb{X})$ is defined as follows:

$$\|\varphi\|_{\mathcal{B}} := \sup\{\|\varphi(\theta)\| : -r \leq \theta \leq 0\} + \Big(\int_{-\infty}^{-r} \rho(\theta)\|\varphi(\theta)\|^p d\theta\Big)^{1/p}.$$

The space $\mathcal{B} = C_r \times L^p(\rho, \mathbb{X})$ satisfies axioms (A), (A-1), and (B). Moreover, when $r = 0$ and $p = 2$, one can then take $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + \big(\int_{-t}^0 \rho(\theta)\, d\theta\big)^{1/2}$ for $t \geq 0$ (see [25, Theorem 1.3.8] for details).

It is worth mentioning that if the conditions [25, (g-5)-(g-7)] hold, then $\mathcal{B}$ is a uniform fading memory.

5. Existence of Almost Auotomorphic Solutions

This section is devoted to the search of an almost automorphic solution to the neutral functional differential equation (1.2).

Definition 5.1. A continuous function $u : [\sigma, \sigma + a) \to \mathbb{X}$ for $a > 0$ is said to be a mild solution to the neutral system (1.2) on $[\sigma, \sigma + a)$ whenever the function $s \to AT(s)f(s, u_s)$ is integrable on $[\sigma, t)$ for every $\sigma < t < \sigma + a$, and

$$\begin{aligned} u(t) = T(t-\sigma)(\varphi(0) + f(\sigma, \varphi)) - f(t, u_t) - \int_\sigma^t AT(t-s)f(s, u_s)ds \\ + \int_\sigma^t T(t-s)g(s, u_s)ds, \quad t \in [\sigma, \sigma + a). \end{aligned}$$

Let $p > 1$ and let $q \geq 1$ such that $1/p + 1/q = 1$. Motivated by Definition 5.1, in the sequel we introduce the technical tools needed for the proof of our main result. From now on, we let $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ denote a Banach space continuously embedded into $\mathbb{X}$ and require:

(H1) The function $s \to AT(t-s)$ defined from $(-\infty, t)$ into $L(\mathbb{Y}, \mathbb{X})$ is strongly measurable and there exist a non-increasing function $H : [0, \infty) \to [0, \infty)$ and $\gamma > 0$ with $s \mapsto e^{-\gamma s}H(s) \in L^1[0, \infty) \cap L^q[0, \infty)$ such that

$$\|AT(s)\|_{L(\mathbb{Y},\mathbb{X})} \leq e^{-\gamma s}H(s), \quad s > 0.$$

(H2) The functions $f, g \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X}) \cap C(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, f is $\mathbb{Y}$-valued, $f : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{Y}$ is continuous and there are a constant $L_f \in (0, 1)$ and a continuous function $L_g : \mathbb{R} \to (0, \infty)$ satisfying $L_g := \sup_{t\in\mathbb{R}} L_g(t) < \infty$ and such that

$$\begin{aligned} \|f(t, y_1) - f(t, y_2)\|_{\mathbb{Y}} \leq L_f\|y_1 - y_2\|, \quad t \in \mathbb{R},\ y_1, y_2 \in \mathbb{X}, \\ \|g(t, y_1) - g(t, y_2)\| \leq L_g(t)\|y_1 - y_2\|, \quad t \in \mathbb{R},\ y_1, y_2 \in \mathbb{X}. \end{aligned}$$

Remark 5.2. Note that the assumption on f and (H1) are linked to the integrability of the function $s \to AT(t-s)f(s,u_s)$ over $[0,t)$. Observe for instance, that except trivial cases, the operator function $s \to AT(s)$ is not integrable over $[0,a]$. If we assume that $AT(\cdot) \in L^1([0,t])$, then from the relation

$$T(t)x - x = A\int_0^t T(s)ds = \int_0^t AT(s)ds$$

it follows that the semigroup is uniformly continuous and as consequence that A is a bounded linear operator on $\mathbb{X}$, which is not interesting, especially for applications. On the other hand, if we assume that (H1) is valid, then from the Bochner's criterion for integrable functions and the estimate

$$\|AT(t-s)f(s,u_s)\| \le H(s)e^{-\gamma(t-s)}\|f(s,u_s)\|_{\mathbb{Y}},$$

it follows that the function $s \mapsto AT(t-s)f(s,u_s)$ is integrable over $(-\infty,t)$ for each $t > 0$.

Lemma 5.3 ([7])**.** *Let $u \in AA_u(\mathbb{X}) \subset AS^p(\mathbb{X})$. Then the function $t \to u_t$ belongs to $AA_u(\mathcal{B}) \subset AS^p(\mathcal{B})$.*

Proof. For a given sequence $(s'_n)_{n\in\mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n\in\mathbb{N}}$ of $(s'_n)_{n\in\mathbb{N}}$ and a function $v \in BC(\mathbb{R},\mathbb{X})$ such that $u(s+s_n) \to v(s)$ uniformly on compact subsets of $\mathbb{R}$. Since $\mathcal{B}$ satisfies axiom $\mathbf{C}_2$, from [25, Proposition 7.1.1], we infer that $u_{s+s_n} \to v_s$ in $\mathcal{B}$ for each $s \in \mathbb{R}$. Let $\Omega \subset \mathbb{R}$ be an arbitrary compact and let $L > 0$ such that $\Omega \subset [-L,L]$. For $\varepsilon > 0$, fix $N_{\varepsilon,L} \in \mathbb{N}$ such that

$$\begin{gathered}\|u(s+s_n) - v(s)\| \le \varepsilon, \quad s \in [-L,L],\\ \|u_{-L+s_n} - v_{-L}\| \le \varepsilon,\end{gathered}$$

whenever $n \ge N_{\epsilon,L}$.

In view of the above, for $t \in \Omega$ and $n \ge N_{\varepsilon,L}$ we get

$$\begin{aligned}&\|u_{t+s_n} - v_t\|_{\mathcal{B}}\\ &\le M(L+t)\|u_{-L+s_n} - v_{-L}\|_{\mathcal{B}} + K(L+t)\sup_{\theta\in[-L,L]}\|u(\theta+s_n) - v(\theta)\|\\ &\le 2Q'\varepsilon,\end{aligned}$$

where Q' is the constant appearing in Remark 4.2.

In view of the above, u_{t+s_n} converges to v_t uniformly on Ω. Similarly, one can prove that v_{t-s_n} converges to u_t uniformly on Ω. Thus, the function $s \mapsto u_s$ belongs to $AA_c(\mathcal{B})$. $\square$

Lemma 5.4. *Under assumption* (H1)*, define the function Φ, for $u \in AS^p(\mathbb{Y})$, by*

$$\Phi(t) := \int_{-\infty}^t AT(t-s)u(s)ds$$

for each $t \in \mathbb{R}$ and suppose

$$h_q^{\gamma,H} := \sum_{n=1}^{\infty}\Big[\int_{n-1}^n e^{-q\gamma r}H^q(r)dr\Big]^{1/q} < \infty.$$

Then $\Phi \in AA(\mathbb{X})$.

Remark 5.5. Note that there are several functions H for which the assumption "$h_q^{\gamma,H} < \infty$" appearing in Lemma 5.4 is achieved. For instance when $H_0(s) = e^{-\beta s}$ for all $\beta > 0$, then $h_q^{\gamma,H_0} < \infty$.

Proof of Lemma 5.4. Define for all $n = 1, 2, \dots,$ the sequence of integral operators

$$\Phi_n(t) := \int_{n-1}^{n} AT(s)u(t-s)ds$$

for each $t \in \mathbb{R}$. Now letting $r = t - s$, it follows that

$$\Phi_n(t) = \int_{t-n}^{t-n+1} AT(t-r)u(r)dr \quad \text{for all } t \in \mathbb{R}.$$

From the Bochner's criterion on integrable functions and the estimate

$$\begin{aligned} \|AT(t-r)u(r)\| &\leq \|AT(t-r)\|_{L(\mathbb{Y},\mathbb{X})}\|u(r)\|_{\mathbb{Y}} \\ &\leq e^{-\gamma(t-r)}H(t-r)\|u(r)\|_{\mathbb{Y}} \end{aligned} \tag{5.1}$$

it follows that the function $s \mapsto AT(t-r)u(r)$ is integrable over $(-\infty, t)$ for each $t \in \mathbb{R}$, by assumption (H1).

Using the Hölder's inequality, it follows that

$$\begin{aligned} \|\Phi_n(t)\| &\leq \int_{t-n}^{t-n+1} e^{-\gamma(t-r)}H(t-r)\|u(r)\|_{\mathbb{Y}}dr \\ &\leq \Big(\int_{t-n}^{t-n+1} e^{-q\gamma(t-r)}H^q(t-r)dr\Big)^{1/q}\Big(\int_{t-n}^{t-n+1} \|u(r)\|_{\mathbb{Y}}^p dr\Big)^{1/p} \\ &\leq \Big(\int_{t-n}^{t-n+1} e^{-q\gamma(t-r)}H^q(t-r)dr\Big)^{1/p} \|u\|_{\mathbf{S}^p} \\ &= \Big(\int_{n-1}^{n} e^{-q\gamma s}H^q(s)ds\Big)^{1/q} \|u\|_{\mathbf{S}^p}. \end{aligned}$$

Using the assumption $h_q^{\gamma,H} < \infty$, we then deduce from the well-known Weirstrass theorem that the series $\sum_{n=1}^{\infty} \Phi_n(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$\Phi(t) = \sum_{n=1}^{\infty} \Phi_n(t),$$

$\Phi \in C(\mathbb{R}, \mathbb{Y})$, and

$$\|\Phi(t)\| \leq \sum_{n=1}^{\infty} \|\Phi_n(t)\| \leq h_q^{\gamma,H}\ \|u\|_{\mathbf{S}^p} \quad \text{for each } t \in \mathbb{R}.$$

The next step consists of showing that $\Phi_n \in AA(\mathbb{X})$. Indeed, let $(s_m)_{m\in\mathbb{N}}$ be a sequence of real numbers. Since $u \in AS^p(\mathbb{Y})$, there exists a subsequence $(s_{m_k})_{k\in\mathbb{N}}$ of $(s_m)_{m\in\mathbb{N}}$ and a function $v \in AS^p(\mathbb{Y})$ such that

$$\Big[\int_t^{t+1} \|u(s_{m_k} + \sigma) - v(\sigma)\|_{\mathbb{Y}}^p d\sigma\Big]^{1/p} \to 0 \quad \text{as } k \to \infty.$$

Define

$$\Psi_n(t) = \int_{n-1}^{n} AT(\xi)v(t-\xi)d\xi.$$

Then using the Hölder's inequality we get

$$\begin{aligned}\|\Phi_n(t+s_{m_k})-\Psi_n(t)\| &= \Big\| \int_{n-1}^{n} AT(\xi)[u(t+s_{m_k}-\xi)-v(t-\xi)]d\xi \Big\| \\ &\le \int_{n-1}^{n} e^{-\gamma\xi} H(\xi)\|u(t+s_{m_k}-\xi)-v(t-\xi)\|_{\mathbb{Y}} d\xi \\ &\le g_q^{\gamma,H}\Big[\int_{n-1}^{n} \|u(t+s_{m_k}-\xi)-v(t-\xi)\|_{\mathbb{Y}}^p d\xi\Big]^{1/p}\end{aligned}$$

where $g_q^{\gamma,H} = \sup_n \big[\int_{n-1}^n e^{-q\gamma s} H^q(s)ds\big]^{1/q} < \infty$, as $h_q^{\gamma,H} < \infty$. Obviously,

$$\|\Phi_n(t+s_{m_k})-\Psi_n(t)\| \to 0 \quad \text{as } k\to\infty.$$

Similarly, we can prove that

$$\|\Psi_n(t+s_{m_k})-\Phi_n(t)\| \to 0 \quad \text{as } k\to\infty.$$

Therefore each $\Phi_n \in AA(\mathbb{X})$ for each n and hence their uniform limit $\Phi \in AA(\mathbb{X})$, by using [30, Theorem 2.1.10]. □

Lemma 5.6. *If $u \in AS^p(\mathbb{X})$ and if Ξ is the function defined by*

$$\Xi(t) := \int_{-\infty}^{t} T(t-s)u(s)ds$$

for each $t\in\mathbb{R}$, then $\Xi \in AA(\mathbb{X})$.

Proof. Define the sequence of operators

$$\Xi_n(t) = \int_{n-1}^{n} T(s)u(t-s)ds \quad \text{for each } t\in\mathbb{R}.$$

Letting $r = t-s$ one obtains

$$\Xi_n(t) = \int_{t-n}^{t-n+1} T(t-r)u(r)dr \quad \text{for each } t\in\mathbb{R}.$$

From the asymptotic stability of $T(t)$, it follows that the function $s \mapsto T(t-r)u(r)$ is integrable over $(-\infty,t)$ for each $t\in\mathbb{R}$. Furthermore, using the Hölder's inequality, it follows that

$$\begin{aligned}\|\Xi_n(t)\| &\le M\int_{t-n}^{t-n+1} e^{-\delta(t-r)}\|u(r)\|dr \\ &\le M\Big(\int_{t-n}^{t-n+1} e^{-q\delta(t-r)}dr\Big)^{1/q}\Big(\int_{t-n}^{t-n+1}\|u(r)\|^p dr\Big)^{1/p} \\ &\le \Big(\int_{n-1}^{n} e^{-q\delta s}ds\Big)^{1/q}\|u\|_{\mathbf{S}^p} \\ &\le \Big(e^{-\delta n}\ M\sqrt[q]{\frac{1+e^{q\delta}}{q\delta}}\Big)\|u\|_{\mathbf{S}^p}.\end{aligned}$$

Now since $M\sqrt[q]{\frac{1+e^{q\delta}}{q\delta}}\sum_{n=1}^{\infty} e^{-\delta n} < \infty$, we deduce from the well-known Weirstrass theorem that the series $\sum_{n=1}^{\infty}\Xi_n(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$\Xi(t) = \sum_{n=1}^{\infty}\Xi_n(t),$$

$\Xi \in C(\mathbb{R}, \mathbb{Y})$, and

$$\|\Xi(t)\| \le \sum_{n=1}^{\infty} \|\Xi_n(t)\| \le k_q^{\delta,M} \|u\|_{\mathbf{S}^p},$$

where $k_q^{\delta,M} > 0$ is a constant, which depends on the parameters q, δ, and M only.

The next step consists of showing that $\Xi_n \in AA(\mathbb{X})$. Indeed, let $(s_m)_{m\in\mathbb{N}}$ be a sequence of real numbers. Since $u \in AS^p(\mathbb{X})$, there exists a subsequence $(s_{m_k})_{k\in\mathbb{N}}$ of $(s_m)_{m\in\mathbb{N}}$ and a function $v \in AS^p(\mathbb{X})$ such that

$$\Big[\int_t^{t+1} \|u(s_{m_k} + \sigma) - v(\sigma)\|^p d\sigma\Big]^{1/p} \to 0 \quad \text{as } k \to \infty.$$

Define

$$\Omega_n(t) = \int_{n-1}^{n} T(\xi) v(t-\xi) d\xi.$$

Then using the Hölder's inequality we get

$$\begin{aligned}
\|\Xi_n(t + s_{m_k}) - \Omega_n(t)\| &= \| \int_{n-1}^{n} T(\xi)[u(t + s_{m_k} - \xi) - v(t-\xi)] d\xi \| \\
&\le M \int_{n-1}^{n} e^{-\delta\xi} \|u(t + s_{m_k} - \xi) - v(t-\xi)\| d\xi \\
&\le m_q^{\gamma,M} \Big[\int_{n-1}^{n} \|u(t + s_{m_k} - \xi) - v(t-\xi)\|^p d\xi\Big]^{1/p}
\end{aligned}$$

where $m_q^{\delta,M} = M \sqrt[q]{\frac{1+e^{q\delta}}{q\delta}}$. Obviously,

$$\|\Xi_n(t + s_{m_k}) - \Omega_n(t)\| \to 0 \quad \text{as } k \to \infty.$$

Similarly, we can prove that

$$\|\Omega_n(t + s_{m_k}) - \Xi_n(t)\| \to 0 \quad \text{as } k \to \infty.$$

Therefore, each $\Xi_n \in AA(\mathbb{X})$ for each n and hence their uniform limit $\Xi(t) \in AA(\mathbb{X})$, by using [30, Theorem 2.1.10]. $\square$

Definition 5.7. A function $u \in AA(\mathbb{X})$ is a mild solution to the neutral system (1.2) provided that the function $s \to AT(t-s)f(s, u_s)$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$ and

$$u(t) = -f(t, u_t) - \int_{-\infty}^{t} AT(t-s)f(s, u_s)ds + \int_{-\infty}^{t} T(t-s)g(s, u_s)ds,$$

for each $t \in \mathbb{R}$.

Theorem 5.8. *Under previous assumptions and if* (H1)–(H2) *hold, then there exist a unique almost automorphic solution to* (1.2) *whenever*

$$C = \Big(L_f + L_f \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\gamma(t-s)} H(t-s)ds + M \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\delta(t-s)} L_g(s)ds\Big) Q < 1,$$

where Q is the constant appearing in Remark 4.2.

Proof. In $AS^p(\mathbb{X})$, define the operator $\Gamma : AS^p(\mathbb{X}) \to C(\mathbb{R}, \mathbb{X})$ by setting

$$\Gamma u(t) := -f(t, u_t) - \int_{-\infty}^{t} AT(t-s)f(s, u_s)ds + \int_{-\infty}^{t} T(t-s)g(s, u_s)ds,$$

for each $t \in \mathbb{R}$.

From previous assumptions one can easily see that Γu is well-defined and continuous. Moreover, from Lemmas 5.3, 5.4, and 5.6 we infer that Γ maps $AS^p(\mathbb{X})$ into $AA(\mathbb{X})$. In particular, Γ maps $AA(\mathbb{X}) \subset AS^p(\mathbb{X})$ into $AA(\mathbb{X})$. Next, we prove that Γ is a strict contraction on $AA(\mathbb{X})$. Indeed, if Q is the constant appearing in Remark 4.2, for $u, v \in AA(\mathbb{X})$, we get

$$\begin{aligned}
\|\Gamma u(t) - \Gamma v(t)\| &\le L_f \|u_t - v_t\|_{\mathcal{B}} + L_f \int_{-\infty}^{t} e^{-\gamma(t-s)} H(t-s) \|u_s - v_s\|_{\mathcal{B}} ds \\
&\quad + M \int_{-\infty}^{t} e^{-\delta(t-s)} L_g(t) \|u_s - v_s\|_{\mathcal{B}} ds \\
&\le L_f \Big(1 + \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\gamma(t-s)} H(t-s) ds\Big) Q \|u - v\|_\infty \\
&\quad + \Big(M \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\delta(t-s)} L_g(s) ds\Big) Q \|u - v\|_\infty \\
&\le C \|u - v\|_\infty .
\end{aligned}$$

The assertion is now a consequence of the classical Banach fixed-point principle. $\square$

6. Examples

In this section we provide with an example to illustrate our main result. We study the existence of almost automorphic solutions to a nonautonomous integrodifferential equation which was considered in Diagana et al. [11] in the pseudo almost periodic case. Consider

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big[\varphi(t,x) + \int_{-\infty}^{t} \int_{0}^{\pi} b(t-s,\eta,x)\varphi(s,\eta) d\eta ds\Big] \\
&= \frac{\partial^2}{\partial x^2}\varphi(t,x) + V\varphi(t,x) + \int_{-\infty}^{t} a_1(t-s)\varphi(s,x) ds + a_2(t,x),
\end{aligned} \tag{6.1}$$

$$\varphi(t,0) = \varphi(t,\pi) = 0, \tag{6.2}$$

for $t \in \mathbb{R}$ and $x \in I = [0,\pi]$.

It is worth mentioning that systems of the type (6.1)-(6.2) arise in control systems described by abstract retarded functional differential equations with feedback control governed by proportional integro-differential law [17].

The existence and qualitative properties of the solutions to (6.1)-(6.2) was recently described in [17, 19] for the existence and regularity of mild solutions, [18] for the existence of periodic solutions, [20] for the existence of almost periodic and asymptotically almost periodic solutions, [11] for pseudo almost periodic solutions, and [7] for the existence of almost automorphic solutions. For similar works we refer the reader to Hernández [22] and Diagana et al. [9, 10].

To establish the existence of almost automorphic solutions to Eqns. (6.1)-(6.2), we need to introduce the required technical tools.

Let $\mathbb{X} = L^2[0,\pi]$ and $\mathcal{B} = C_0 \times L^2(\rho,\mathbb{X})$ (see Example 4.4). Define the linear operator A by

$$\begin{gathered}
D(A) := \{\varphi \in L^2[0,\pi] : \varphi'' \in L^2[0,\pi],\ \varphi(0) = \varphi(\pi) = 0\}, \\
A\varphi = \varphi'' + V\varphi \quad \text{for all } \varphi \in D(A),
\end{gathered}$$

where V is a constant satisfying $V < 1$.

The operator A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on $L^2[0,\pi]$ satisfying

$$\|T(s)\| \leq e^{-(1-V)s} \quad \text{for every } s \geq 0.$$

For the rest of the paper, we assume that the following conditions hold:

(i) The functions $b(\cdot), \frac{\partial^i}{\partial \zeta^i}, b(\tau,\eta,\zeta)$, for $i=1,2$, are Lebesgue measurable, $b(\tau,\eta,\pi)=0$, $b(\tau,\eta,0)=0$ for every (τ,η), and

$$L_f := \max\Big\{\int_0^\pi \int_{-\infty}^0 \int_0^\pi \Big(\frac{\partial^i}{\partial\zeta^i} b(\tau,\eta,\zeta)\Big)^2 d\eta d\tau d\zeta : i=0,1,2\Big\} < \infty.$$

(ii) The functions a_1, a_2, b are continuous, $\mathbf{S}^p$-almost automorphic and

$$L_g = \Big(\int_{-\infty}^0 \frac{a_1^2(-\theta)}{\rho(\theta)} d\theta\Big)^{1/2} < \infty.$$

Additionally, we define the operators $f, g : \mathcal{B} \to L^2[0,\pi]$ by setting

$$f(t,\psi)(x) := \int_{-\infty}^0 \int_0^\pi b(s,\eta,x)\psi(s,\eta) d\eta ds, \tag{6.3}$$

$$g(t,\psi)(x) := \int_{-\infty}^0 a_1(s)\psi(s,x) ds + a_2(t,x), \tag{6.4}$$

which enable us to transform the system (6.1)-(6.2) into an equation of the form (1.2). Obviously, f, g are continuous. Moreover, using a straightforward estimation, which can be obtained with the help of both (i) and (ii), it is then easy to see that f has values in $\mathbb{Y} = (D(A), \|\cdot\|_1)$, where the norm $\|\cdot\|_1$ defined by: $\|\varphi\|_1 = \|A\varphi\|$ for each $\varphi \in D(A)$). Furthermore, f a $\mathbb{Y}$-valued bounded linear operator with $\|f\|_{L(\mathcal{B},\mathbb{Y})} \leq L_f$. Note also that g is Lipschitz with respect to the second variable ψ whose Lipschitz constant is L_g.

The next result is a direct consequence of Theorem 5.8.

Theorem 6.1. *Under the previous assumptions, the system* (6.1)-(6.2) *has a unique almost automorphic solution whenever*

$$Q\big[L_f\big(1+\frac{1}{1-V}\big) + L_g\big] < 1.$$

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