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MULTIPLE SOLUTIONS FOR A ELLIPTIC SYSTEM IN EXTERIOR DOMAIN

HUIJUAN GU, JIANFU YANG, XIAOHUI YU

Abstract. In this paper, we study the existence of solutions for the nonlinear elliptic system

$$
\begin{gathered}
-\Delta u+u=|u|^{p-1}u+\lambda v \quad \text{in } \Omega,\\
-\Delta v+v=|v|^{p-1}v+\lambda u \quad \text{in } \Omega,\\
u=v=0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

where Ω is a exterior domain in $\mathbb{R}^N$, $N\geq 3$. We show that the system possesses at least one nontrivial positive solution.

1. Introduction

This article concerns the existence of solutions to the semilinear elliptic problem

$$
\begin{gathered}
-\Delta u+u=|u|^{p-1}u+\lambda v \quad \text{in } \Omega,\\
-\Delta v+v=|v|^{p-1}v+\lambda u \quad \text{in } \Omega,\\
u=v=0 \quad \text{on } \partial\Omega,
\end{gathered}
\tag{1.1}
$$

where $\Omega\subset\mathbb{R}^N, N\geq 3$, is an exterior domain, $0<\lambda<1$ is a real parameter, $\partial\Omega\neq\emptyset$ and $1<p<\frac{N+2}{N-2}$. In general, in a unbounded domain Ω, the inclusion of $H_0^1(\Omega)\hookrightarrow L^p(\Omega), 2\leq p<\frac{2N}{N-2}$, is not compact, the (PS) condition in critical point theory does not satisfy for related functionals. In some special cases, for instance, if $\Omega=\mathbb{R}^N$, $H_r^1(\Omega)$ is compactly embedded in $L^p(\Omega), 2\leq p<\frac{2N}{N-2}$. Using the fact, it was proved in [4] that the problem

$$
-\Delta u+u=|u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N \tag{1.2}
$$

possesses a positive solution and infinitely many solutions respectively. The general case was considered in [10]; i.e., problem

$$
\begin{gathered}
-\Delta u+a(x)u=b(x)|u|^{p-1}u \quad \text{in } \mathbb{R}^N,\\
u=0 \quad \text{on } \partial\Omega.
\end{gathered}
\tag{1.3}
$$

Suppose $a(x)\geq 0, b(x)\geq 0$ and $\lim_{|x|\to\infty}a(x)=\bar{a}, \lim_{|x|\to\infty}b(x)=\bar{b}$, let c_Ω be the mountain pass level of problem (1.3) and c_∞ be the mountain pass level of the

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limiting problem

$$
\begin{gathered}
-\Delta u+\bar{a}u=\bar{b}|u|^{p-1}u \quad \text{in } \mathbb{R}^N, \\
u\in H^1(\mathbb{R}^N).
\end{gathered} \tag{1.4}
$$

It was showed in [10] that the $(PS)_c$ condition holds for the associated functional of (1.3) provided that $c\in(0,c_\infty)$. However, for problems defined in an exterior domain, it was proved in [2] that $c_\Omega=c_\infty$. One then has to look for solutions with higher energy. Using barycenter function lifting critical values up, a solution of (1.2) with the critical value belonging in $(c_\infty,2c_\infty)$ was found in [2]. The uniqueness of positive solution, up to a translation, of problem (1.4) and the behavior of the solution at infinity play crucial roles in insuring that there are no solutions with energy in between c_∞ and $2c_\infty$.

In this paper, we are interested in finding solutions of problem (1.1). The limiting problem of (1.1) is

$$
\begin{gathered}
-\Delta u+u=|u|^{p-1}u+\lambda v \quad \text{in } \mathbb{R}^N, \\
-\Delta v+v=|v|^{p-1}v+\lambda u \quad \text{in } \mathbb{R}^N.
\end{gathered} \tag{1.5}
$$

In a recent paper [1], Ambrosetti, Cerami and Ruiz showed that solutions of problem (1.5) bifurcating from the semi-trivial solutions if λ is sufficiently small. We will show that ground state solutions of problem (1.5) are obstacles preventing the global compactness of the associated functional of problem (1.1), and furthermore, problem (1.1) has no ground state solutions. So we have to find solutions at higher energy levels. It is not known whether problem (1.5) has unique positive solution or not. This brings difficulties in finding solutions. Fortunately, it was showed in [1] that ground state levels of (1.5) are isolated if λ is sufficiently small or $\lambda<1$ and sufficiently close to 1.

Our main result is the following.

Theorem 1.1. *There exist $\delta>0$ and a constant $\bar\rho=\bar\rho(\lambda)$ such that if $\lambda\in(0,\delta)$ and*

$$
\mathbb{R}^N\setminus\Omega\subset B_{\bar\rho}(x_0)=\{x\in\mathbb{R}^N:|x-x_0|\le\bar\rho\},
$$

problem (1.1) *has at least three pairs of nontrivial solutions.*

Theorem 1.1 will be proved by finding critical points of the corresponding functional of problem (1.1)

$$
\begin{aligned}
I(u,v)=&\frac12\int_\Omega|\nabla u|^2+u^2\,dx+\frac12\int_\Omega|\nabla v|^2+v^2\,dx \\
&-\frac{1}{p+1}\int_\Omega|u|^{p+1}+|v|^{p+1}\,dx-\lambda\int_\Omega uv\,dx,
\end{aligned} \tag{1.6}
$$

where $(u,v)\in E=H_0^1(\Omega)\times H_0^1(\Omega)$. In section 2, we show that ground state solutions are exponentially decaying at infinity and that problem (1.1) has no ground state solution. In final section, we prove Theorem 1.1.

2. Preliminaries

It was proved in [1] that problem (1.5) has a ground state solution (u_λ,v_λ) for $0<\lambda<1$, which is positive and radially symmetric.

Lemma 2.1. *There exist $\delta = \delta(\lambda) > 0$ and $C > 0$ such that*
$$|D^\alpha u_\lambda(x)| \le Ce^{-\delta|x|}, \quad |D^\alpha v_\lambda(x)| \le Ce^{-\delta|x|} \quad \forall x \in \mathbb{R}^N \tag{2.1}$$
for $|\alpha| \le 2$.

Proof. Let $w_\lambda = u_\lambda + v_\lambda$, then w_λ satisfies
$$-\Delta w_\lambda + w_\lambda = (u_\lambda^p + v_\lambda^p) + \lambda w_\lambda, \quad \text{in} \mathbb{R}^N. \tag{2.2}$$
Since $w = w(r)$ is radially symmetric, let $\phi(r) = r^{\frac{N-1}{2}} w_\lambda$, then ϕ satisfies
$$\phi_{rr} = [q(r) + \frac{b}{r^2}]\phi \tag{2.3}$$
with $q(r) = \frac{(1-\lambda)w_\lambda - (u_\lambda^p + v_\lambda^p)}{w_\lambda}$ and $b = \frac{(N-1)(N-3)}{4}$. Since u_λ and v_λ are radially symmetric, $u_\lambda(r), v_\lambda(r) \to 0$ as $|x| \to \infty$. There is $r_0 > 0$ such that $q(r) \ge \frac{1-\lambda}{2}$ if $r \ge r_0$. Set $\psi = \phi^2$, then ψ satisfies
$$\frac{1}{2}\psi_{rr} = \phi_r^2 + (q(r) + \frac{b}{r^2})\psi, \tag{2.4}$$
this implies that $\psi_{rr} \ge (1-\lambda)\psi$ for $r \ge r_0$. Let $z = e^{-\sqrt{1-\lambda}r}[\psi_r + \sqrt{1-\lambda}\psi]$, we have
$$z_r = e^{-\sqrt{1-\lambda}r}[\psi_{rr} - (1-\lambda)\psi] \ge 0 \tag{2.5}$$
for $r \ge r_0$. So z is nondecreasing on $(r_0, +\infty)$. If there exists $r_1 > r_0$ such that $z(r_1) > 0$, then $z(r) \ge z(r_1) > 0$ for $r \ge r_1$, that is
$$\psi_r + \sqrt{1-\lambda}\psi \ge (z(r_1))e^{\sqrt{1-\lambda}r}, \tag{2.6}$$
implying that $\psi_r + \sqrt{1-\lambda}\psi$ is not integrable, a contradiction to the fact that both ψ and ψ_r are integrable. Hence, there holds
$$(e^{\sqrt{1-\lambda}r}\psi)_r = e^{\sqrt{1-\lambda}r}\psi_r + \sqrt{1-\lambda}e^{\sqrt{1-\lambda}r}\psi = e^{2\sqrt{1-\lambda}r}z \le 0 \tag{2.7}$$
for $r \ge r_0$. This implies
$$\psi(r) \le Ce^{-\sqrt{1-\lambda}r}; \tag{2.8}$$
i.e.,
$$\phi(r) \le Ce^{-\frac{\sqrt{1-\lambda}}{2}r}. \tag{2.9}$$
By the definition of ϕ, w and the fact that $u_\lambda, v_\lambda > 0$ we have
$$u_\lambda, v_\lambda \le Cr^{-\frac{N-1}{2}}e^{-\frac{\sqrt{1-\lambda}}{2}r}. \tag{2.10}$$
This proves (2.1) with $\alpha = 0$. Next we estimate the derivatives of u_λ, v_λ. Since
$$(r^{N-1}(u_\lambda)_r)_r = -r^{N-1}[-u_\lambda + u_\lambda^p + \lambda v_\lambda], \tag{2.11}$$
we have
$$\begin{aligned}\int_s^R |(r^{N-1}(u_\lambda)_r)_r|\,dr &= \int_s^R r^{N-1}[-u_\lambda + u_\lambda^p + \lambda v_\lambda]\,dr \\ &\le C\int_s^\infty r^{\frac{N-1}{2}}e^{-\frac{\sqrt{1-\lambda}}{2}r}\,dr \\ &\le Ce^{-\frac{\sqrt{1-\lambda}}{4}s},\end{aligned} \tag{2.12}$$

this means that $r^{N-1}u_r$ has a limit as $r \to \infty$ and this limit can only be 0 by (2.12). Integrating (2.11) on (r, ∞) we get

$$-r^{N-1}(u_\lambda)_r \le Ce^{-\frac{\sqrt{1-\lambda}}{4}r}. \tag{2.13}$$

Similarly, $-r^{N-1}(v_\lambda)_r \le Ce^{-\frac{\sqrt{1-\lambda}}{4}r}$. Finally the exponential decay of $(u_\lambda)_{rr}$ and $(v_\lambda)_{rr}$ follows from equation (1.5). This completes the proof. $\square$

Now we consider the variational problem

$$m_\lambda = \inf_{(u,v)\in\mathcal{N}} I(u,v), \tag{2.14}$$

where

$$\mathcal{N} = \{(u,v) \in E \setminus \{(0,0)\} : \langle I'(u,v),(u,v)\rangle = 0\} \tag{2.15}$$

is the Nehari manifold related to I. Minimizers of m_λ are ground state solutions of (1.1). By a ground state solution of (1.1) we mean a nontrivial solution of (1.1) with the least energy among all nontrivial solutions of (1.1). Correspondingly, for the limiting problem (1.5), the associated functional

$$\begin{aligned} I_\infty(u,v) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + u^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + v^2\,dx \\ &\quad - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} + |v|^{p+1}\,dx - \lambda\int_{\mathbb{R}^N} uv\,dx \end{aligned} \tag{2.16}$$

is well defined in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. We define

$$m_\infty^\lambda = \inf_{(u,v)\in\mathcal{N}_\infty} I_\infty(u,v), \tag{2.17}$$

where

$$\mathcal{N}_\infty = \{(u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \setminus \{(0,0)\} : \langle I'_\infty(u,v),(u,v)\rangle = 0\} \tag{2.18}$$

is the Nehari manifold for I_∞.

Lemma 2.2. *Problem* (1.1) *has no ground state solution.*

Proof. First we show that $m_\lambda = m_\infty^\lambda$. The fact $H_0^1(\Omega) \subset H^1(\mathbb{R}^N)$ implies $m_\lambda \ge m_\infty^\lambda$. Let $\bar\xi$ be a cutoff function such that $0 \le \bar\xi(t) \le 1$, $\bar\xi(t) = 0$ for $t \le 1$, $\bar\xi(t) = 1$ for $t \ge 2$ and $|\bar\xi'(t)| \le 2$. Set $\xi(x) = \bar\xi(\frac{|x|}{\rho})$, where ρ is the smallest positive number such that $\mathbb{R}^N \setminus \Omega \subset B_\rho(0)$. Consider the sequence $\{(\phi_n,\psi_n)\} \subset E$ defined by

$$(\phi_n,\psi_n) = (\xi(x)u_\lambda(x-y_n), \xi(x)v_\lambda(x-y_n)), \tag{2.19}$$

where $\{y_n\} \subset \Omega$ is a sequence of points such that $|y_n| \to \infty$. We may verify that there exists a sequence $\{t_n\} \in \mathbb{R}^+$ such that $t_n(\xi(x)u_\lambda(x-y_n), \xi(x)v_\lambda(x-y_n)) \in \mathcal{N}$. In fact, we may choose t_n so that

$$t_n^{p-1} = \frac{\int_\Omega |\nabla\phi_n|^2 + \phi_n^2 + |\nabla\psi_n|^2 + \psi_n^2 - \lambda\phi_n\psi_n\,dx}{\int_\Omega |\phi_n|^{p+1} + |\psi_n|^{p+1}\,dx}. \tag{2.20}$$

Hence, for $2 \le q < \frac{2N}{N-2}$,

$$\|\phi_n(x) - u_\lambda(x-y_n)\|_{L^q}^q \le 2\int_{B_\rho} |u_\lambda(x-y_n)|^q\,dx \to 0,$$

$$\|\psi_n(x) - v_\lambda(x-y_n)\|_{L^q}^q \le 2\int_{B_\rho} |v_\lambda(x-y_n)|^q\,dx \to 0,$$

$$\|\nabla\phi_n(x) - \nabla u_\lambda(x-y_n)\|_{L^2}^2 \le C\int_{B_\rho} |\nabla u_\lambda(x-y_n)|^2\,dx \to 0,$$

$$\|\nabla\psi_n(x) - \nabla v_\lambda(x-y_n)\|_{L^2}^2 \le C\int_{B_\rho} |\nabla v_\lambda(x-y_n)|^2\,dx \to 0$$

and

$$\int_{\mathbb{R}^N} \phi(x)\psi(x) - u_\lambda(x-y_n)v_\lambda(x-y_n)\,dx \to 0$$

as $n\to\infty$. It follows that $t_n \to 1$ as $n\to\infty$ since $(u_\lambda, v_\lambda) \in \mathcal{N}$. By the definition of m_λ, we have

$$m_\lambda \le I(t_n(\phi_n,\psi_n)) = m_\infty^\lambda + o(1) \tag{2.21}$$

as $n\to\infty$, which implies $m_\lambda = m_\infty^\lambda$.

Suppose now that m_λ is achieved by $(\bar u, \bar v)$. Extending $(\bar u,\bar v)$ to $\mathbb{R}^N$ by setting $(\bar u,\bar v) = (0,0)$ outside Ω, we see that $(\bar u,\bar v)$ is a minimizer of m_∞. Since we may assume that $\bar u \ge 0, \bar v \ge 0$, we obtain a contradiction by the strong maximum principle. This completes the proof. $\square$

3. Proof of Theorem 1.1

Problem (1.1) is setting in a unbounded, in general, (PS) condition does not hold for I. In spirit of [2, Lemma 3.1] and [1, Lemma 4.1], we have the following global compact result.

Lemma 3.1. *Let $\{(u_n, v_n)\} \subset E$ be a sequence such that $I(u_n,v_n) \to c$ and $I'(u_n,v_n) \to 0$ as $n\to\infty$. Then there are a number $K \in \mathbb{N}$, K sequences of points $\{y_n^j\}$ such that $|y_n^j| \to \infty$ as $n\to\infty$, $1\le j\le K$, $K+1$ sequences of functions $(u_n^j, v_n^j) \subset H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N), 0\le j\le K$ such that up to a subsequence,*

(i) *$u_n(x) = u_n^0(x) + \sum_{j=1}^K u_n^j(x-y_n^j), v_n(x) = v_n^0(x) + \sum_{j=1}^K v_n^j(x-y_n^j)$.*

(ii) *$u_n^0(x) \to u^0(x), v_n^0(x)\to v^0(x)$ as $n\to\infty$ strongly in $H_0^1(\Omega)$.*

(iii) *$u_n^j(x) \to u^j(x), v_n^j(x) \to v^j(x)$ as $n\to\infty$ strongly in $H^1(\mathbb{R}^N)$, where $1\le j\le K$.*

(iv) *(u^0,v^0) is a solution of (1.1) and (u^j,v^j) is a solution of (1.5) for $1\le j\le K$. Moreover, when $n\to\infty$*

$$\|u_n\|^2 \to \|u^0\|^2 + \sum_{j=1}^K \|u^j\|^2, \|v_n\|^2 \to \|v^0\|^2 + \sum_{j=1}^K \|v^j\|^2, \tag{3.1}$$

$$I(u_n,v_n) \to I(u_0,v_0) + \sum_{j=1}^K I_\infty(u^j,v^j). \tag{3.2}$$

Proof. We sketch the proof for reader's convenience. We may verify that (u_n,v_n) is bounded. Suppose that $u_n \rightharpoonup u^0, v_n \rightharpoonup v^0$ in $H_0^1(\Omega)$ and $u_n \to u^0, v_n\to v^0$ a.e in

Ω. Then, (u^0, v^0) solves (1.1). If $(u_n, v_n) \to (u^0, v^0)$, then we are done. Otherwise, let

$$z_n^1(x) = \begin{cases} u_n - u^0(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \qquad w_n^1(x) = \begin{cases} v_n - v^0(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

then

$$\|u_n\|^2 = \|u^0\|^2 + \|z_n^1\|^2 + o(1), \quad \|v_n\|^2 = \|v^0\|^2 + \|w_n^1\|^2 + o(1).$$

By Brezis-Lieb's Lemma [9], we deduce

$$\|u_n\|_{L^{p+1}}^{p+1} = \|u^0\|_{L^{p+1}}^{p+1} + \|z_n^1\|_{L^{p+1}}^{p+1} + o(1), \quad \|v_n\|_{L^{p+1}}^{p+1} = \|v^0\|_{L^{p+1}}^{p+1} + \|w_n^1\|_{L^{p+1}}^{p+1} + o(1).$$

Thus,

$$I(z_n^1, w_n^1) = I(u_n, v_n) - I(u^0, v^0) + o(1),$$
$$I'(z_n^1, w_n^1) = I'(u_n, v_n) - I'(u^0, v^0) + o(1) = o(1).$$

Suppose now that $(z_n^1, w_n^1) \not\to (0,0)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, we define

$$\delta_z = \limsup_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |z_n^1|^{p+1}\,dx, \quad \delta_w = \limsup_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |w_n^1|^{p+1}\,dx.$$

We may verify that $\delta_z + \delta_w > 0$ since $(z_n^1, w_n^1) \not\to (0,0)$. We may suppose $\delta_z > 0$, then there is a sequence $\{y_n^1\} \subset \mathbb{R}^N$ such that $\int_{B_1(y_n^1)} |z_n^1|^{p+1} \ge \frac{\delta_z}{2}$. Let us consider now the sequence $(z_n^1(x + y_n^1), w_n^1(x + y_n^1))$. We assume that $(z_n^1(x + y_n^1), w_n^1(x + y_n^1)) \rightharpoonup (u^1, v^1)$, then (u^1, v^1) is a nontrivial solution of (1.5). By the fact that $z_n^1 \rightharpoonup 0$ we see that $|y_n^1| \to \infty$. Set

$$z_n^2(x) = z_n^1(x) - u^1(x - y_n^1), w_n^2(x) = w_n^1(x) - v^1(x - y_n^1),$$

and repeat above procedure, it will stop at finite steps. The lemma follows. $\square$

By [1, Lemmas 7.8 and 7.9], there exist $0 < \lambda_1 \le \lambda_2 < 1$ such that m_∞ is an isolated critical value of I_∞ for $\lambda \in (0, \lambda_1) \cup (\lambda_2, 1)$. Denote $m_0 = \inf\{\alpha > m_\infty^\lambda :$ α is a critical value of $I_\infty\}$ and $\bar{m} = \min\{m_0, 2m_\lambda\}$, then we have the following result.

Corollary 3.2. *The functional I satisfies the $(PS)_c$ condition for $c \in (m_\lambda, \bar{m})$.*

Proof. Let $\{(u_n, v_n)\} \subset E$ be such that $I(u_n, v_n) \to c$ and $I'(u_n, v_n) \to 0$ with $c \in (m_\lambda, \bar{m})$. Since $\{(u_n, v_n)\}$ is bounded, we may assume that $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$. By Lemma 3.1,

$$(u_n, v_n) - \sum_{j=1}^{K} (u^j(x - y_n^j), v^j(x - y_n^j)) \to (u, v),$$

where (u, v) is a solution of (1.1) and (u^j, v^j) is a solution of (1.5), $\{y_n^j\}(1 \le j \le K)$ are K sequences of points in $\mathbb{R}^N$. Moreover,

$$I(u_n, v_n) = I(u, v) + \sum_{j=1}^{K} I_\infty(u^j, v^j) + o(1).$$

To prove that $u_n \to u, v_n \to v$ in $H_0^1(\Omega)$, we need only to show $K = 0$. Since $c < 2m_\lambda$, we have $K < 2$. We claim that $K = 0$. Indeed, if $K = 1$, we have either $(u, v) \ne (0,0)$ or $(u, v) = (0,0)$. If $(u, v) \ne (0,0)$, then $I(u_n, v_n) \ge 2m_\lambda + o(1)$, which contradicts to the fact that $c < 2m_\lambda$; if $(u,v) = (0,0)$, then $I_\infty(u^1, v^1) = c$, which contradicts the definition of $\bar{m}$. The assertion follows. □

Now we introduce a function $\Phi_\rho : \mathbb{R}^N \to H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ defined by

$$\Phi_\rho(y) = t_\rho(\xi(\frac{|x|}{\rho})u_\lambda(x-y), \xi(\frac{|x|}{\rho})v_\lambda(x-y)), \tag{3.3}$$

where (u_λ, v_λ) is a ground state solution of equation (1.5), t_ρ is chosen such that $t_\rho(\xi(\frac{|x|}{\rho})u_\lambda(x-y), \xi(\frac{|x|}{\rho})v_\lambda(x-y)) \in \mathcal{N}$.

Lemma 3.3. (i) *$\Phi_\rho(y)$ is continuous in y for every $\rho > 0$.*
(ii) *$\Phi_\rho(y) \to (u_\lambda(x-y), v_\lambda(x-y))$ strongly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ uniformly in y as $\rho \to 0$.*
(iii) *$I(\Phi_\rho(y)) \to m_\lambda$ as $|y| \to \infty$ uniformly for every ρ.*

Proof. (i) is obvious since $\Phi_\rho(\cdot)$ is the composition of continuous functions. (iii) follows from the same argument of Lemma 2.2. It remains to prove (ii). We claim that

$$\|\xi(\frac{|x|}{\rho})u_\lambda(x-y)\|_{L^{p+1}} \to \|u_\lambda(x)\|_{L^{p+1}}, \quad \|\xi(\frac{|x|}{\rho})v_\lambda(x-y)\|_{L^{p+1}} \to \|v_\lambda(x)\|_{L^{p+1}},$$
$$\|\xi(\frac{|x|}{\rho})u_\lambda(x-y)\| \to \|u_\lambda(x)\|, \quad \|\xi(\frac{|x|}{\rho})v_\lambda(x-y)\| \to \|u_\lambda(x)\|,$$
$$\int_{\mathbb{R}^N} \xi(\frac{|x|}{\rho})u_\lambda(x-y)\xi(\frac{|x|}{\rho})v_\lambda(x-y)\,dx \to \int_{\mathbb{R}^N} u_\lambda(x-y)v_\lambda(x-y)\,dx.$$

Indeed,

$$\begin{aligned}\|\xi(\frac{|x|}{\rho})u_\lambda(x-y) - u_\lambda(x-y)\|_{L^{p+1}}^{p+1} &\le 2^{p+1}\int_{B_{2\rho}} |u_\lambda(x-y)|^{p+1}\,dx \\ &\le 2^{p+1}|\max u_\lambda|^{p+1} \operatorname{meas}(B_{2\rho}) \to 0.\end{aligned} \tag{3.4}$$

Similarly, we have

$$\|\xi(\frac{|x|}{\rho})v_\lambda(x-y) - v_\lambda(x-y)\|_{L^{p+1}} \to 0 \tag{3.5}$$

and

$$\begin{aligned}&\|\xi(\frac{|x|}{\rho})u_\lambda(x-y) - u_\lambda(x-y)\|^2 \\ &= \int_{\mathbb{R}^N} |\frac{1}{\rho}\nabla\xi(\frac{|x|}{\rho})u_\lambda(x-y) - \xi(\frac{|x|}{\rho})\nabla u_\lambda(x-y) - \nabla u_\lambda(x-y)|^2\,dx + k_2 \operatorname{meas}(B_{2\rho}) \\ &\le 2\int_{\rho\le|x|\le 2\rho} |\nabla\xi(\frac{|x|}{\rho})u_\lambda(x-y)|^2\,dx \\ &\quad + 2\int_{\rho\le|x|\le 2\rho} |\xi(\frac{|x|}{\rho})\nabla u_\lambda(x-y) - \nabla u_\lambda(x-y)|^2\,dx + k_2 \operatorname{meas}(B_{2\rho}) \\ &\le k_3\rho^{N-2} + k_4\rho^N \to 0\end{aligned} \tag{3.6}$$

as well as

$$\begin{aligned}
&|\int_{\mathbb{R}^N} \xi(\frac{|x|}{\rho})u_\lambda(x-y)\xi(\frac{|x|}{\rho})v_\lambda(x-y) - u_\lambda(x-y)v_\lambda(x-y)\,dx| \\
&\le \int_{\mathbb{R}^N} |\xi(\frac{|x|}{\rho})u_\lambda(x-y)\xi(\frac{|x|}{\rho})v_\lambda(x-y) - u_\lambda(x-y)v_\lambda(x-y)|\,dx \\
&\le k_5\rho^N \to 0.
\end{aligned} \tag{3.7}$$

This proves the claim. The definition of t_ρ and the claim yield that $t_\rho \to 1$ as $\rho \to 0$. This together with equation (3.6) imply (ii). □

Since $I^\lambda_\infty(u_\lambda(x-y), v_\lambda(x-y)) = m_\lambda$, the following result is a consequence of (ii) in Lemma 3.3.

Corollary 3.4. *For $0 < \lambda < \lambda_1$ or $\lambda_2 < \lambda < 1$, there exists a $\bar\rho = \bar\rho(\lambda)$ such that for $\rho \le \bar\rho$, there holds*

$$\sup_{y\in\mathbb{R}^N} I(\Phi_\rho(y)) < \bar m. \tag{3.8}$$

From now on we will suppose that Ω is fixed in such a way that $\rho < \bar\rho$. Now we define a function $\beta : H^1(\mathbb{R}^N) \to R^N$ as follows

$$\beta(u) = \int_{\mathbb{R}^N} u(x)\chi(|x|)x\,dx,$$

where

$$\chi(t) = \begin{cases} 1 & \text{if } 0 \le t \le R, \\ R/tt & \text{if } t > R \end{cases}$$

and R is chosen such that $\mathbb{R}^N \setminus \Omega \subset B_R$.

Let $\mathcal{B}_0 := \{(u,v) \in \mathcal{N} : \beta(u) = 0 \text{ or } \beta(v) = 0\}$ and let $c_0 = \inf_{(u,v)\in\mathcal{B}_0} I(u,v)$.

Lemma 3.5. *There holds $c_0 > m_\lambda$, and there is an $R_0 > \rho$ such that*

(a) *if $|y| \ge R_0$, then $I(\Phi_\rho(y)) \in (m_\lambda, \frac{m_\lambda + c_0}{2})$;*
(b) *if $|y| = R_0$, then $\langle \beta\circ P_1\circ\Phi_\rho(y), y\rangle > 0$ or $\langle \beta\circ P_2\circ\Phi_\rho(y), y\rangle > 0$, where $P_i(u,v)$ is the projection of (u,v) on the ith coordinate.*

Proof. It is obvious that $c_0 \ge m_\lambda$. Now suppose that $c_0 = m_\lambda$, then there exists a sequence $(u_n, v_n) \in \mathcal{N}$ with $\beta(u_n) = 0$ or $\beta(v_n) = 0$ such that $I(u_n, v_n) \to m_\lambda$. We may assume that $\beta(u_n) = 0$. By Lemma 3.1, $u_n(x) = u_0(x-y_n) + o(1), v_n = v_0(x-y_n) + o(1)$ with $|y_n| \to \infty$. Denote $(\mathbb{R}^N)^+_n = \{x \in \mathbb{R}^N : \langle x, y_n\rangle > 0\}, (\mathbb{R}^N)^-_n = \mathbb{R}^N \setminus (\mathbb{R}^N)^+_n$, then for n large we have $B_{\hat r}(y_n) := \{x : |x-y_n| < \hat r\} \subset (\mathbb{R}^N)^+_n$ for some fixed $\hat r > 0$ and $u_0(x-y_n) \ge \delta_0 > 0, v_0(x-y_n) \ge \delta_0 > 0$ for $x \in B_{\hat r}(y_n)$ and some $\delta_0 > 0$. Lemma 2.1 implies

$$u_0(x-y_n) \le \frac{K}{e^{\delta|x-y_n|}|x-y_n|^{\frac{N-1}{2}}}, \quad v_0(x-y_n) \le \frac{K}{e^{\delta|x-y_n|}|x-y_n|^{\frac{N-1}{2}}}$$

for $x \in B_{\hat{r}}(y_n)$. So we have

$$\begin{aligned}
&\langle \beta(u_0(x-y_n)), y_n\rangle \\
&= \int_{(\mathbb{R}^N)_n^+} u_0(x-y_n)\chi(|x|)\langle x, y_n\rangle\,dx + \int_{(\mathbb{R}^N)_n^-} u_0(x-y_n)\chi(|x|)\langle x, y_n\rangle\,dx \\
&\geq \int_{B_{\hat{r}}(y_n)} \delta_0\chi(|x|)\langle x, y_n\rangle\,dx - \int_{(\mathbb{R}^N)_n^-} \frac{KR|y_n|}{e^{\delta|x-y_n|}|x-y_n|^{\frac{N-1}{2}}}\,dx \\
&\geq \alpha - o(\frac{1}{|y_n|}) > 0,
\end{aligned} \tag{3.9}$$

where $\alpha > 0$ is a constant. Since β is continuous, we have $\beta(u_n) \neq 0$. This contradicts to the fact that $\beta(u_n) = 0$.

(a) can be proved in the same way as the proof of Lemma 2.2 and (b) can be proved as (3.9). □

Now let us consider the set Σ given by

$$\Sigma := \{t_\rho \Phi_\rho(y) : |y| \leq R_0\},$$

where t_ρ is chosen such that $t_\rho\Phi_\rho(y) \in \mathcal{N}$. We define

$$H = \{h \in C(\mathcal{N}, \mathcal{N}) : h(u,v) = (u,v) \text{ for } \forall (u,v) \in \mathcal{N} \text{ with } I(u,v) \leq \frac{c_0+m}{2}\}$$

and $\Gamma = \{A \subset \mathcal{N}, A = h(\Sigma)\}$.

Lemma 3.6. *If $A \in \Gamma$, then $A \cap \mathcal{B}_0 \neq \emptyset$.*

Proof. The proof of the lemma is equivalent to prove that for $\forall h \in H$, there is $\bar{y} \in \mathbb{R}^N$ with $|\bar{y}| \leq R_0$ such that $\beta \circ h \circ P_1 \circ \Phi_\rho(y) = 0$ or $\beta \circ h \circ P_2 \circ \Phi_\rho(y) = 0$. By Lemma 3.5, we have $\langle \beta \circ P_1 \circ \Phi_\rho(y), y\rangle > 0$ or $\langle \beta \circ P_2 \circ \Phi_\rho(y), y\rangle > 0$ for $|y| = R_0$. Assume that $\langle \beta \circ P_1 \circ \Phi_\rho(y), y\rangle > 0$ without of loss generality and define

$$\begin{gathered}
f(y) = \beta \circ h \circ P_1 \circ \Phi_\rho(y), \\
F(t,y) = tf(y) + (1-t)id.
\end{gathered}$$

(b) of Lemma 3.5 implies $0 \notin F(t, \partial B_{R_0})$, hence, $deg(F, B_{R_0}, 0) = \deg(id, B_{R_0}, 0) = 1$. This yields that there exists $\bar{y} \in B_{R_0}$ such that $\beta \circ h \circ P_1 \circ \Phi_\rho(y) = 0$.

If $\langle \beta \circ P_2 \circ \Phi_\rho(y), y\rangle > 0$, we may show that there exists a $\bar{y} \in B_{R_0}$ such that $\beta \circ h \circ P_2 \circ \Phi_\rho(y) = 0$ in the same way. This proves the Lemma. □

Proof of Theorem 1.1. For $\lambda \in (0, \delta)$, obviously, problem (1.1) has two pair of positive solutions $(U_{1-\lambda}, U_{1-\lambda})$ and $(\pm U_{1+\lambda}, \mp U_{1+\lambda})$, where $U_{1-\lambda}$ and $U_{1+\lambda}$ are positive solutions of

$$\begin{gathered}
-\Delta u + (1-\lambda)u = |u|^{p-1}u \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{3.10}$$

and

$$\begin{gathered}
-\Delta u + (1+\lambda)u = |u|^{p-1}u \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{3.11}$$

respectively. It is proved in [2] that problem (3.10) and problem (3.11) have nontrivial solutions. Define

$$c_\lambda = \inf_{A \in \Gamma} \sup_{(u,v) \in A} I(u,v), \tag{3.12}$$

then we have $\bar{m} > c_\lambda \geq c_0 > m_\lambda$ since $id \in H$ and $A \cap \mathcal{B}_0 \neq \emptyset$. A standard deformation argument implies that c_λ is a critical value of I. Now, we claim that $c_\lambda < I(U_{1-\lambda}, U_{1-\lambda}) < I(\pm U_{1+\lambda}, \mp U_{1+\lambda})$ for $\bar{\rho}$ small sufficiently. Then the critical points corresponding to c_λ are different from trivial solutions $(U_{1-\lambda}, U_{1-\lambda})$ and $(\pm U_{1+\lambda}, \mp U_{1+\lambda})$. In fact, we note that $U_0(x) = (1-\lambda)^{-\frac{1}{p-1}} U_{1-\lambda}(\frac{x}{\sqrt{1-\lambda}})$ is a solution of

$$
\begin{gathered}
-\Delta u + u = |u|^{p-1}u \quad \text{in } \Omega_{\sqrt{1-\lambda}}, \\
u \in H_0^1(\Omega_{\sqrt{1-\lambda}})
\end{gathered}
\tag{3.13}
$$

and extend $U_{1-\lambda}$ to $\mathbb{R}^N$ by setting $U_{1-\lambda} = 0$ outside Ω. Denote by $J_\lambda(u)$ the functional corresponding to problem (3.10) and let (u_λ, v_λ) be a ground state solution of (1.5), since $(U_0, 0) \in \mathcal{N}$ is not a ground state solution of (1.5), for λ small, we have

$$
\begin{aligned}
I_\infty(u_\lambda, v_\lambda) &\leq I_\infty(U_0, 0) = J_0(U_0) \\
&< 2(1-\lambda)^{\frac{p+1}{p-1} - \frac{N}{2}} J_0(U_0) \\
&= 2J_\lambda(U_{1-\lambda}) \\
&= I(U_{1-\lambda}, U_{1-\lambda}).
\end{aligned}
$$

By (ii) of Lemma 3.3, $c_\lambda \to I_\infty(u_\lambda, v_\lambda)$ as $\rho \to 0$, for fixed $\lambda_0 > 0$ small, there exists $\bar{\rho} = \bar{\rho}(\lambda_0)$ such that $c_{\lambda_0} < I(U_{1-\lambda_0}, U_{1-\lambda_0})$. Noticing that c_λ and $I(U_{1-\lambda}, U_{1-\lambda})$ are continuous in λ, applying compact argument to $[0, \lambda_0]$, we may find $\bar{\rho}_1 \leq \bar{\rho}$ such that for $\lambda \in [0, \lambda_0]$ we have $c_\lambda < I(U_{1-\lambda}, U_{1-\lambda})$ if $0 < \rho \leq \bar{\rho}_1$. On the other hand, by [1] we have $I(U_{1-\lambda}, U_{1-\lambda}) < I(\pm U_{1+\lambda}, \mp U_{1+\lambda})$, the proof is completed. □

Remark 3.7. For λ close to 1, we may also obtain a critical value c_λ of I as the proof of Theorem 1.1. However, c_λ and $I(U_{1-\lambda}, U_{1-\lambda})$ are close to each other if $\rho \to 0$. Hence, we may not obtain nontrivial solutions in this way.

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