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REMARK ON THE SPATIAL REGULARITY FOR THE NAVIER-STOKES EQUATIONS

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ABSTRACT. Let u be a Leray-Hopf weak solution to the Navier-Stokes equations. We will show that the set of possible singular points of the vector field resulting from integrating the velocity u with respect to time has Hausdorff dimension zero.

1. INTRODUCTION

Let us consider the viscous incompressible fluid flow moving within a region Ω of the three dimensional space $\mathbb{R}^3$, which can be described by the Navier-Stokes equations

$$\begin{gathered} \partial_t u - \nu\Delta u + (u\cdot\nabla)u = -\nabla\pi, \quad \text{in } \Omega\times(0,\infty), \\ \nabla\cdot u = 0, \quad \text{in } \Omega\times(0,\infty) \end{gathered} \tag{1.1}$$

with the homogeneous boundary condition

$$u = 0 \quad \text{on } \partial\Omega\times(0,\infty) \tag{1.2}$$

when the boundary is not empty, and the initial condition

$$u(x,0) = u_0(x) \quad \text{in } \Omega. \tag{1.3}$$

Here $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ denotes the unknown velocity vector field, and $\pi = \pi(x,t)$ denotes the scalar pressure; ν is the viscosity; and $u_0(x)$ is the initial velocity vector field. For simplicity, the viscosity ν is normalized to 1.

For the initial value problem and the initial boundary value problem to the Navier-Stokes equations, the existence of a class of global weak solutions was shown by Leray and Hopf in their pioneering works [8] and [7] a long time ago. Since then, much effort has been made to try to establish uniqueness and regularity of weak solutions. However, these two remarkable questions remain open. It is still not known whether or not a weak solution can develop singularities at finite time, even for sufficiently smooth initial data. A lot of attention has been turned to the study of partial regularity of weak solutions to the Navier-Stokes equations. The first analysis about the possible singular set was done by Leray.

Following Caffarelli, Kohn and Nirenberg [1], a point (x,t) (or t) is called a singular point of a weak solution u to the Navier-Stokes equations if and only if u is not essentially bounded in any neighborhood of (x,t) (or t). Leray [8] showed that

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the singularities, if exist, can occur at most on a set of t with Lebesgue measure zero.

Scheffer [12]-[16] began the development of the analysis about the set of possible singular points, and established various partial regularity results for a class of weak solutions. Scheffer's results showed that the set of possible time singular points of the weak solution has 1/2-dimensional Hausdorff measure zero, and that the set of possible space-time singular points of the weak solution has 5/3-dimensional Hausdorff measure zero. See also [2, 18]. Later, Caffarelli, Kohn and Nirenberg [1] improved Scheffer's results and showed that the set of possible space-time singular points of a class of special weak solutions, named as suitable weak solutions, has one-dimensional Hausdorff measure zero. Note that suitable weak solutions differ essentially from the usual weak solutions in the sense that they should satisfy a generalized energy inequality. A simplified proof of the main results of [1] was presented in [9]; see also [19, 5, 10].

It is well-known that there exists a large time T such that after the time T the weak solution is smooth and the interval $(0, T)$ can be expressed as $\cup_{i\in I}(a_i, b_i)\cup\mathcal{T}$, where the set I is at most a countable set, (a_i, b_i) with $i \in I$ are disjoint open intervals in $(0, T)$, the set $\mathcal{T}$ has 1/2-dimensional Hausdorff measure zero, and u belongs to C^∞ for $(x, t) \in \Omega\times(a, b)$ for each interval (a, b) whose closure is contained in some of the intervals (a_i, b_i) (cf. Fioas and Temam [2], Heywood [6], Leray [8], Scheffer [12], sohr and W. von Wahl [18], Miyakawa and Sohr [11]).

Applying their own local regularity theory, Caffarelli, Kohn and Nirenberg [1] showed that the suitable weak solution is regular in the region $\{(x, t) : |x|^2 t > k\}$ if the initial velocity $u_0 \in L^2(\mathbb{R}^3)$ and $|x|^{1/2}u_0 \in L^2(\mathbb{R}^3)$, or in the region $\{(x, t) : t > 0, |x| > R'\}$ for some $R' > R$ if the initial velocity $u_0 \in L^2(R^3)$ and $\int_{|x|>R} |\nabla u_0|^2 dx < \infty$. Similar results has been obtained by Maremonti [10] in the case of an exterior domain. To the best of our knowledge, up to now, the set of possible singular points is still not fully understood.

In this paper, we try to estimate the Hausdorff dimension of the set of possible spatial singular points of weak solutions in some sense. As is well-known, the set of possible time singular points of a class of weak solutions has 1/2-dimensional Hausdorff measure zero (see [2, 3, 8, 12, 17]). As for the set of possible spatial-time singular points, it is known that the 1-dimensional Hausdorff measure vanishes (cf. [1, 5, 9, 10, 20]). However, as far as we know, no such results are available for the set of possible spatial singular points.

It is difficult to study directly the set of possible spatial singular points of the weak solutions. So we will study the partial regularity of the weak solution by integrating the solution in time. For this purpose, let u be a weak solution to the Navier-Stokes equations (1.1) and define $U(x) = \int_0^T u(x, s)ds$ for some $T > 0$. By the definition of weak solutions, $U(x) = \int_0^T u(x, s)ds$ is well-defined in the sense of Bochner (See below). Then following ideas in [2], we will show that the set of points at any neighborhood of which U is essentially unbounded has Hausdorff dimension zero. This implies the corresponding estimate of the Hausdorff dimension of the set of possible spatial singular points of the weak solution u in some sense.

We conclude this introduction by introducing some function spaces used in this paper. Let $L^p(\Omega)$, $1 \le p \le \infty$, represent the usual Lesbegue space of scalar functions as well as that of vector-valued functions with norm denoted by $\|\cdot\|_p$. Let $C^\infty_{0,\sigma}(\Omega)$ denote the set of all C^∞ vector functions with compact support in Ω

such that $\operatorname{div}\phi = 0$. Let $L^p(0,T;X)$, $1 \le p \le \infty$, be the set of function $f(t)$ defined on $(0,T)$ with values in X such that $\int_0^T \|f(t)\|_X^p dt < \infty$ for a given Banach space X with norm $\|\cdot\|_X$.

2. Main Results

In this article, we only consider four types of the domains: (1) $\mathbb{R}^3$, (2) a bounded domain in $\mathbb{R}^3$, (3) a half-space in $\mathbb{R}^3_+$, and (4) an exterior domain in $\mathbb{R}^3$.

We will consider the Leray-Hopf weak solutions defined as follows:

Definition. A *Leray-Hopf weak solution* of the system (1.1)-(1.3) in $Q_\infty \equiv \Omega \times (0,\infty)$ is a vector field $u : Q_\infty \to \mathbb{R}^3$ such that

$$u \in L^\infty(0,\infty; L^2_\sigma(\Omega)) \quad \text{and } \nabla u \in L^2(0,\infty; L^2(\Omega)), \tag{2.1}$$
e2.1

$$\int_{Q_\infty} \Big(u \cdot \partial_t w + u \otimes u : \nabla w - \nabla u : \nabla w \Big)\, dx\, dt = 0 \tag{2.2}$$
e2.2

for any $w \in C^\infty_{0,\sigma}(Q_\infty)$ and any $t \in [0,\infty)$, u satisfies the energy inequality

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(\tau)\|_2^2 d\tau \le \|u_0\|_2^2, \tag{2.3}$$
e2.3

and u takes the initial value in the sense that

$$\|u(\cdot,t) - u_0(\cdot)\|_2 \to 0 \quad \text{as } t \to 0. \tag{2.4}$$
e2.4

It is well-known now that Leray [8] and Hopf [7] constructed a global Leray-Hopf weak solution . Here we intend to study the spatial partial regularity of the Leray-Hopf weak solution, in some sense. In fact, we are interested in the estimate of the Hausdorff dimension of the set of possible singular points of the vector resulting from integrating the velocity u with respect to time. For simplicity, assume that $u_0 \in C^\infty_{0,\sigma}(\Omega)$. The argument can be applied to general initial data u_0.

First we introduce the following result which was obtained by Giga and Sohr [4].

lem2.1 **Lemma 2.1.** *Let $u_0 \in C^\infty_{0,\sigma}(\Omega)$. Then there exists a weak solution (u,π) such that*

$$u \in L^\infty(0,\infty; L^2(\Omega)), \quad \nabla u \in L^2(0,\infty; L^2(\Omega)), \tag{2.5}$$
e2.5

$$\partial_t u,\ \partial_x^2 u,\ \nabla\pi \in L^p(0,\infty; L^q(\Omega)) \tag{2.6}$$
e2.6

for any $1 < p, q < \infty$ with $1/p + 3/2q = 2$. Also u satisfies the energy inequality

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(\tau)\|_2^2 d\tau \le \|u_0\|_2^2. \tag{2.7}$$
e2.7

So π can be chosen such that

$$\nabla u,\ \pi \in L^p(0,\infty; L^{q^*}(\Omega)), \quad \frac{1}{q^*} = \frac{1}{q} - \frac{1}{3}. \tag{2.8}$$
e2.8

As stated in the introduction, there is a time T such that, after T, a Leray-Hopf weak solution u is smooth. For this $T > 0$, it is easy to see that

$$U(x) := \int_0^T u(x,t)dt, \quad \Pi(x) := \int_0^T \pi(x,t)dt$$

are well-defined in $L^{q^*}(\Omega)$. Then (U,Π) satisfies the equations

$$-\Delta U = f, \quad f = u_0 - u(T) - \int_0^T (u\cdot\nabla u)(t)dt - \nabla\Pi. \tag{2.9}$$
e2.9

Define

$$\Omega_0 =: \{x \in \Omega : U(x) \quad \text{is essentially unbounded in any neighborhood of } x\}.$$

Now our main result can be stated as follows:

thmA **Theorem 2.2.** *Let $u_0 \in C^\infty_{0,\sigma}(\Omega)$. Then Ω_0 has Hausdorff dimension zero.*

To prove our main theorem, we need the following lemma established by Foias and Temam [2].

lem3.1 **Lemma 2.3** ([2, Lemma 4.2])**.** *For $a > 0$ and $f \in L^1(\mathbb{R}^n)$, let $\Lambda_a(f)$ be the set of $x \in \mathbb{R}^n$ such that there exists m_x with*

$$\int_{|y-x|\le 2^{-m}} |f(y)|dy \le 2^{-am} \quad \text{for all } m \ge m_x.$$

Then $\mathbb{R}^n \setminus \Lambda_a(f)$ has Hausdorff dimension less than or equal to a.

Plase see addendum

Proof of Theorem 2.2. We will follow the ideas in [2]. Since $U \in L^{q^*}(\Omega)$,

$$U(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y)dy =: U_0 \quad \text{when } \partial\Omega = \emptyset, \tag{2.10}$$ e3.1

and

$$U(x) = W(x) + \frac{1}{4\pi}\int_{\Omega} \frac{1}{|x-y|} f(y)dy \quad \text{when } \partial\Omega \ne \emptyset \text{ with a harmonic function } W. \tag{2.11}$$ e3.2

It is well-known that the function W is smooth in the interior of Ω and is bounded in the subdomain with positive distance away from the boundary. Extend f to the outside of Ω by zero. So we only consider the partial regularity of U_0. For any $x_0 \in \mathbb{R}^3$, we have

$$\begin{aligned}\frac{1}{r^3}\int_{|x-x_0|\le r} |U_0(y)|dy &\le \frac{1}{4\pi}\frac{1}{r^3}\int_{|x-x_0|\le r}\int_{\mathbb{R}^3} \frac{1}{|x-y|}|f(y)|dydx \\ &\le \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x_0-x|}\cdot\frac{1}{r^3}\int_{|x-y|\le r} |f(y)|dydx \\ &\le \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x_0-x|} f^*(x)dx\end{aligned} \tag{2.12}$$ e3.3

with

$$f^*(x) = \sup_r \frac{1}{r^3}\int_{|x-y|\le r} |f(y)|dy.$$

By (2.6), we know that $f \in L^q(\mathbb{R}^3)$ for any $1 < q < 3/2$. So it follows from the inequality on maximal functions that $f^* \in L^q(\mathbb{R}^3)$. Let

$$M_j = \{x \in \mathbb{R}^3 : \ |x - x_0| \le 2^{-j}\}.$$

For any $x_0 \in \mathbb{R}^3$, we have

$$
\begin{aligned}
&\frac{1}{r^3}\int_{|x-x_0|\le r} |U_0(x)|dx \\
&\le \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x_0-x|} f^*(x)dx \\
&\le \frac{1}{4\pi}\int_{\mathbb{R}^3\setminus M_1} \frac{1}{|x_0-x|} f^*(x)dx + \sum_1^\infty \frac{1}{4\pi}\int_{M_j\setminus M_{j+1}} \frac{1}{|x_0-x|} f^*(x)dx \\
&\le C\|f^*\|_q\Big(\int_{\mathbb{R}^3\setminus M_1} \frac{1}{|x-x_0|^{\frac{q}{q-1}}}dx\Big)^{1-\frac{1}{q}} \\
&\quad + C\sum_1^\infty \Big(\int_{M_j\setminus M_{j+1}} \frac{1}{|x-x_0|^{\frac{q}{q-1}}}dx\Big)^{1-\frac{1}{q}} \Big(\int_{M_j} |f^*(x)|^q dx\Big)^{\frac{1}{q}} \\
&\le C + C\sum_1^\infty 2^{j-3j(1-\frac{1}{q})}\cdot \Big(\int_{M_j} |f^*(x)|^q dx\Big)^{\frac{1}{q}}.
\end{aligned} \tag{2.13}
$$

e3.4

It is obvious that $|f^*|^q \in L^1(\mathbb{R}^3)$. Let $\Lambda_a(f^*)$ be the set of these $x_0 \in R^3$ such that there exists j_{x_0} with

$$
\int_{|x_0-x|\le 2^{-j}} |f^*(x)|^q dx \le 2^{-aj}
$$

for all $j \ge j_{x_0}$. Thus, for any $x_0 \in \Lambda_a(f^*)$, (2.13) gives us

$$
\frac{1}{r^3}\int_{|x-x_0|\le r} |U_0(x)|dx \le C + C\sum_1^\infty 2^{j-3j(1-\frac{1}{q})}\cdot 2^{-\frac{aj}{q}} \le C_1 \tag{2.14}
$$

e3.5

provided that $a > 3-2q$. Note that the constant C_1 is independent of x_0. Let

$$
\overline{U}_0 = \frac{3}{4\pi r^3}\int_{|x-x_0|\le r} U_0(x)dx
$$

denote the average of U_0 in the ball centered at x_0 with radius r. Then, by (2.14), we have

$$
\frac{1}{r^3}\int_{|x-x_0|\le r} \big|U_0(x)-\overline{U}_0\big|dx \le \frac{2}{r^3}\int_{|x-x_0|\le r} \big|U_0(x)\big|dx \le 2C_1
$$

provided that $a > 3-2q$. This implies

$$
\sup_{x_0\in\Lambda_a(f^*),\ r>0} \frac{1}{r^3}\int_{|x-x_0|\le r} \big|U_0(x)-\overline{U}_0\big|dx \le 2C_1 \tag{2.15}
$$

e3.6

provided that $a > 3-2q$.

Then, for any $x_0 \in \Lambda_a(f^*)$ with $a > 3-2q$, (2.15) tells us that

$$
|U_0(x_0)| < \infty.
$$

Therefore,

$$
\Omega_0 = \big\{x \in \mathbb{R}^3 :\ |U_0(x)| = \infty\big\} \subset \mathbb{R}^3 \setminus \Lambda_a(f^*).
$$

Applying Lemma 2.3, we deduce that the Hausdorff dimension of Ω_0 is less or equal to a. Letting $a \to 3-2q$, we deduce that the Hausdorff dimension of Ω_0 does not exceed $3-2q$. Since $q \in (1, 3/2)$ is arbitrary, we deduce that the Hausdorff dimension of Ω_0 is zero. This completes the proof $\square$

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Addendum posted on September 30, 2008

Following a suggestion from the anonymous refeee (to whom the author wants to express his gratitude), the proof of the main theorem is rewritten as follows:

Proof of Theorem 2.2. We will follow the ideas in [2]. Since $U \in L^{q^*}(\Omega)$,

$$U(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y) dy =: U_0 \quad \text{when } \partial\Omega = \emptyset, \tag{2.16}$$

e4.10

and

$$U(x) = W(x) + \frac{1}{4\pi} \int_{\Omega} \frac{1}{|x-y|} f(y) dy \quad \text{when } \partial\Omega \neq \emptyset \text{ with a harmonic function } W. \tag{2.17}$$

e4.11

It is well-known that the function W is smooth in the interior of Ω and is bounded in the subdomain with positive distance away from the boundary. Extend f to the outside of Ω by zero. So we only consider the partial regularity of U_0. For any $x_0 \in \mathbb{R}^3$, we have

$$\begin{aligned}
\frac{1}{r^3} \int_{|x-x_0| \le r} |U_0(y)| dy &\le \frac{1}{4\pi} \frac{1}{r^3} \int_{|x-x_0| \le r} \int_{\mathbb{R}^3} \frac{1}{|x-y|} |f(y)| dy\, dx \\
&\le \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x_0 - x|} \cdot \frac{1}{r^3} \int_{|x-y| \le r} |f(y)| dy\, dx \qquad (2.18) \\
&\le \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x_0 - x|} f^*(x) dx := F(x_0)
\end{aligned}$$

e4.12

with

$$f^*(x) = \sup_r \frac{1}{r^3} \int_{|x-y| \le r} |f(y)| dy.$$

By (2.6), we know that $f \in L^q(\mathbb{R}^3)$ for any $1 < q < 3/2$. So it follows from the inequality on maximal functions that $f^* \in L^q(\mathbb{R}^3)$. Since $U_0(x)$ is continuous in x, from (2.18), we deduce that

$$|U(x_0)| \le F(x_0) \quad \forall x_0 \in \mathbb{R}^3.$$

Let

$$M_j = \{ x \in \mathbb{R}^3 : |x - x_0| \le 2^{-j} \}.$$

For each $x_0 \in \mathbb{R}^3$,

$$\begin{aligned}
F(x_0) &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x_0 - x|} f^*(x) dx \\
&\le \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus M_1} \frac{1}{|x_0 - x|} f^*(x) dx + \sum_1^\infty \frac{1}{4\pi} \int_{M_j \setminus M_{j+1}} \frac{1}{|x_0 - x|} f^*(x) dx \\
&\le C \|f^*\|_q \Big(\int_{\mathbb{R}^3 \setminus M_1} \frac{1}{|x - x_0|^{\frac{q}{q-1}}} dx \Big)^{1 - \frac{1}{q}} \qquad (2.19) \\
&\quad + C \sum_1^\infty \Big(\int_{M_j \setminus M_{j+1}} \frac{1}{|x - x_0|^{\frac{q}{q-1}}} dx \Big)^{1 - \frac{1}{q}} \Big(\int_{M_j} |f^*(x)|^q dx \Big)^{1/q} \\
&\le C + C \sum_1^\infty 2^{j - 3j(1 - \frac{1}{q})} \Big(\int_{M_j} |f^*(x)|^q dx \Big)^{1/q}.
\end{aligned}$$

e4.13

It is obvious that $|f^*|^q \in L^1(\mathbb{R}^3)$. Let $\Lambda_a(f^*)$ be the set of these $x_0 \in \mathbb{R}^3$ such that there exists j_{x_0} with

$$\int_{|x_0 - x| \le 2^{-j}} |f^*(x)|^q dx \le 2^{-aj}$$

for all $j \geq j_{x_0}$. Thus, for each $x_0 \in \Lambda_a(f^*)$, (2.19) gives us

$$\begin{aligned} |F(x_0)| &\leq C + C\sum_{1}^{j_{x_0}-1} 2^{j-3j(1-\frac{1}{q})}\Big(\int_{M_j} |f^*(x)|^q dx\Big)^{\frac{1}{q}} + C\sum_{j_{x_0}}^{\infty} 2^{j-3j(1-\frac{1}{q})}2^{-\frac{aj}{q}} \\ &\leq C_1(x_0) \end{aligned} \tag{2.20}$$

e4.14

provided that $a > 3-2q$. Then, for each $x_0 \in \Lambda_a(f^*)$ with $a > 3-2q$, (2.20) tells us that

$$|U_0(x_0)| \leq F(x_0) \leq C_1(x_0) < \infty.$$

It is obvious that

$$\Omega_0 = \big\{x \in \mathbb{R}^3 :\ |U_0(x)| = \infty\big\} \subset \mathbb{R}^3 \setminus \Lambda_a(f^*).$$

Applying Lemma 2.3, we deduce that the Hausdorff dimension of Ω_0 is less or equal to a. Letting $a \to 3-2q$, we deduce that the Hausdorff dimension of Ω_0 does not exceed $3-2q$. Since $q \in (1, 3/2)$ is arbitrary, we deduce that the Hausdorff dimension of Ω_0 is zero. This completes the proof □

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