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AMBROSETTI-PRODI TYPE RESULTS IN A SYSTEM OF SECOND AND FOURTH-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, by the variational method, we study the existence, nonexistence, and multiplicity of solutions of an Ambrosetti-Prodi type problem for a system of second and fourth order ordinary differential equations.

1. Introduction

Lazer and McKenna [1] presented the following (one-dimensional) mathematical model for the suspension bridge:

$$\begin{gathered} y_{tt} + y_{xxxx} + \delta_1 y_t + k(y-z)^+ = W(x), \quad \text{in } (0,L)\times\mathbb{R}, \\ z_{tt} - z_{xx} + \delta_2 z_t - k(y-z)^+ = h(x,t), \quad \text{in } (0,L)\times\mathbb{R}, \\ y(0,t) = y(L,t) = y_{xx}(0,t) = y_{xx}(L,t) = 0, \quad t\in\mathbb{R}, \\ z(0,t) = z(L,t) = 0, \quad t\in\mathbb{R}. \end{gathered} \tag{1.1}$$

Where the variable z measures the displacement from equilibrium of the cable and the variable y measures the displacement of the road bed. The constant k is spring constant of the ties.

When the motion of the cable is ignored, the coupled system (1.1) can be simplified into a single equation which describes the motion of the road bed of suspension bridge, as follows

$$\begin{gathered} y_{tt} + y_{xxxx} + \delta y_t + k y^+ = W(x,t), \quad \text{in } (0,L)\times\mathbb{R}, \\ y(0,t) = y(L,t) = y_{xx}(0,t) = y_{xx}(L,t) = 0, \quad t\in\mathbb{R}. \end{gathered} \tag{1.2}$$

This Problem have been studied by many authors. In [2, 3, 4], the authors, using degree theory and the variational method, investigated the multiplicity of some symmetrical periodic solutions when $\delta = 0$ and $W(x,t) = 1 + \epsilon h(x,t)$ or $W(x,t) = \alpha\cos x + \beta\cos 2t\cos x\epsilon$. In [5], the similar results for (1.2) are obtained in case of $\delta \neq 0$ and $W(x,t) = h(x,t) = \alpha\cos x + \beta\cos 2t\cos x + \gamma\sin 2t\cos x$. Those results give the conditions impose on the spring constant k which guarantees the existence of multiple periodic solutions, especially the sign-changing periodic

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solutions in the case of $W(x,t)$ is single-sign. It is notable that the functions $\cos x, \cos 2t \cos x, \sin 2t \cos x$ are the eigenfunctions of linear principal operator of (1.2) in some function spaces.

When we consider only the steady state solutions of problem (1.1), we arrive at the system

$$\begin{gathered} y_{xxxx} + k(y-z)^+ = h_1(x), \quad \text{in } (0,\pi), \\ -z_{xx} - k(y-z)^+ = h_2(x), \quad \text{in } (0,\pi), \\ y(0) = y(\pi) = y_{xx}(0) = y_{xx}(\pi) = 0, \\ z(0) = z(\pi) = 0. \end{gathered} \tag{1.3}$$

This problem has little been studied in [12, 13]. In [6, 15], the analogous partial differential systems have been considered when the nonlinearities $k(y-z)^+, -k(y-z)^+$ are replaced by general $f_1(y,z), f_2(y,z)$. And also, in recently, literature [16] studied the system

$$\begin{gathered} y_{xx} + k_1 y^+ + \epsilon z^+ = \sin x, \quad \text{in } (0,\pi), \\ z_{xx} + \epsilon y^+ + k_2 z^+ = \sin x, \quad \text{in } (0,\pi), \\ y(0) = y(\pi) = 0, \\ z(0) = z(\pi) = 0. \end{gathered} \tag{1.4}$$

Where $u^+ = \max\{u, 0\}$, the constant ϵ is small enough such that the matrix

$$\begin{pmatrix} k_1 & \epsilon \\ \epsilon & k_2 \end{pmatrix}$$

is a near-diagonal matrix and the positive numbers k_1, k_2 satisfy

$$m_1^2 < k_1 < (m_1+1)^2, \ m_2^2 < k_2 < (m_2+1)^2 \quad \text{for some } m_1, m_2 \in \mathbf{N}.$$

This is a first work in the direction of extending to systems some of well-known results established on nonlinear equation with an asymmetric nonlinearity. Meanwhile in [16] there are two open questions to be interesting:

Question 1. Can one obtain corresponding results if the second-order differential operator is replaced with a fourth-order differential operator with corresponding boundary conditions?

Question 2. Can one replace the near-diagonal matrix with something more general and use information on the eigenvalues of matrix?

Following the above works and questions, we consider the system

$$\begin{gathered} -u'' = f_1(x,u,v) + t_1 \sin x + h_1(x), \quad \text{in } (0,\pi) \\ v'''' = f_2(x,u,v) + t_2 \sin x + h_2(x), \quad \text{in } (0,\pi) \\ u(0) = u(\pi) = 0, \\ v(0) = v(\pi) = v''(0) = v''(\pi) = 0, \end{gathered} \tag{1.5}$$

where t_1, t_2 are parameters and $(f_1, f_2) : [0,\pi] \times \mathbb{R}^2 \to \mathbb{R}^2$ is asymptotically linear.

On the other hand, the second order elliptic systems as follows

$$\begin{gathered} -\Delta u = f_1(u,v) + t_1 \varphi_1 + h_1(x), \quad \text{in } \Omega, \\ -\Delta v = f_2(u,v) + t_2 \varphi_1 + h_2(x), \quad \text{in } \Omega, \\ u = v = 0, \quad \text{on } \partial\Omega \end{gathered} \tag{1.6}$$

have been widely studied. Here we mention the papers [7, 8, 9, 10] and the references therein. If $(f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is asymptotically linear and the asymptotic matrixes at $-\infty$ and $+\infty$ are

$$\begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix}, \quad \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$$

Under some growth conditions on (f_1, f_2), in those papers, the Ambrosetti-Prodi type results for (1.6) have been given respectively.

We remind that let $g \in C^\alpha(\overline{\Omega} \times \mathbb{R})$ be a given function such that

$$\limsup_{s\to-\infty} \frac{g(x,s)}{s} < \lambda_1 < \liminf_{s\to+\infty} \frac{g(x,s)}{s}$$

uniformly in $x \in \Omega$, where λ_1 is the first eigenvalue of the Laplacian on a bounded domain Ω under the Dirichlet condition and φ_1 is the associated eigenfunction. The Ambrosetti-Prodi type result in a Cartesian version states that for a given $h \in C^\alpha(\overline{\Omega})$ there exists a real number t_0 such that the problem

$$\begin{gathered} -\Delta u = g(x,u) + t\varphi_1 + h, \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega \end{gathered}$$

(i) has no solution if $t > t_0$;
(ii) has at least two solutions if $t < t_0$.

With different variants and formulations this problem has been extensively studied.

Inspired, we consider the Ambrosetti-Prodi type problem for system (1.5). This paper is organized as follows: in Section 2, we prepare the proper variational framework and prove (PS) condition to the Euler-Lagrange functional associated to our problem. In Section 3, we prove the main theorem. Finally, a piecewise linear problem is considered as an example in Section 4.

2. Preliminaries

In this section, we prepare the proper variational frame work for (1.5), that is

$$\begin{gathered} -u'' = f_1(x,u,v) + t_1 \sin x + h_1(x), \quad \text{in } (0,\pi) \\ v'''' = f_2(x,u,v) + t_2 \sin x + h_2(x), \quad \text{in } (0,\pi) \\ u(0) = u(\pi) = 0, \\ v(0) = v(\pi) = v''(0) = v''(\pi) = 0. \end{gathered}$$

Where t_1, t_2 are parameters, $h_1, h_2 \in C[0,\pi]$ are fixed functions with $\int_0^\pi h_1 \sin x = \int_0^\pi h_2 \sin x = 0$.

We shall need some assumptions on the nonlinearities, which are necessary to settle the existence or not of solutions in the case of the Ambrosetti-Prodi type problem and to establish (PS) condition.

Let us order $\mathbb{R}^2$ with the order defined by

$$\xi = (\xi_1, \xi_2) \geq 0 \Longleftrightarrow \xi_1, \xi_2 \geq 0.$$

and denote $W = (u,v)$ and $F(x,W) = (f_1(x,u,v), f_2(x,u,v))$.

We will use the following hypotheses in this article.

(H1) $F = (f_1, f_2) : [0, \pi] \times \mathbb{R}^2 \to \mathbb{R}^2$ is locally Lipschitzian function respect to u, v, and there exists a function $H : [0, \pi] \times \mathbb{R}^2 \to \mathbb{R}$ such that

$$\nabla H(x, u, v) = (\frac{\partial H}{\partial u}, \frac{\partial H}{\partial v}) = (f_1(x, u, v), f_2(x, u, v)).$$

(H2) For $\xi = (\xi_1, \xi_2) > 0$ large enough,

$$F(x, \xi) \geq 0. \tag{2.1}$$

(H3) F satisfies

$$|F(x, \xi)| \leq c(|\xi_1| + |\xi_2| + 1), \quad \forall \xi \in \mathbb{R}^2,\ x \in (0, \pi) \tag{2.2}$$

where $c > 0$ is constant.

(H4) For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $x \in (0, \pi)$ there holds

$$F(x, \xi) \geq \underline{A}\xi - ce, \tag{2.3}$$

for some constant $c > 0$. Where $e = (1, 1)$ and the matrix $\underline{A} = \begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix}$ satisfies

$$\underline{b}, \underline{c} \geq 0, \tag{2.4}$$

$$(\underline{A}\xi, \xi) \leq \underline{\mu}|\xi|^2, \quad \text{for some } 0 < \underline{\mu} < 1. \tag{2.5}$$

(H5) For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $x \in (0, \pi)$ there holds

$$F(x, \xi) \geq \overline{A}\xi - ce, \tag{2.6}$$

for some constant $c > 0$. Where $e = (1, 1)$ and the matrix $\overline{A} = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}$ satisfies

$$\overline{b}, \overline{c} \leq 0, \tag{2.7}$$

$$(\overline{A}\xi, \xi) \geq \overline{\mu}|\xi|^2, \quad \text{for some } \overline{\mu} > 1. \tag{2.8}$$

(If not mentioned, c will always denote a generic positive constant.)

Remark 2.1. With a simple computation it is easy to show that (2.4)-(2.5) and (2.7)-(2.8) imply, respectively,

$$\begin{gathered} (1 - \underline{a})(1 - \underline{d}) - \underline{bc} > 0, \quad \underline{a}, \underline{d} < 1, \\ (\underline{A} - I)^{-1}\xi \leq 0, \quad \forall \xi \in \mathbb{R}^2,\ \xi \geq 0, \end{gathered} \tag{2.9}$$

and

$$\begin{gathered} (1 - \overline{a})(1 - \overline{d}) - \overline{bc} > 0, \quad \overline{a}, \overline{d} > 1, \\ (\overline{A} - I)^{-1}\xi \geq 0, \quad \forall \xi \in \mathbb{R}^2,\ \xi \geq 0, \end{gathered} \tag{2.10}$$

where I is the identity matrix.

Let $X = H_0^1(0, \pi) \times (H_0^1(0, \pi) \cap H^2(0, \pi))$ be Hilbert space with the inner product

$$\langle W, \Psi \rangle = \int_0^\pi (u'\psi_1' + v''\psi_2''), \quad \forall W = (u, v),\ \Psi = (\psi_1, \psi_2) \in X,$$

and the corresponding norm

$$\|W\|_X^2 = \int_0^\pi (u'^2 + v''^2).$$

Consider the second-order ordinary differential eigenvalue problem

$$-u'' = \lambda u, \quad \text{in } (0,\pi),$$
$$u(0) = u(\pi) = 0,$$

and the fourth-order ordinary differential eigenvalue problem

$$v'''' = \lambda v, \quad \text{in } (0,\pi),$$
$$v(0) = v(\pi) = v''(0) = v''(\pi) = 0.$$

It is well known that $\lambda_1 = 1$ and $\varphi_1 = \sin x$ are the positive first eigenvalue and the associated eigenfunction, respectively. Hence, it follows from the Poincare inequality that, for all $W \in X$,

$$\int_0^\pi |W|^2 \le \|W\|_X^2. \tag{2.11}$$

A vector $W \in X$ is a weak solution of (1.5) if, and only if, it is a critical point of the associated Euler-Lagrange functional

$$J(W) = \frac{1}{2}\int_0^\pi (u'^2 + v''^2) - \int_0^\pi H(x,u,v) - \int_0^\pi [(t_1 \sin x + h_1)u + (t_2 \sin x + h_2)v] \tag{2.12}$$

It is standard to show that the functional $J(W)$ is well defined, $J(W) \in C^1(X,\mathbb{R})$ and $X \to \mathbb{R}$; $W \to \int_0^\pi H(x,u,v) + \int_0^\pi [(t_1 \sin x + h_1)u + (t_2 \sin x + h_2)v]$ has compact derivative under the assumptions (H1) and (H3).

Lemma 2.2. *Assume that* (H1)-(H5) *hold. Then J satisfies the (PS) condition.*

Proof. Let $\{W_n = (u_n, v_n)\} \subset X$ be a sequence such that $|J(W_n)| \le c$ and $J'(W_n) \to 0$. This implies

$$\begin{aligned}&\Big|\int_0^\pi (u_n'\psi_1' + v_n''\psi_2'') - \int_0^\pi \Big[(f_1\psi_1 + f_2\psi_2) + (t_1 \sin x + h_1)\psi_1 + (t_2 \sin x + h_2)\psi_2\Big]\Big| \\ &\le \varepsilon_n \|\Psi\|_X\end{aligned} \tag{2.13}$$

for all $\Psi = (\psi_1, \psi_2) \in X$, where $\varepsilon_n \to 0 (n \to \infty)$. Then by the above discussion it suffices to prove that $\{W_n\}$ is bounded.

Step 1: Show the boundedness of $\{W_n^-\}$. Let $W_n^- = (u_n^-, v_n^-)$, $w^- = \max\{0, -w\}$. Since h_1, h_2 are bounded, there exists $M_1, M_2 \ge 0$ such that

$$|t_1 \sin x + h_1| \le M_1, \quad |t_2 \sin x + h_2| \le M_2. \tag{2.14}$$

Moreover, from (2.3) and (2.4), we have

$$f_1(x,u_n,v_n)(-u_n^-) \le \underline{a}(u_n^-)^2 + \underline{b}u_n^- v_n^- + cu_n^-,$$
$$f_2(x,u_n,v_n)(-v_n^-) \le \underline{d}(v_n^-)^2 + \underline{c}u_n^- v_n^- + cv_n^-.$$

Choosing $c > \max\{M_1, M_2\}$ and taking $\psi_1 = -u_n^-, \psi_2 = -v_n^-$ in (2.13), then using the above inequalities and (2.5), we obtain

$$\begin{aligned}\|W_n^-\|_X^2 &\le \int_0^\pi (\underline{A}W_n^-, W_n^-) + \int_0^\pi (cu_n^- - M_1u_n^- + cv_n^- - M_2v_n^-) + c\|W_n^-\|_X \\ &\le \underline{\mu}\int_0^\pi |W_n^-|^2 + d\int_0^\pi (u_n^- + v_n^-) + c\|W_n^-\|_X.\end{aligned}$$

Where $d \geq \max\{c-M_1, c-M_2\}$ is constant. Using Hölder inequality and Poincare inequality, we get

$$\int_0^\pi |u_n^-| \leq c(\int_0^\pi |u_n^-|^2)^{1/2} \leq c(\int_0^\pi |u_n^{-\prime}|^2)^{1/2},$$
$$\int_0^\pi |v_n^-| \leq c(\int_0^\pi |v_n^-|^2)^{1/2} \leq c(\int_0^\pi |v_n^{-\prime\prime}|^2)^{1/2}.$$

Then from these two inequalities and (2.11) we have

$$(1-\underline{\mu})\|W_n^-\|_X^2 \leq c\|W_n^-\|_X,$$

since $0 < \underline{\mu} < 1$, $\|W_n^-\|$ is bounded.

Step 2: Show the boundedness of $\{W_n\}$. Suppose by contradiction that $\{W_n\}$ is unbounded, then there exists a subsequence (still denote $\{W_n\}$) such that $\|W_n\|_X \to \infty$ as $n \to \infty$. Setting $V_n = (x_n, y_n) = W_n/\|W_n\|_X$, then $\|V_n\|_X = 1$ and there exists a subsequence such that

$$V_n \rightharpoonup V_0 = (x_0, y_0), \quad \text{in } X, \tag{2.15}$$
$$V_n \to V_0, \quad \text{in } L^2(0,\pi) \times L^2(0,\pi), \tag{2.16}$$
$$V_n \to V_0, \quad \text{a.e. in } (0,\pi),$$
$$\text{with } |x_n(x)|, |y_n(x)| \leq h(x) \in L^2,\ x \in (0,\pi). \tag{2.17}$$

By step 1 we may assume that $V_n^- \to 0$ in $L^2 \times L^2$ and $V_n^- \to 0$ a.e.in $(0,\pi)$. Clearly, $V_0 \geq 0$. Denote

$$\begin{aligned} G_n(x) &= (g_n^1(x), g_n^2(x)) \\ &= \frac{(f_1(x, W_n(x)) + t_1 \sin x + h_1, f_2(x, W_n(x)) + t_2 \sin x + h_2)}{\|W_n\|_X}. \end{aligned}$$

We claim that

$$G_n \to \gamma = (\gamma_1, \gamma_2) \geq 0 \quad \text{in } L^2 \times L^2. \tag{2.18}$$

In fact, let $A_n = \{x \in (0,\pi); u_n(x) \leq 0 \ \textit{and}\ v_n(x) \leq 0\}$ and let χ_n denotes its characteristic function, then $G_n = \chi_n G_n + (1-\chi_n)G_n$. By (H3), (2.16), (2.17) and using the Lebesgue Dominated Convergence Theorem, we get

$$\chi_n \frac{F(x, W_n)}{\|W_n\|_X} \to 0 \quad \text{in } L^2 \times L^2.$$

Moreover, from (2.14) we have

$$\chi_n \frac{(t_1 \sin x + h_1, t_2 \sin x + h_2)}{\|W_n\|_X} \to 0 \quad \text{in } L^2 \times L^2.$$

Hence $\chi_n G_n \to 0$ in $L^2 \times L^2$. With the same reasoning $(1-\chi_n)G_n \to \gamma' = (\gamma_1', \gamma_2')$ in $L^2 \times L^2$. Therefore, we only need to prove that $\gamma' \geq 0$.

(i) If $u_n(x) \geq 0$ and $v_n(x) \leq 0$, since $\overline{a} > 1$, from (2.6) we have

$$(1-\chi_n)g_n^1(x) + \bar{b}(y_n^-(x)) + \frac{c}{\|W_n\|_X} - (1-\chi_n)\frac{t_1 \sin x + h_1}{\|W_n\|_X} \geq \overline{a}x_n^+(x) \geq 0$$

and from (2.3) and (2.4), we obtain

$$(1-\chi_n)g_n^2(x) + \underline{d}(y_n^-(x)) + \frac{c}{\|W_n\|_X} - (1-\chi_n)\frac{t_2 \sin x + h_2}{\|W_n\|_X} \geq \underline{c}x_n^+(x) \geq 0$$

Since $V_n^- \to 0$ in $L^2 \times L^2$ and

$$(1-\chi_n)g_n^1(x) + \overline{b}(y_n^-(x)) + \frac{c}{\|W_n\|_X} - (1-\chi_n)\frac{t_1 \sin x + h_1}{\|W_n\|_X} \to \gamma_1',$$
$$(1-\chi_n)g_n^2(x) + \underline{d}(y_n^-(x)) + \frac{c}{\|W_n\|_X} - (1-\chi_n)\frac{t_2 \sin x + h_2}{\|W_n\|_X} \to \gamma_2'$$

we get $\gamma' \geq 0$.

(ii) If $u_n(x) \leq 0$ and $v_n(x) \geq 0$, we can handle in the same way to obtain that $\gamma' \geq 0$.

(iii) If $u_n(x) \geq 0$ and $v_n(x) \geq 0$, the assertion $\gamma' \geq 0$ can be inferred from (H2).

Now dividing (2.13) by $\|W_n\|_X$, using (2.15), (2.18) and passing to the limit we obtain

$$\int_0^\pi (x_0'\psi_1' + y_0''\psi_2'') = \int_0^\pi (\gamma_1\psi_1 + \gamma_2\psi_2), \quad \forall \Psi = (\psi_1, \psi_2) \in X. \tag{2.19}$$

From (2.6) we have

$$\frac{(f_1(x, W_n(x)) + t_1 \sin x + h_1, f_2(x, W_n(x)) + t_2 \sin x + h_2)}{\|W_n\|_X} \geq \overline{A}V_n - \frac{ce}{\|W_n\|_X}.$$

Passing to the limit in this inequality we get

$$\gamma \geq \overline{A}V_0. \tag{2.20}$$

Taking $\psi_1 = \sin x, \psi_2 = 0$ and then $\psi_1 = 0$, $\psi_2 = \sin x$ in (2.19) and using (2.20), it is achieved that

$$(\overline{A} - I)\begin{pmatrix} \int_0^\pi x_0 \sin x \\ \int_0^\pi y_0 \sin x \end{pmatrix} \leq 0. \tag{2.21}$$

From Remark 2.1, applying $(\overline{A} - I)^{-1}$ to (2.21) we get $(\int_0^\pi x_0 \sin x, \int_0^\pi y_0 \sin x) \leq 0$. Hence $x_0 = y_0 = 0$ a.e. So, from (2.19), $\int_0^\pi (\gamma, \Psi) = 0$ and taking $\Psi > 0$ we have $\gamma = 0$.

Finally, consider $\psi_1 = x_n, \psi_2 = y_n$ in (2.13). Dividing the resulting expression by $\|W_n\|_X$, and passing to the limit we obtain $1 \leq 0$, that is impossible. $\square$

Lemma 2.3. *Suppose* (H5) *hold. Then*

$$\lim_{s \to +\infty} J(s \sin x, s \sin x) = -\infty. \tag{2.22}$$

Proof. From (2.6) we have

$$H(x, u, v) \geq \frac{\overline{a}}{2}u^2 + \overline{b}uv - cu + H(x, 0, v) \quad \text{as } u \geq 0, \forall v, \tag{2.23}$$
$$H(x, u, v) \geq \frac{\overline{d}}{2}v^2 + \overline{c}uv - cv + H(x, u, 0) \quad \text{as } v \geq 0, \forall u. \tag{2.24}$$

Adding (2.23), (2.24) and using them again,

$$\begin{aligned} 2H(x, u, v) &\geq \frac{\overline{a}}{2}u^2 + (\overline{b} + \overline{c})uv + \frac{\overline{d}}{2}v^2 - cu - cv + H(x, 0, v) + H(x, u, 0) \\ &\geq \overline{a}u^2 + (\overline{b} + \overline{c})uv + \overline{d}v^2 - 2cu - 2cv + 2H(x, 0, 0) \\ &\geq \overline{a}u^2 + (\overline{b} + \overline{c})uv + \overline{d}v^2 - 2cu - 2cv + 2c, \quad \text{for } u, v \geq 0. \end{aligned}$$

Then by (2.8) we have

$$H(x, W) \geq \frac{\overline{\mu}}{2}|W|^2 - cu - cv + c. \tag{2.25}$$

Taking $W = (s\sin x, s\sin x)$, where $s > 0$, from (2.14) and (2.25) we get

$$\begin{aligned} J(s\sin x, s\sin x) &\le \frac{\pi s^2}{2}(1-\overline{\mu}) + (c+M_1)\int_0^\pi s\sin x + (c+M_2)\int_0^\pi s\sin x - c \\ &\le \frac{\pi s^2}{2}(1-\overline{\mu}) + cs - c \end{aligned}$$

since $\overline{\mu} > 1$, (2.22) holds. ◻

3. The Ambrosetti-Prodi type result

In this section, we state and prove the Ambrosetti-Prodi type result for system (1.5). We need the following concepts.

Definition 3.1. (1) We say that a vector function $W \in X$ is a weak subsolution of (1.5) if

$$J'(W)(\Psi) \le 0, \quad \forall \Psi \in X,\ \Psi \ge 0.$$

(2) $W = (u, v) \in C^2 \times C^4$ is a subsolution (classical) of (1.5) if

$$\begin{gathered} -u'' \le f_1(x,u,v) + t_1\sin x + h_1, \quad \text{in } (0,\pi),\\ v'''' \le f_2(x,u,v) + t_2\sin x + h_2, \quad \text{in } (0,\pi),\\ u(0) = u(\pi) = 0,\\ v(0) = v(\pi) = v''(0) = v''(\pi) = 0. \end{gathered}$$

(3) Weak supersolutions and supersolutions (classical) are defined likewise by reversing the above inequalities.

We can easily show that each a subsolution or a supersolution of (1.5) is indeed also a weak subsolution or a weak supersolution, respectively.

For to present the subsolution and supersolution for (1.5), we firstly show a maximum principle as follows.

Lemma 3.2. *Let A be a matrix-function with entries in $C[0,\pi]$ satisfy* (2.4) *and* (2.5)*. If $W = (u, v) \in X$ is such that*

$$\int_0^\pi (u'\psi_1' + v''\psi_2'') \ge \int_0^\pi (AW, \Psi), \quad \forall \Psi = (\psi_1, \psi_2) \in X, \tag{3.1}$$

then $W \ge 0$.

Proof. Let $\Psi = W^- = (u^-, v^-)$ in (3.1), by (2.4) and (2.5), we obtain

$$\begin{aligned} \int_0^\pi (|{u^-}'|^2 + |{v^-}''|^2) &\le \int_0^\pi (AW^-, W^-) - \int_0^\pi (AW^+, W^-) \\ &\le \underline{\mu}\int_0^\pi |W^-|^2 \le \underline{\mu}\|W^-\|_X^2. \end{aligned}$$

Therefore, $W^- = 0$, i.e. $W \ge 0$. ◻

Remark 3.3. In the classical sense, (2.4) and (2.5) are also sufficient conditions for having a maximum principle for the problem

$$\begin{gathered} -u'' = \underline{a}u + \underline{b}v + g_1(x), \quad \text{in } (0,\pi),\\ v'''' = \underline{c}u + \underline{d}v + g_2(x), \quad \text{in } (0,\pi),\\ u(0) = u(\pi) = 0, \end{gathered}$$

$$v(0) = v(\pi) = v''(0) = v''(\pi) = 0.$$

This is, $W = (u, v) \geq 0$ if $g_1 \geq 0, g_2 \geq 0$.

Lemma 3.4. *Assume condition* (H4), *i.e.* (2.3), (2.4) *and* (2.5) *hold. Then, for all* $t = (t_1, t_2) \in \mathbb{R}^2$, *system* (1.5) *has a subsolution* W_t *such that, if* W^t *is any supersolution we have*

$$W_t \leq W^t \quad \text{in } (0, \pi). \tag{3.2}$$

Proof. We consider the system

$$\begin{gathered} -u'' = \underline{a}u + \underline{b}v - c + t_1 \sin x + h_1, \quad \text{in } (0, \pi), \\ v'''' = \underline{c}u + \underline{d}v - c + t_2 \sin x + h_2, \quad \text{in } (0, \pi), \\ u(0) = u(\pi) = 0, \\ v(0) = v(\pi) = v''(0) = v''(\pi) = 0, \end{gathered} \tag{3.3}$$

where c is the constant in (2.3) and (2.6). From the hypotheses on $\underline{A}$ and h_1, h_2, (3.3) has a unique solution $W_t \in C^2 \times C^4$. Then, using (2.3) we conclude that W_t is in fact a subsolution of (1.5).

Finally, suppose that W^t is any supersolution of (1.5), from (2.3) and applying Lemma 3.2 directly we can get the assertion (3.2). □

Lemma 3.5. *Suppose* (H1) *holds and* $(h_1, h_2) \in C[0, \pi] \times C[0, \pi]$. *Then there exists* $t^0 \in \mathbb{R}^2$ *such that, for all* $t \leq t^0$, *system* (1.5) *has a supersolution* W^t.

Proof. Let $\overline{u}, \overline{v}$ be the solution of the system

$$\begin{gathered} -\overline{u}'' = f_1(x, 0, 0) + h_1(x), \quad \text{in } (0, \pi), \\ \overline{v}'''' = f_2(x, 0, 0) + h_2(x), \quad \text{in } (0, \pi), \\ u(0) = u(\pi) = 0, \\ v(0) = v(\pi) = v''(0) = v''(\pi) = 0. \end{gathered} \tag{3.4}$$

Due to the locally Lipschitzian condition on f_1, f_2, it is possible to choose $t^0 = (t_1^0, t_2^0) < 0$ such that

$$\begin{gathered} f_1(x, \overline{u}, \overline{v}) - f_1(x, 0, 0) + t_1^0 \sin x \leq 0, \\ f_2(x, \overline{u}, \overline{v}) - f_2(x, 0, 0) + t_2^0 \sin x \leq 0. \end{gathered}$$

Hence, from these inequalities and the system (3.4), for all $t \leq t^0$, $W^{t^0} = (\overline{u}, \overline{v})$ is a supersolution for (1.5). □

Lemma 3.6. *Let* (H4), (H5) *hold. Then for a given* h_1, h_2, *there exists an unbounded domain* $\Re$ *in the plane such that if* $t \in \Re$, *system* (1.5) *has no supersolution.*

Proof. Suppose $W = (u, v)$ is a supersolution for (1.5). Multiplying both equations of this system by $\sin x$, integration them by parts and using (2.3), (2.6) we deduce that

$$(\underline{A} - I) \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} \leq \frac{\pi}{2} \begin{pmatrix} -s_1 \\ -s_2 \end{pmatrix}, \tag{3.5}$$

$$(\overline{A} - I) \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} \leq \frac{\pi}{2} \begin{pmatrix} -s_1 \\ -s_2 \end{pmatrix}. \tag{3.6}$$

Where $\rho_1 = \int_0^\pi u \sin x, \rho_2 = \int_0^\pi v \sin x, s_1 = t_1 - c, s_2 = t_2 - c$ and c is the constant in (2.3) and (2.6). From remark 2.1, applying $(\underline{A} - I)^{-1}$ and $(\overline{A} - I)^{-1}$ to (3.5) and (3.6), respectively, we obtain that

(i) If $\rho_1 \leq 0$, then $s_2 \leq \frac{\underline{d}-1}{\underline{b}} s_1$ when $\underline{b} \neq 0$, or $s_1 \leq 0$ when $\underline{b} = 0$.

(ii) If $\rho_1 \geq 0$, then $s_2 \leq \frac{\overline{d}-1}{\overline{b}} s_1$ when $\overline{b} \neq 0$, or $s_1 \leq 0$ when $\overline{b} = 0$.

Therefore, independently of the sign of ρ_1, the pair (s_1, s_2) is in a region composed of the union of two half-planes passing through the origin, each of them bounded above by a straight-line of negative or infinity slope. $\Re$ is the complement of this region in the original variables t_1 and t_2. □

Now, we are at a position to prove the Ambrosetti-Prodi type result for system (1.5).

Theorem 3.7. *Suppose that conditions* (H1)–(H5) *are satisfied and that there exists a matrix*

$$A(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix},$$

with $b(x), c(x) \geq 0$ *(cooperativeness condition on* $A(x)$*) satisfies* (2.5) *such that*

$$F(x, \xi) - F(x, \eta) \geq A(x)(\xi - \eta), \quad \text{for } \xi, \eta \in \mathbb{R}^2, \ \xi \geq \eta. \tag{3.7}$$

Then there exists a continuous curve Γ *splitting* $\mathbb{R}^2$ *into two unbounded components* N *and* E *such that:*

(1) *for each* $t = (t_1, t_2) \in N$, (1.5) *has no solution;*

(2) *for each* $t = (t_1, t_2) \in E$, (1.5) *has at least two solutions.*

Proof. For each $\theta \in \mathbb{R}$, define

$$L_\theta = \{(t_1, t_2) \in \mathbb{R}^2; t_2 + \theta = t_1\},$$

and $R(\theta) = \{t_1 \in \mathbb{R}; (1.5)$ has a supersolution with $t \in L_\theta$ for some $t_2 \in \mathbb{R}\}$.

Lemmas 3.5 and 3.6 allows us to define the continuous curve

$$\Gamma(\theta) = (\sup R(\theta), \sup R(\theta) - \theta),$$

which splits the plane into two disjoints unbounded domains N and E such that for all $t \in N$ no supersolution exists for (1.5), while for all $t \in E$ (1.5) has a supersolution.

Obviously, for all $t \in N$, no solution exists for (1.5), result (1) is proved.

To prove result (2), now we use the abstract variational theorems to find the solutions of (1.5) when $t \in E$. We write

$$\begin{aligned} &\langle J'(W), \Psi\rangle \\ &= \langle W, \Psi\rangle - \int_0^\pi [(f_1(x,u,v) + t_1 \sin x + h_1)\psi_1 + (f_2(x,u,v) + t_2 \sin x + h_2)\psi_2]. \end{aligned}$$

Given $t \in E$ there exists a supersolution $W^t = (u^t, v^t)$ and a subsolution $W_t = (u_t, v_t)$ of (1.5) such that $W_t \leq W^t$ in $(0, \pi)$. Let

$$M = [W_t, W^t] = \{W \in X; W_t \leq W \leq W^t\},$$

since $W_t, W^t \in L^\infty$ by assumption, also $M \subset L^\infty$ and $H(x, W(x)) + (t_1 \sin x + h_1)u + (t_2 \sin x + h_2)v \leq c$ for all $W \in M$ and almost every $x \in (0, \pi)$.

Clearly, M is a closed and convex subset of X, hence weakly closed. Since M is essentially bounded, $J(W) \geq \frac{1}{2}\|W\|_X^2 - c$ is coercive on M. On the other hand, if

$W_n \rightharpoonup W$ weakly in X, where $W_n, W \in M$, we may assume that $W_n \to W$ pointwise almost everywhere; moreover, $|H(x, W_n) + (t_1 \sin x + h_1)u_n + (t_2 \sin x + h_2)v_n| \le c$ uniformly, using Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}&\int_0^\pi H(x, W_n) + \int_0^\pi [(t_1 \sin x + h_1)u_n + (t_2 \sin x + h_2)v_n]\\ &\to \int_0^\pi H(x, W) + \int_0^\pi [(t_1 \sin x + h_1)u + (t_2 \sin x + h_2)v].\end{aligned}$$

Hence J is weakly lower semi-continuous on M. Then we can use [17, Theorem 1.2] to find a vector function $W_0 = (u_0, v_0) \in X$ such that $W_0 \in M$ is the infimum of the functional J restricted to M.

To see that W_0 is a weak solution of (1.5), for $\varphi = (\varphi_1, \varphi_2) \in C_0^\infty(0, \pi)$ and $\varepsilon > 0$ let

$$\begin{aligned}u_\varepsilon &= \min\{u^t, \max\{u_t, u_0 + \varepsilon\varphi_1\}\} = u_0 + \varepsilon\varphi_1 - \varphi_1^\varepsilon + \varphi_{1\varepsilon}\\ v_\varepsilon &= \min\{v^t, \max\{v_t, v_0 + \varepsilon\varphi_2\}\} = v_0 + \varepsilon\varphi_2 - \varphi_2^\varepsilon + \varphi_{2\varepsilon}\end{aligned}$$

with

$$\begin{aligned}\varphi_1^\varepsilon &= \max\{0, u_0 + \varepsilon\varphi_1 - u^t\} \ge 0,\\ \varphi_2^\varepsilon &= \max\{0, v_0 + \varepsilon\varphi_2 - v^t\} \ge 0,\end{aligned}$$

and

$$\begin{aligned}\varphi_{1\varepsilon} &= -\min\{0, u_0 + \varepsilon\varphi_1 - u_t\} \ge 0,\\ \varphi_{2\varepsilon} &= -\min\{0, v_0 + \varepsilon\varphi_2 - v_t\} \ge 0.\end{aligned}$$

Note that $W_\varepsilon = (u_\varepsilon, v_\varepsilon) \in M$ and $\varphi^\varepsilon = (\varphi_1^\varepsilon, \varphi_2^\varepsilon), \varphi_\varepsilon = (\varphi_{1\varepsilon}, \varphi_{2\varepsilon}) \in X \cap L^\infty(0, \pi)$.

The functional J is differentiable in direction $W_\varepsilon - W_0$. Since W_0 minimizes J in M we have

$$0 \le \langle W_\varepsilon - W_0, J'(W_0)\rangle = \varepsilon\langle\varphi, J'(W_0)\rangle - \langle\varphi^\varepsilon, J'(W_0)\rangle + \langle\varphi_\varepsilon, J'(W_0)\rangle,$$

so that

$$\langle\varphi, J'(W_0)\rangle \ge \frac{1}{\varepsilon}[\langle\varphi^\varepsilon, J'(W_0)\rangle - \langle\varphi_\varepsilon, J'(W_0)\rangle].$$

Now, from W^t is a supersolution to (1.5), we get

$$\begin{aligned}&\langle\varphi^\varepsilon, J'(W_0)\rangle\\ &= \langle\varphi^\varepsilon, J'(W^t)\rangle + \langle\varphi^\varepsilon, J'(W_0) - J'(W^t)\rangle\\ &\ge \langle\varphi^\varepsilon, J'(W_0) - J'(W^t)\rangle\\ &= \int_\Omega [(u_0 - u^t)'(u_0 + \varepsilon\varphi_1 - u^t)' + (v_0 - v^t)''(v_0 + \varepsilon\varphi_2 - v^t)'']\\ &\quad - \int_\Omega [f_1(x, W_0) - f_1(x, W^t)](u_0 + \varepsilon\varphi_1 - u^t)\\ &\quad - \int_\Omega [f_2(x, W_0) - f_2(x, W^t)](v_0 + \varepsilon\varphi_2 - v^t)\\ &\ge \varepsilon\int_\Omega [(u_0 - u^t)'\varphi_1' + (v_0 - v^t)''\varphi_2'']\\ &\quad - \varepsilon\int_\Omega |f_1(x, W_0) - f_1(x, W^t)||\varphi_1| - \varepsilon\int_\Omega |f_2(x, W_0) - f_2(x, W^t)||\varphi_2|\end{aligned}$$

where $\Omega = \{x \in (0,\pi); W_0(x) + \varepsilon\varphi(x) \geq W^t(x) > W_0(x)\}$. Note that meas$(\Omega) \to 0$ as $\varepsilon \to 0$. Hence by absolute continuity of the Lebesgue integral we obtain that

$$\langle \varphi^\varepsilon, J'(W_0)\rangle \geq o(\varepsilon)$$

where $o(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$. Similarly, we conclude that $\langle \varphi_\varepsilon, J'(W_0)\rangle \leq o(\varepsilon)$; thus

$$\langle \varphi, J'(W_0)\rangle \geq 0$$

for all $\varphi \in C_0^\infty(0,\pi)$. Reversing the sign of φ and since $C_0^\infty(0,\pi)$ is dense in X we finally get that $J'(W_0) = 0$, i.e. W_0 is a weak solution to (1.5). Then using (3.7) and a Maximum Principle Lemma 3.2, we claim that W_0 is a local minimum of J.

Suppose by contradiction that W_0 is not a local minimum, then for every $\varepsilon > 0$ there is $\widetilde{W_\varepsilon} \in \overline{B_\varepsilon(W_0)}$ (a ball of radius ε around $W_0 \in X$) such that $J(\widetilde{W_\varepsilon}) < J(W_0)$. We know that $\overline{B_\varepsilon(W_0)}$ is weaker sequentially compact in X and J is weakly lower semi-continuous, therefore there is $\widehat{W_\varepsilon} \in \overline{B_\varepsilon(W_0)}$ such that

$$J(\widehat{W_\varepsilon}) = \inf_{\overline{B_\varepsilon(W_0)}} J \leq J(\widetilde{W_\varepsilon}) < J(W_0),$$

and $\langle J'(\widehat{W_\varepsilon}), \widehat{W_\varepsilon} - W_0\rangle \leq 0$, or

$$J'(\widehat{W_\varepsilon}) = \lambda_\varepsilon(\widehat{W_\varepsilon} - W_0) \quad \text{with } \lambda_\varepsilon \leq 0,$$

namely

$$\begin{aligned} &\int_0^\pi (\widehat{u_\varepsilon}'\psi_1' + \widehat{v_\varepsilon}''\psi_2'') - \int_0^\pi [f_1(x,\widehat{u_\varepsilon},\widehat{v_\varepsilon})\psi_1 + f_2(x,\widehat{u_\varepsilon},\widehat{v_\varepsilon})\psi_2] \\ &- \int_0^\pi [(t_1\sin x + h_1)\psi_1 + (t_2\sin x + h_2)\psi_2] \\ &= \lambda_\varepsilon[(\widehat{u_\varepsilon} - u_0)\psi_1 + (\widehat{v_\varepsilon} - v_0)\psi_2]. \end{aligned} \tag{3.8}$$

On the other hand, from Definition 3.1 we have

$$\begin{aligned} &\int_0^\pi (u_t'\psi_1' + v_t''\psi_2'') - \int_0^\pi [f_1(x,u_t,v_t)\psi_1 + f_2(x,u_t,v_t)\psi_2] \\ &- \int_0^\pi [(t_1\sin x + h_1)\psi_1 + (t_2\sin x + h_2)\psi_2] \leq 0, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &\int_0^\pi ({u^t}'\psi_1' + {v^t}''\psi_2'') - \int_0^\pi [f_1(x,u^t,v^t)\psi_1 + f_2(x,u^t,v^t)\psi_2] \\ &- \int_0^\pi [(t_1\sin x + h_1)\psi_1 + (t_2\sin x + h_2)\psi_2] \geq 0. \end{aligned} \tag{3.10}$$

From (3.8)–(3.9), we obtain

$$\begin{aligned} &\int_0^\pi [(\widehat{u_\varepsilon}' - u_t')\psi_1' + (\widehat{v_\varepsilon}'' - v_t'')\psi_2''] \\ &- \int_0^\pi [(f_1(x,\widehat{W_\varepsilon}) - f_1(x,W_t))\psi_1 + (f_2(x,\widehat{W_\varepsilon}) - f_2(x,W_t))\psi_2] \\ &\geq \lambda_\varepsilon[(\widehat{u_\varepsilon} - u_t + u_t - u_0)\psi_1 + (\widehat{v_\varepsilon} - v_t + v_t - v_0)\psi_2]. \end{aligned}$$

This implies

$$\begin{gathered} -(\widehat{u_\varepsilon} - u_t)'' \geq f_1(x,\widehat{W_\varepsilon}) - f_1(x,W_t) + \lambda_\varepsilon(\widehat{u_\varepsilon} - u_t) + \lambda_\varepsilon(u_t - u_0), \\ (\widehat{v_\varepsilon} - v_t)^{(4)} \geq f_2(x,\widehat{W_\varepsilon}) - f_2(x,W_t) + \lambda_\varepsilon(\widehat{v_\varepsilon} - v_t) + \lambda_\varepsilon(v_t - v_0). \end{gathered}$$

Then from (3.7) we obtain

$$\begin{pmatrix} -(\widehat{u_\varepsilon} - u_t)'' \\ (\widehat{v_\varepsilon} - v_t)^{(4)} \end{pmatrix} \geq A(x)(\widehat{W_\varepsilon} - W_t) + \lambda_\varepsilon(\widehat{W_\varepsilon} - W_t),$$

note that $\lambda_\varepsilon \leq 0$, and by using Lemma 3.2 we obtain

$$\widehat{W_\varepsilon} - W_t \geq 0, \quad \text{or} \quad W_t \leq \widehat{W_\varepsilon}.$$

Similarly, from (3.10)−(3.8), we can obtain

$$\widehat{W_\varepsilon} \leq W^t.$$

Which contradicts $J(W_0) = \inf_M J(W)$.

Finally, since J is not bounded from below, a weaker form of the Mountain Pass Theorem can be used to find another solution $W_1 \neq W_0$ of (1.5). Then result (2) is proved. □

4. Example: A piecewise linear problem

Consider the system

$$\begin{gathered} -u'' = k_1 u^+ + \epsilon v^+ + t_1 \sin x + h_1(x), \quad \text{in } (0,\pi), \\ v^{(4)} = \epsilon u^+ + k_2 v^+ + t_2 \sin x + h_2(x), \quad \text{in } (0,\pi), \\ u(0) = u(\pi) = 0, \\ v(0) = v(\pi) = v''(0) = v''(\pi) = 0. \end{gathered} \tag{4.1}$$

Where ϵ and k_1, k_2 are constants, t_1, t_2 are parameters and $h_1, h_2 \in C[0,\pi]$ are fixed functions with $\int_0^\pi h_1 \sin x = \int_0^\pi h_2 \sin x = 0$. This problem is similar to system (1.4).

Theorem 4.1. *Suppose that $k_1 > 1$, $k_2 > 1$ and $\epsilon \geq 0$. Then there exists a curve Γ splitting $\mathbb{R}^2$ into two unbounded components N and E such that:*

1. *for each $t = (t_1, t_2) \in N$, (4.1) has no solution;*
2. *for each $t = (t_1, t_2) \in E$, (4.1) has at least two solutions.*

Proof. Let

$$\overline{A} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \quad \underline{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then we can easily verify that the conditions of Theorem 3.7 hold and therefore the results are follow. □

Remark 4.2. (1) Denote by μ_i $(i = 1, 2)$ the eigenvalues of matrix

$$A = \begin{pmatrix} k_1 & \epsilon \\ \epsilon & k_2 \end{pmatrix}$$

and let $\mu_1 \leq \mu_2$. It can be shown that $\mu_2 > 1$ since $k_1 > 1$ and $k_2 > 1$.

(2) This result gives a partial answer to Question 1 and Question 2 that were posted in [16] and stated in Section 1.

References

[1] A. C. Lazer, P. J. McKenna; *Large-amplitude periodic oscillations in suspension bridge: some new connections with nonlinear analysis*, SIAM Rev. 32 (1990) 537-578.

[2] P. J. McKenna and W. Walter; *Nonlinear oscillation in a suspension bridge*, Arch. Rational Mech. Anal. 98 (1987) 167-177.

[3] Q. H. Choi, T. Jung and P. J. McKenna; *The study of a nonlinear suspension bridge equation by a variational reduction method*, Appl. Anal. 50 (1995) 71-90.

[4] Q. H. Choi and T. Jung; *A nonlinear suspension bridge equation with nonconstant load*, Nonli.Anal. 35 (1999) 649-668.

[5] Y. K. An and C. K. Zhong; *Periodic solutions of Nonlinear suspension bridge equation with damping and nonconstant load*, J. Math. Anal. Appl. 279 (2003) 569-579.

[6] A. Ambrosetti, G. Prodi; *On the inversion of some differentiable mappings with singularities between Banach spaces*, Annali Mat. pura appl. 93 (1972) 231-246.

[7] D. G. de Figueiredo; *Lectures on boundary value problems of Ambrosetti-Prodi type, 12th Brazilian Seminar of Analysis*, São Paulo, Brazil (1980).

[8] D. G. de Figueiredo; *On the superlinear Ambrosetti-Prodi problem*, MRC Tech Rep # 2522, May(1983).

[9] D. C. de Morais Filho; *A variational approach to an Ambrosetti-Prodi type problem for a system of elliptic equations*, Nonlinear Anal. TMA 26 (1996) 1655-1668.

[10] D. C. de Morais Filho, F.R. Pereira, Critical Ambrosetti-Prodi type results for systems of Elliptic Equations, Nonlinear Anal. TMA, in prees.

[11] K. C. Chang; *Ambrosetti-Pfodi type results in elliptic systems*, Nonlinear Anal. 51 (2002) 553-566.

[12] P. Drabek, H. Leinfelder and G. Tajcova; *Coupled string-beam equations as a model of suspension bridges*, Appl. Math. 44 (1999) 97-142.

[13] Y. K. An; *Nonlinear Perturbations of a Coupled System of Steady State Suspension Bridge Equations*, Nonli. Anal. 51 (2002) 1285-1292.

[14] Y. K. An, X. L. Fan; *On the coupled systems of second and fourth order elliptic equations*, Appl. Math. Comput. 140 (2003) 341-351.

[15] Y.K. An; *Mountain pass solutions for the coupled systems of second and fourth order elliptic equations*, Nonli. Anal. 63 (2005) 1034-1041.

[16] F. Dalbono, P. J. McKenna; *Multiplicity results for a class of asymmetric weakly coupled systems of second-order ordinary differential equations*, Boundary Value Problems. 2 (2005) 129-151.

[17] M. Struwe; *Variational Methods*, Springer-Verlag, 2000.

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