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ON THE EXISTENCE OF WEAK SOLUTIONS FOR p, q-LAPLACIAN SYSTEMS WITH WEIGHTS

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Abstract. This paper studies degenerate quasilinear elliptic systems involving p, q-superlinear and critical nonlinearities with singularities. Existence results are obtained by using properties of the best Hardy-Sobolev constant together with an approach developed by Brezis and Nirenberg.

1. Introduction

In a well-known paper, Brezis and Nirenberg [11] proved that, under certain conditions, the elliptic problem with Dirichlet boundary condition

$$\begin{gathered} -\Delta u = \lambda u^q + u^{2^*-1} \qquad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

possesses at least a solution, for all $\lambda > 0$, where $1 < q < 2^* = 2N/(N-2)$, $N \geq 3$, 2^* is said to be the critical Sobolev exponent, and $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded smooth domain. In general, the main difficulty in this type of problem is the lack of compactness of the injection $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$.

We recall that the perturbation λu^q is essential in this kind of the problem. By Pohozaev identity [30], problem (1.1) does not possess any solution when $\lambda \leq 0$.

García and Peral in [19] studied the existence of nontrivial solution for a class of problems involving the p-laplacian operator, namely,

$$\begin{gathered} -\Delta_p u \equiv -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{q-2}u + \mu |u|^{p^*-2}u \quad \text{in } \Omega, \\ u \geq 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N (N > p)$, with $1 < p \leq q < p^* = Np/(N-p)$. When $p < q < p^*$, we say that the above problem is p-superlinear. These type of problems, which are related to the Brezis and Nirenberg problem [11] (problem (1.1) with $p = 2$), have been widely treated by several authors and we would like

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to mention some of them, e.g., [14, 20, 21] for $1 < p < N$ and [26, 28, 29] for $p = 2$, see also references cited there.

Caffarelli, Kohn and Nirenberg in [12] proved that if $1 < p < N$, $-\infty < a < (N-p)/p$, $a \le c_1 \le a+1$, $d_1 = 1 + a - c_1$, and $p^* = p^*(a, c_1, p) := Np/(N - d_1 p)$, there exists $C_{a,p} > 0$ such that the following Hardy-Sobolev type inequality with weights is satisfied

$$\Big(\int_{\mathbb{R}^N} |x|^{-c_1 p^*} |u|^{p^*} dx\Big)^{p/p^*} \le C_{a,p}\Big(\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx\Big), \quad \forall u \in C_0^\infty(\mathbb{R}^N).$$

Note that several papers have been appeared on this subject, mainly, the works about the existence of solution for a class of quasilinear elliptic problems of the type

$$-Lu_{ap} = g(x,u) + |x|^{-e_1 p^*} |u|^{q-2} u \quad \text{in } \Omega,$$

where $Lu_{ap} = \operatorname{div}(|x|^{-ap} |\nabla u|^{p-2} \nabla u)$, under certain suppositions on the exponents $1 < p < N$, $-\infty < a < (N-p)/p$, $a \le e_1 < a+1$, $d = 1 + a - e_1$, and $p^* = Np/(N - dp)$, and on the function $g : \Omega \times \mathbb{R} \to \mathbb{R}$. See, for instance, [4, 7, 13, 16, 34, 35] and references therein. The lack of compactness is overcame proving that all the Palais Smale sequence at the level c, ($(PS)_c$-sequence, in short), with $c < (d/N)(C_{a,p}^*)^{N/dp}$, is relatively compact. $(d/N)(C_{a,p}^*)^{N/dp}$ is so called the critical level and $C_{a,p}^*$ is the best Hardy-Sobolev constant and it is characterized by

$$C_{a,p}^* = C_{a,p}^*(\Omega) := \inf_{u \in W_0^{1,p}(\Omega, |x|^{-ap}) \setminus \{0\}} \Big\{ \frac{\int_\Omega |x|^{-ap} |\nabla u|^p dx}{\big(\int_\Omega |x|^{-e_1 p^*} |u|^{p^*} dx\big)^{p/p^*}} \Big\}.$$

Besides the great number of the applications known for the scalar case, for instance, in fluid mechanics, in newtonian fluids, in flow through porous media, reaction-diffusion problems, nonlinear elasticity, petroleum extraction, astronomy, glaciology, etc, see [15], the above systems can involve another phenomena, such as competition model in population dynamics, see [18] and reference therein. For the systems case we would like to mention the papers [2, 32] and a survey paper [17] as well as in the references therein.

In our work, we will use a version of the well-known mountain pass theorem [6] to establish conditions for the existence of a nontrivial solution for a quasilinear elliptic system involving the above operator and a p,q-superlinear nonlinear perturbation

$$\begin{gathered} -Lu_{ap} = \lambda\theta |x|^{-\beta_1} |u|^{\theta-2} |v|^\delta u + \mu\alpha |x|^{-\beta_2} |u|^{\alpha-2} |v|^\gamma u \quad \text{in } \Omega, \\ -Lv_{bq} = \lambda\delta |x|^{-\beta_1} |u|^{\theta} |v|^{\delta-2} v + \mu\gamma |x|^{-\beta_2} |u|^{\alpha} |v|^{\gamma-2} v \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where

$$\Omega \text{ is a bounded smooth domain of } \mathbb{R}^N \text{ with } 0 \in \Omega, \tag{1.3}$$

the parameters λ, μ are positive real numbers and the exponents satisfy

$$\begin{gathered} 1 < p, \quad q < N, \quad -\infty < a < (N-p)/p, \quad -\infty < b < (N-q)/q, \\ a \le c_1 < a+1, \quad b \le c_2 < b+1, \quad d_1 = 1 + a - c_1, \quad d_2 = 1 + b - c_2, \\ p^* = Np/(N - d_1 p), \quad q^* = Nq/(N - d_2 q), \\ \alpha, \gamma, \theta, \delta > 1, \quad \beta_1, \beta_2 \in \mathbb{R}, \end{gathered} \tag{1.4}$$

with one of the following two sets of conditions satisfied:

$$
\begin{gathered}
\frac{\theta}{p}+\frac{\delta}{q},\ \frac{\alpha}{p}+\frac{\gamma}{q}>1 \quad (p,q\text{-superlinear}) \\
\frac{\theta}{p^*}+\frac{\delta}{q^*},\ \frac{\alpha}{p^*}+\frac{\gamma}{q^*}<1 \quad (p,q\text{-subcritical}),
\end{gathered} \tag{1.5}
$$

or

$$
\frac{\theta}{p^*}+\frac{\delta}{q^*}<1<\frac{\theta}{p}+\frac{\delta}{q} \text{ and } \frac{\alpha}{p^*}+\frac{\gamma}{q^*}=1 \quad p,q\text{-superlinear/critical case)} \tag{1.6}
$$

However, the variational systems behave, in a certain sense, like in the scalar case, there exist some additional difficulties mainly coming from the mutual actions of the variables u and v, see e. g. [23, 33]. Another difficulty, even in the regular case, are the systems involving p-laplacian and q-laplacian operators and their respective critical exponents. In this situation, it is hard to find a well appropriated critical level, mainly, when $p \neq q$. This open question was pointed out in Adriouch and Hamidi [1]. But, recently Silva and Xavier in [31] were able to prove, in a certain context and in the regular case, the existence of weak solution for a system involving p-laplacian and q-laplacian operators with $p \neq q$. Still in the regular case and $p = q$, we would like to mention the papers [2, 5, 27, 32, 36], also a survey paper [17]. In particular, Morais and Souto in [27] defined the following critical level number S_H/p, where

$$
S_H = \inf_{W\setminus\{0\}} \Big\{ \frac{\int_\Omega |\nabla u|^p + |\nabla v|^p dx}{\big(\int_\Omega H(u,v)dx\big)^{p/p^*}} \Big\},
$$

$W = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and H is homogeneous nonlinearity of degree p^*. In this work, we will improve the critical level by proving that all the Palais Smale sequences at the level c are relatively compact provided that

$$
c < \big(\frac{1}{p}-\frac{1}{p^*}\big)(\mu p^*)^{\frac{-p}{p^*-p}} \tilde{S}^{\frac{p^*}{p^*-p}} + \lambda\big(\frac{1}{p}-\frac{1}{p_1}\big)M,
$$

where $\tilde{S}$ depends of $C^*_{a,p}$ and $M = M(u_n,v_n) \geq 0$ depends of Palais Smale sequence.

Our first result deals with p,q-superlinear and subcritical nonlinear perturbation.

Theorem 1.1. *In addition to* (1.3), (1.4)*, and* (1.5)*, assume that $p_i \in (p,p^*)$, $q_i \in (q,q^*)$, $i = 1,2$, with $\theta/p_1 + \delta/q_1 = \alpha/p_2 + \gamma/q_2 = 1$ and*

$$
\beta_i < \min\big\{(a+1)p_i + N\big(1-\frac{p_i}{p}\big), (b+1)q_i + N\big(1-\frac{q_i}{q}\big)\big\}, \quad i=1,2. \tag{1.7}
$$

Then system (1.2) *possesses a weak solution, where each component is nontrivial and nonnegative, for each $\lambda \geq 0$ and $\mu > 0$.*

The next result treats the p,q-superlinear and critical case.

Theorem 1.2. *Assume* (1.3), (1.4) *and* (1.6)*, with $p = q$ and $a = b \geq 0$. Suppose also $p_1 = q_1 \in (p,p^*)$, with $\theta/p_1 + \delta/q_1 = 1$, $p^* = q^*$, $\beta_2 = c_1 p^*$, and $\beta_1 = (a+1)p_1 - c$ with*

$$
-N\left[1-(p_1/p)\right] < c < \frac{(p_1-p+1)N-(a+1)p_1}{p-1} - \frac{(N-p-ap)(p_1-p)}{p(p-1)}.
$$

Then, system (1.2) *possesses a weak solutions, where each component is nontrivial and nonnegative, for each $\lambda,\mu > 0$.*

The p,q-superlinear and critical case with $p \neq q$ is studied in the following result.

Theorem 1.3. *In addition to* (1.3), (1.4), *and* (1.6), *assume that* $p_1 \in (p, p^*)$, $q_1 \in (q, q^*)$, *with* $\theta/p_1 + \delta/q_1 = 1$, $\beta_2 = c_1 p^* = c_2 q^*$, *and* β_1 *as in* (1.7). *Then there exists* μ_0 *sufficiently small such that system* (1.2) *posesses a weak solution, where each component is nontrivial and nonnegative, for each* $\lambda > 0$ *and* $0 < \mu < \mu_0$.

2. Preliminaries

We will set some spaces and their norms. If $\alpha \in \mathbb{R}$ and $l \geq 1$, we define $L^l(\Omega, |x|^\alpha)$ as being the subspace of $L^l(\Omega)$ of the Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ satisfying

$$\|u\|_{L^l(\Omega,|x|^\alpha)} := \Big(\int_\Omega |x|^\alpha |u|^l dx \Big)^{1/l} < \infty.$$

If $1 < p < N$ and $-\infty < a < (N-p)/p$, we define $W_0^{1,p}(\Omega, |x|^{-ap})$ as being the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|$ defined by

$$\|u\| := \Big(\int_\Omega |x|^{-ap} |\nabla u|^p dx \Big)^{1/p}.$$

First of all, from the Caffarelli, Kohn and Nirenberg inequality (see [12]) and by the boundedness of Ω, it is easy to see that there exists $C > 0$ such that

$$\Big(\int_\Omega |x|^{-\delta} |u|^r dx \Big)^{p/r} \leq C \Big(\int_\Omega |x|^{-ap} |\nabla u|^p dx \Big), \quad \forall u \in W_0^{1,p}(\Omega, |x|^{-ap}),$$

where $1 \leq r \leq Np/(N-p)$ and $\delta \leq (a+1)r + N[1 - (r/p)]$.

Lemma 2.1. *Suppose that* Ω *is a bounded smooth domain of* $\mathbb{R}^N$ *with* $0 \in \Omega$, $1 < p < N$, $-\infty < a < (N-p)/p$, $a \leq e_1 < a+1$, $d_1 = 1 + a - e_1$, $p^* = Np/(N - d_1 p)$, *and* $\alpha + \gamma = p^*$, *then*

$$\tilde{S} := \inf_{(u,v) \in \tilde{W}} \Big\{ \frac{\int_\Omega |x|^{-ap} (|\nabla u|^p + |\nabla v|^p) dx}{\big(\int_\Omega |x|^{-e_1 p^*} |u|^\alpha |v|^\gamma dx \big)^{p/p^*}} \Big\},$$

where

$$\tilde{W} = \big\{ (u,v) \in \big(W_0^{1,p}(\Omega, |x|^{-ap}) \big)^2 : |u||v| \not\equiv 0 \big\},$$

satisfies

$$\tilde{S} = \big[(\alpha/\gamma)^{\gamma/p^*} + (\alpha/\gamma)^{-\alpha/p^*} \big] C^*_{a,p}.$$

The proof of the above lemma is similar to the proof of [5, Theorem 5] (see also [27, Lemma 3] for $p \neq 2$).

Let us consider Ω a smooth domain of $\mathbb{R}^N$ (not necessarily bounded), $0 \in \Omega$, $1 < p < N$, $0 \leq a < (N-p)/p$, $a \leq c_1 < a+1$, $d_1 = 1 + a - c_1$, and $p^* = Np/(N - d_1 p)$. We define the space

$$W^{1,p}_{a,c_1}(\Omega) = \big\{ u \in L^{p^*}(\Omega, |x|^{-c_1 p^*}) : |\nabla u| \in L^p(\Omega, |x|^{-ap}) \big\},$$

equipped with the norm

$$\|u\|_{W^{1,p}_{a,c_1}(\Omega)} = \|u\|_{L^{p^*}(\Omega, |x|^{-c_1 p^*})} + \|\nabla u\|_{L^p(\Omega, |x|^{-ap})}.$$

We consider the best Hardy-Sobolev constant given by

$$\tilde{S}_{a,p} = \inf_{W^{1,p}_{a,c_1}(\mathbb{R}^N) \setminus \{0\}} \Big\{ \frac{\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx}{\big(\int_{\mathbb{R}^N} |x|^{-c_1 p^*} |u|^{p^*} dx \big)^{p/p^*}} \Big\}.$$

Also, we define

$$R^{1,p}_{a,c_1}(\Omega) = \{u \in W^{1,p}_{a,c_1}(\Omega) : u(x) = u(|x|)\},$$

endowed with the norm

$$\|u\|_{R^{1,p}_{a,c_1}(\Omega)} = \|u\|_{W^{1,p}_{a,c_1}(\Omega)}.$$

Actually, Horiuchi in [24] proved that

$$\tilde{S}_{a,p,R} = \inf_{R^{1,p}_{a,c_1}(\mathbb{R}^N)\setminus\{0\}} \Big\{ \frac{\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx}{\big(\int_{\mathbb{R}^N} |x|^{-c_1 p^*}|u|^{p^*}dx\big)^{p/p^*}} \Big\} = \tilde{S}_{a,p}$$

and it is achieved by functions of the form

$$y_\epsilon(x) := k_{a,p}(\epsilon)U_{a,p,\epsilon}(x), \quad \forall \epsilon > 0,$$

where

$$k_{a,p}(\epsilon) = c_0\epsilon^{(N-d_1p)/d_1p^2} \quad \text{and} \quad U_{a,p,\epsilon}(x) = \Big(\epsilon + |x|^{\frac{d_1p(N-p-ap)}{(p-1)(N-d_1p)}}\Big)^{-(\frac{N-d_1p}{d_1p})}.$$

Moreover, y_ϵ satisfies

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla y_\epsilon|^p dx = \int_{\mathbb{R}^N} |x|^{-c_1p^*}|y_\epsilon|^{p^*}dx. \tag{2.1}$$

See also Clément, Figueiredo and Mitidieri [16, Proposition 1.4].

The next lemma can be proved arguing as in [11] (see also [35, Lemma 5.1]). For the sake of the completeness we will give the proof in the appendix.

Lemma 2.2. *In addition to* (1.3) *and* (1.4)*, assume that* $p_1 = q_1 \in (p, p^*)$*,* $\theta/p_1 + \delta/q_1 = 1$*,* $\beta_2 = c_1p^* = c_2q^*$*, and* $\beta_1 = (a+1)p_1 - c$ *with*

$$-N\,[1 - (p_1/p)] < c.$$

Let $R_0 \in (0,1)$ *be such that* $B(0,2R_0) \subset \Omega$ *and* $\psi \in C_0^\infty(B(0,2R_0))$ *with* $\psi \geq 0$ *in* $B(0,2R_0)$ *and* $\psi \equiv 1$ *in* $B(0,R_0)$*, then the function*

$$u_\epsilon(x) = \frac{\psi(x)U_{a,p,\epsilon}(x)}{\|\psi U_{a,p,\epsilon}\|_{L^{p^*}(\Omega,|x|^{-c_1p^*})}}$$

satisfies

$$\|u_\epsilon\|^{p^*}_{L^{p^*}(\Omega,|x|^{-c_1p^*})} = 1, \quad \|\nabla u_\epsilon\|^p_{L^p(\Omega,|x|^{-ap})} \leq \tilde{S}_{a,p,R} + O(\epsilon^{(N-d_1p)/d_1p}),$$

and

$$\|u_\epsilon\|^{p_1}_{L^{p_1}(\Omega,|x|^{-\beta_1})} \geq \begin{cases} O(\epsilon^{(N-d_1p)p_1/d_1p^2}) \text{ if } c > \frac{(p_1-p+1)N-(a+1)p_1}{p-1}, \\ O(\epsilon^{(N-d_1p)p_1/d_1p^2}|ln(\epsilon)|) \text{ if } c = \frac{(p_1-p+1)N-(a+1)p_1}{p-1}, \\ O\Big(\epsilon^{\frac{(N-d_1p)(p-1)(N-p_1-ap_1+c)}{d_1p(N-p-ap)} - \frac{(N-d_1p)(p-1)p_1}{d_1p^2}}\Big) \\ \text{if } c < \frac{(p_1-p+1)N-(a+1)p_1}{p-1}. \end{cases} \tag{2.2}$$

The following result, which will be useful in the proof of our results, was proved by Kavian in [25, Lemma 4.8].

Lemma 2.3. *Let* Ω *be an open subset of* $\mathbb{R}^N$*,* $\{f_n\} \in L^r(\Omega)$*, for some* $1 < r < \infty$*, a bounded sequence such that* $f_n(x) \to f(x)$*, for a.e.* $x \in \Omega$*, as* $n \to \infty$*. Then,* $f \in L^r(\Omega)$ *and* $f_n \rightharpoonup f$ *weakly in* $L^r(\Omega)$ *as* $n \to \infty$*.*

Definition. Let us consider $\{(u_n, v_n)\}$ in $W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$. We say that the sequence $\{(u_n, v_n)\}$ is a Palais Smale sequence for operator I at the level c (or simply, $(PS)_c$-sequence) if

$$I(u_n, v_n) \to c \quad \text{and} \quad I'(u_n, v_n) \to 0, \quad \text{as } n \to \infty.$$

Our approach will be to use variational techniques; that is, we have to find the critical points of the Euler-Lagrange functional

$$I : W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq}) \to \mathbb{R}$$

given by

$$\begin{aligned} I(u, v) = &\frac{1}{p}\int_\Omega |x|^{-ap}|\nabla u|^p \, dx + \frac{1}{q}\int_\Omega |x|^{-bq}|\nabla v|^q \, dx \\ &- \lambda\int_\Omega |x|^{-\beta_1} u_+^\theta v_+^\delta \, dx - \mu\int_\Omega |x|^{-\beta_2} u_+^\alpha v_+^\gamma \, dx, \end{aligned}$$

which is well defined and is of class C^1, with the Gâteaux derivative

$$\begin{aligned} \langle I'(u, v), (w, z)\rangle = &\int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla w \, dx + \int_\Omega |x|^{-bq}|\nabla v|^{q-2}\nabla v \nabla z \, dx \\ &- \lambda\theta\int_\Omega |x|^{-\beta_1} u_+^{\theta-1} v_+^\delta w \, dx - \lambda\delta\int_\Omega |x|^{-\beta_1} u_+^\theta v_+^{\delta-1} z \, dx \\ &- \mu\alpha\int_\Omega |x|^{-\beta_2} u_+^{\alpha-1} v_+^\gamma w \, dx - \mu\gamma\int_\Omega |x|^{-\beta_2} u_+^\alpha v_+^{\gamma-1} z \, dx, \end{aligned}$$

where $u_\pm = \max\{0, \pm u\}$ which is in $W_0^{1,p}(\Omega, |x|^{-ap})$ (Similarly $v_\pm = \max\{0, \pm v\}$ which is in $W_0^{1,q}(\Omega, |x|^{-bq})$; see [3]).

First of all, we are going to show the geometric conditions of the mountain pass theorem.

Lemma 2.4. *In addition to* (1.3) *and* (1.4), *assume that one of the following conditions hold:*

- (i) *the case* (1.5), $p_i \in (p, p^*)$, $q_i \in (q, q^*)$, *with* $\theta/p_1 + \delta/q_1 = \alpha/p_2 + \gamma/q_2 = 1$, *and* β_i *as in* (1.7), *for* $i = 1, 2$.
- (ii) *the case* (1.6), $p_1 \in (p, p^*)$, $q_1 \in (q, q^*)$, *with* $\theta/p_1 + \delta/q_1 = 1$, β_1 *as in* (1.7), $p_2 = p^*$, $q_2 = q^*$, *and* $\beta_2 = c_1 p^* = c_2 q^*$.

Then the Euler-Lagrange functional I *satisfies:*

- (a) *There exist* $\sigma, \rho > 0$ *such that*
$$I(u, v) \geq \sigma \ \text{ if } \ \|(u, v)\| = \rho. \tag{2.3}$$
- (b) *There exists* $e \in W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$ *such that*
$$I(e) \leq 0, \quad \|e\| \geq R \quad \text{for some } R > \rho.$$

Proof. Part (a). For $(u, v) \in W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$ with $\|(u, v)\| \leq 1$, we have

$$\begin{aligned} I(u, v) \geq &\Big(\frac{1}{p}\|u\|^p - \lambda\frac{\theta C^{p_1/p}}{p_1}\|u\|^{p_1} - \mu\frac{\alpha C^{p_2/p}}{p_2}\|u\|^{p_2}\Big) \\ &+ \Big(\frac{1}{q}\|v\|^q - \lambda\frac{\delta C^{q_1/q}}{q_1}\|v\|^{q_1} - \mu\frac{\gamma C^{q_2/q}}{q_2}\|v\|^{q_2}\Big) \end{aligned}$$

$$\begin{aligned}&\geq \frac{1}{p}\|u\|^p - \Big(\lambda\frac{\theta C^{p_1/p}}{p_1} + \mu\frac{\alpha C^{p_2/p}}{p_2}\Big)\|u\|^{\min\{p_1,p_2\}}\\ &\quad + \frac{1}{q}\|v\|^q - \Big(\lambda\frac{\delta C^{q_1/q}}{q_1} + \mu\frac{\gamma C^{q_2/q}}{q_2}\Big)\|v\|^{\min\{q_1,q_2\}}.\end{aligned}$$

Hence, as $p < \min\{p_1, p_2\}$ and $q < \min\{q_1, q_2\}$, we can choose $\rho \in (0, 1)$ such that

$$I(u, v) \geq \sigma \quad \text{if } \|(u, v)\| = \rho.$$

Part (b). The proof follows by taking $(u_0, v_0) \in W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$ with $u_{0_+}.v_{0_+} \not\equiv 0$. Then, defining $(u_t, v_t) = (t^{1/p}u_0, t^{\frac{1}{q}}v_0)$, for $t > 0$, we obtain

$$I(u_t, v_t) \leq \Big(\frac{1}{p}\|u_0\|^p + \frac{1}{q}\|v_0\|^q\Big)t - \mu t^{\frac{\alpha}{p}+\frac{\gamma}{q}}\int_\Omega |x|^{-\beta_2}u_{0_+}^\alpha v_{0_+}^\gamma\,dx \to -\infty, \tag{2.4}$$

as $t \to \infty$. $\square$

From the mountain pass theorem [6] we get a $(PS)_c$-sequence $\{(u_n, v_n)\}$ in $W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$, where

$$0 < \sigma \leq c = \inf_{h\in\Gamma}\max_{t\in[0,1]} I(h(t)) \tag{2.5}$$

and

$$\Gamma = \big\{h \in C([0, 1], W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})) : h(0) = 0,\, h(1) = e\big\}, \tag{2.6}$$

with $I(e) \equiv I(t_0u_0, t_0v_0) < 0$.

Lemma 2.5. *In addition to* (1.3) *and* (1.4), *assume that one of the two following conditions hold:*

- (i) *the case* (1.5), $p_i \in (p, p^*)$, $q_i \in (q, q^*)$, *with* $\theta/p_1 + \delta/q_1 = \alpha/p_2 + \gamma/q_2 = 1$, *and* β_i *as in* (1.7), *for* $i = 1, 2$.
- (ii) *the case* (1.6), $p_1 \in (p, p^*)$, $q_1 \in (q, q^*)$, *with* $\theta/p_1 + \delta/q_1 = 1$, β_1 *as in* (1.7), $p_2 = p^*$, $q_2 = q^*$, *and* $\beta_2 = c_1p^* = c_2q^*$.

Let $\{(u_n, v_n)\} \subset W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$ *be a* $(PS)_c$*-sequence. Then* $\{(u_{n_+}, v_{n_+})\}$ *is a* $(PS)_c$*-sequence which is bounded uniformly in* $\mu > 0$.

Proof. Let $\theta_1 = \min\{p_1, p_2\}$ and $\theta_2 = \min\{q_1, q_2\}$, we have

$$\begin{aligned}c + \|(u_n, v_n)\| + O_n(1) &\geq I(u_n, v_n) - \langle I'(u_n, v_n), (u_n/\theta_1, v_n/\theta_2)\rangle\\ &\geq (\frac{1}{p} - \frac{1}{\theta_1})\|u_n\|^p + (\frac{1}{q} - \frac{1}{\theta_2})\|v_n\|^q\\ &\quad + \lambda(\frac{\theta}{\theta_1} + \frac{\delta}{\theta_2} - 1)\int_\Omega |x|^{-\beta_1}u_{n_+}^\theta v_{n_+}^\delta\,dx\\ &\quad + \mu(\frac{\alpha}{\theta_1} + \frac{\gamma}{\theta_2} - 1)\int_\Omega |x|^{-\beta_2}u_{n_+}^\alpha v_{n_+}^\gamma\,dx\\ &\geq (\frac{1}{p} - \frac{1}{\theta_1})\|u_n\|^p + (\frac{1}{q} - \frac{1}{\theta_2})\|v_n\|^q.\end{aligned}$$

Therefore, independently of $\lambda \geq 0$ and $\mu > 0$, we conclude that $\{(u_n, v_n)\}$ is a bounded sequence in $W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$. In particular, we have that $\{(u_{n_-}, v_{n_-})\}$ and $\{(u_{n_+}, v_{n_+})\}$ are bounded sequences in $W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$, then

$$-\|u_{n_-}\|^p = \big\langle I'(u_n, v_n), (u_{n_-}, 0)\big\rangle \to 0 \quad \text{as } n \to \infty \tag{2.7}$$

and similarly

$$-\|v_{n_-}\|^q = \big\langle I'(u_n,v_n),(0,v_{n_-})\big\rangle \to 0 \quad \text{as} \ \ n\to\infty. \tag{2.8}$$

Moreover, we get

$$I(u_{n_+},v_{n_+}) = I(u_n,v_n) + \frac{1}{p}\|u_{n_-}\|^p + \frac{1}{q}\|v_{n_-}\|^q = I(u_n,v_n) + O_n(1).$$

Therefore, from (2.7) and (2.8), we obtain $I(u_{n_+},v_{n_+}) \to c$ as $n\to\infty$. Similarly, if $(w,z)\in W_0^{1,p}(\Omega,|x|^{-ap})\times W_0^{1,q}(\Omega,|x|^{-bq})$, we prove that

$$\langle I'(u_{n+},v_{n+}),(w,z)\rangle = \langle I'(u_n,v_n),(w,z)\rangle + O_n(1),$$

hence $I'(u_{n_+},u_{n_+})\to 0$ as $n\to\infty$. □

3. Proof of theorem 1.1

Lemma 3.1. *Suppose that* (1.3) *and* (1.4) *hold. Assume that $p_i\in(p,p^*)$, $q_i\in(q,q^*)$, $i=1,2$, with $\theta/p_1+\delta/q_1=\alpha/p_2+\gamma/q_2=1$, and $\beta_i, i=1,2$, as in* (1.7). *Then, every $(PS)_c$-sequence $\{(u_n,v_n)\}$ with $u_n,v_n\geq 0$, for a.e. in Ω, is precompact.*

Proof. From lemma 2.5, the sequence $\{(u_n,v_n)\}$ is bounded in $W_0^{1,p}(\Omega,|x|^{-ap})\times W_0^{1,q}(\Omega,|x|^{-bq})$. We can assume, passing to a subsequence if necessary, there exists $(u,v)\in W_0^{1,p}(\Omega,|x|^{-ap})\times W_0^{1,q}(\Omega,|x|^{-bq})$ satisfying $u_n\rightharpoonup u$ and $v_n\rightharpoonup v$ weakly, as $n\to\infty$. From the compact embedding theorem [35, Theorem 2.1], we obtain

$$\begin{gathered}
u_n\to u \quad \text{in } L^{p_1}(\Omega,|x|^{-\beta_1})\cap L^{p_2}(\Omega,|x|^{-\beta_2}) \quad \text{as} \ \ n\to\infty,\\
v_n\to v \quad \text{in } L^{q_1}(\Omega,|x|^{-\beta_1})\cap L^{q_2}(\Omega,|x|^{-\beta_2}) \quad \text{as} \ \ n\to\infty.
\end{gathered}$$

Since there exist $f\in L^{p_1}(\Omega,|x|^{-\beta_1})$ and $g\in L^{q_1}(\Omega,|x|^{-\beta_1})$ such that $|u_n|(x)\leq f(x)$ and $|v_n|(x)\leq g(x)$, for a.e. $x\in\Omega$ and all $n\in\mathbb{N}$, applying the Lebesgue's dominated convergence theorem we infer that

$$\lim_{n\to\infty}\int_\Omega |x|^{-\beta_1}u_n^{\theta-1}v_n^{\delta}(u_n-u)dx = 0\,, \tag{3.1}$$

and similarly

$$\lim_{n\to\infty}\int_\Omega |x|^{-\beta_2}u_n^{\alpha-1}v_n^{\gamma}(u_n-u)dx = 0. \tag{3.2}$$

Now, taking the upper limit in the equation

$$\begin{aligned}
&\int_\Omega |x|^{-ap}\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\nabla(u_n-u)dx\\
&= \langle I'(u_n,v_n),(u_n-u,0)\rangle - \int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla(u_n-u)dx\\
&\quad + \lambda\theta\int_\Omega |x|^{-\beta_1}u_n^{\theta-1}v_n^{\delta}(u_n-u)dx + \mu\alpha\int_\Omega |x|^{-\beta_2}u_n^{\alpha-1}v_n^{\gamma}(u_n-u)dx\,.
\end{aligned}$$

Using the definition of $(PS)_c$-sequence, the weak convergence, (3.1), and (3.2), we obtain

$$\limsup_{n\to\infty}\int_\Omega |x|^{-ap}\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\nabla(u_n-u)dx = 0.$$

Consequently, by a well known lemma (see e.g. [20, lemma 4.1]) we achieve, up to a subsequence, that $u_n \to u$ strongly in $W_0^{1,p}(\Omega, |x|^{-ap})$ as $n \to \infty$. Analogously, we get $v_n \to v$ strongly in $W_0^{1,q}(\Omega, |x|^{-bq})$ as $n \to \infty$. □

Proof of theorem 1.1. By combining lemmata 2.4 and 2.5, there exists a $(PS)_c$-sequence $\{(u_n, v_n)\}$ in $W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$ with $u_n, v_n \geq 0$, for a.e. in Ω. Moreover, from lemma 3.1 there exist $(u, v) \in W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$ and a subsequence of $\{(u_n, v_n)\}$, that we will denote by $\{(u_n, v_n)\}$, such that $u_n \to u$ strongly in $W_0^{1,p}(\Omega, |x|^{-ap})$ and $v_n \to v$ strongly in $W_0^{1,q}(\Omega, |x|^{-bq})$, as $n \to \infty$. Then, we conclude that

$$I(u, v) = c > 0 \quad \text{and} \quad I'(u, v) = 0,$$

that is, (u, v) is a nonnegative weak solution of system (1.2). Moreover, it is easy to check that $u, v \not\equiv 0$. □

4. Proof of theorem 1.2

First of all, notice that by lemma 2.4 the geometric conditions of the mountain pass theorem for the functional I are satisfied.

The next three lemmata are crucial in the proof of this theorem.

Lemma 4.1. *Let $\{(u_n, v_n)\} \subset (W_0^{1,p}(\Omega, |x|^{-ap}))^2$ be a bounded $(PS)_c$-sequence such that $u_n, v_n \geq 0$, for a.e. in Ω, and there exists $(u, v) \in (W_0^{1,p}(\Omega, |x|^{-ap}))^2$ satisfying $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega, |x|^{-ap})$, as $n \to \infty$. Then, (u, v) is a weak solution of system* (1.2) *and $u, v \geq 0$ for a.e. in Ω.*

Proof. Arguing as in the proof of lemma 3.1, by combining the compact embedding theorem [35, Theorem 2.1] with the Lebesgue's dominated convergence theorem, we obtain that $u, v \geq 0$ for a.e. in Ω,

$$\lim_{n\to\infty} \int_\Omega |x|^{-\beta_1} u_n^{\theta-1} v_n^{\delta} w dx = \int_\Omega |x|^{-\beta_1} u^{\theta-1} v^{\delta} w dx, \quad \forall w \in W_0^{1,p}(\Omega, |x|^{-ap}), \tag{4.1}$$

and

$$\lim_{n\to\infty} \int_\Omega |x|^{-\beta_1} u_n^{\theta} v_n^{\delta-1} z dx = \int_\Omega |x|^{-\beta_1} u^{\theta} v^{\delta-1} z dx, \quad \forall z \in W_0^{1,p}(\Omega, |x|^{-ap}). \tag{4.2}$$

Notice that $\nabla u_n(x) \to \nabla u(x)$ and $\nabla v_n(x) \to \nabla v(x)$, for a.e. $x \in \Omega$, as $n \to \infty$. These facts can be proved arguing as in [9] (see also [8, 20, 22]).

Since $\{(u_n, v_n)\}$ is bounded in $(W_0^{1,p}(\Omega, |x|^{-ap}))^2$, we have $\{|\nabla u_n|^{p-2} \nabla u_n\}$ and $\{|\nabla v_n|^{p-2} \nabla v_n\}$ are bounded in $(L^{\frac{p}{p-1}}(\Omega, |x|^{-ap}))^N$. On the other hand, since $\alpha + \gamma = p^*$, by the Hölder's inequality, we infer that $\{{u_n}^{\alpha-1} {v_n}^{\gamma}\}$ and $\{{u_n}^{\alpha} {v_n}^{\gamma-1}\}$ are bounded in $L^{\frac{p^*}{p^*-1}}(\Omega, |x|^{-e_1 p^*})$. Therefore, by lemma 2.3 we get

$$\nabla u_n \rightharpoonup \nabla u \quad \text{and } \nabla v_n \rightharpoonup \nabla v \quad \text{weakly in } (L^{\frac{p}{p-1}}(\Omega, |x|^{-ap}))^N \tag{4.3}$$

and

$$u_n^{\alpha} v_n^{\gamma-1} \rightharpoonup u^{\alpha} v^{\gamma-1}, \quad u_n^{\alpha-1} v_n^{\gamma} \rightharpoonup u^{\alpha-1} v^{\gamma} \quad \text{weakly in } L^{\frac{p^*}{p^*-1}}(\Omega, |x|^{-c_1 p^*}), \tag{4.4}$$

as $n \to \infty$. Consequently, using (4.1) – (4.4) we obtain

$$\langle I'(u, v), (w, z)\rangle = \lim_{n\to\infty} \langle I'(u_n, v_n), (w, z)\rangle = 0, \quad \forall (w, z) \in (W_0^{1,p}(\Omega, |x|^{-ap}))^2,$$

that is, (u, v) is a weak solution of system (1.2). □

Lemma 4.2. *In addition to* (1.3), (1.4), *and* (1.6), *assume that* $p = q$, $0 \leq a = b$, $p_1 = q_1 \in (p, p^*)$, *with* $\theta/p_1 + \delta/q_1 = 1$, $p^* = q^*$, *and* $\beta_2 = c_1 p^*$. *Then, all the Palais Smale sequences* $\{(u_n, v_n)\} \subset (W_0^{1,p}(\Omega, |x|^{-ap}))^2$ *for the operator* I *at the level* c, *with* $u_n, v_n \geq 0$ *for a.e. in* Ω, *are precompact provided that*

$$c < (\frac{1}{p} - \frac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}} \tilde{S}^{\frac{p^*}{p^*-p}} + K(\lambda), \tag{4.5}$$

where

$$K(\lambda) = \lambda p_1(\frac{1}{p} - \frac{1}{p_1}) \lim_{n\to\infty} \int_\Omega |x|^{-\beta_1} u_n^\theta v_n^\delta dx.$$

Proof. By Lemma 2.5 the sequence $\{(u_n, v_n)\}$ is bounded in $(W_0^{1,p}(\Omega, |x|^{-ap}))^2$; consequently, there exists $(u, v) \in (W_0^{1,p}(\Omega, |x|^{-ap}))^2$ such that $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega, |x|^{-ap})$, as $n \to \infty$. Then, by combining the compact embedding theorem [35, Theorem 2.1] with the Lebesgue's dominated convergence theorem, we infer that $u_n(x) \to u(x)$, $v_n(x) \to v(x)$, for a.e. in Ω, as $n \to \infty$, and

$$\lim_{n\to\infty} \int_\Omega |x|^{-\beta_1} u_n^\theta v_n^\delta dx = \int_\Omega |x|^{-\beta_1} u^\theta v^\delta dx. \tag{4.6}$$

Moreover, as in Lemma 4.1 we can suppose that $\nabla u_n(x) \to \nabla u(x)$ and $\nabla v_n(x) \to \nabla v(x)$, for a.e. $x \in \Omega$, as $n \to \infty$.

Define $\tilde{u}_n = u_n - u$ and $\tilde{v}_n = v_n - v$. By Brezis and Lieb [10, Theorem 1] we have

(i) $\|u_n\|^p = \|\tilde{u}_n\|^p + \|u\|^p + O_n(1)$, as $n \to \infty$.

(ii) $\|v_n\|^p = \|\tilde{v}_n\|^p + \|v\|^p + O_n(1)$, as $n \to \infty$.

(iii)
$$\begin{aligned} &\int_\Omega |x|^{-c_1 p^*} |u_n|^\alpha |v_n|^\gamma dx - \int_\Omega |x|^{-c_1 p^*} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\gamma dx \\ &= \int_\Omega |x|^{-c_1 p^*} |u|^\alpha |v|^\gamma dx + O_n(1), \quad \text{as } n \to \infty. \end{aligned}$$

We recall that the proof of identity **iii.** follows arguing as in [27, Lemma 8].

By Lemma 4.1 we have that (u, v) is a weak solution of system (1.2), that is, $\langle I'(u, v), (w, z)\rangle = 0$ for all $(w, z) \in (W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}))^2$. By using (4.6) and (i)–(iii), we get

$$\begin{aligned} &\|\tilde{u}_n\|^p - \mu\alpha \int_\Omega |x|^{-c_1 p^*} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\gamma dx \\ &= \|u_n\|^p - \|u\|^p - \mu\alpha \Big[\int_\Omega |x|^{-c_1 p^*} u_n^\alpha v_n^\gamma dx - \int_\Omega |x|^{-c_1 p^*} u^\alpha v^\gamma dx\Big] + O_n(1) \\ &= \langle I'(u_n, v_n), (u_n, 0)\rangle - \langle I'(u, v), (u, 0)\rangle + O_n(1) \\ &= O_n(1), \quad \text{as } n \to \infty. \end{aligned}$$

Analogously, we obtain

$$\|\tilde{v}_n\|^p - \mu\gamma \int_\Omega |x|^{-c_1 p^*} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\gamma dx = O_n(1).$$

Thus, we can take $l \geq 0$ such that

$$l = \lim_{n\to\infty} \frac{\|\tilde{u}_n\|^p}{\alpha} = \lim_{n\to\infty} \frac{\|\tilde{v}_n\|^p}{\gamma} = \mu \lim_{n\to\infty} \int_\Omega |x|^{-c_1 p^*} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\gamma dx.$$

If $l = 0$ the result is proved. Suppose by contradiction that $l > 0$. By the definition of $(PS)_c$-sequence we get

$$
\begin{aligned}
& c + O_n(1) \\
&= I(u_n, v_n) - \frac{1}{p_1}\langle I'(u_n, v_n), (u_n, v_n)\rangle \\
&= (\frac{1}{p} - \frac{1}{p_1})(\|u_n\|^p + \|v_n\|^p) + \mu(\frac{\alpha+\gamma}{p_1} - 1)\int_\Omega |x|^{-c_1 p^*} u_n^\alpha v_n^\gamma dx \\
&= (\frac{1}{p} - \frac{1}{p_1})(\|\tilde{u}_n\|^p + \|\tilde{v}_n\|^p) + (\frac{1}{p} - \frac{1}{p_1})(\|u\|^p + \|v\|^p) \\
&\quad + \mu(\frac{p^*}{p_1} - 1)\Big[\int_\Omega |x|^{-c_1 p^*} \tilde{u}_n^\alpha \tilde{v}_n^\gamma dx + \int_\Omega |x|^{-c_1 p^*} u_n^\alpha v_n^\gamma dx\Big] + O_n(1) \\
&= (\frac{1}{p} - \frac{1}{p_1})p^* l + (\frac{1}{p} - \frac{1}{p_1})(\|u\|^p + \|v\|^p) \\
&\quad + (\frac{1}{p_1} - \frac{1}{p^*})p^*\Big[l + \mu\int_\Omega |x|^{-c_1 p^*} u_n^\alpha v_n^\gamma dx\Big] + O_n(1) \\
&= (\frac{1}{p} - \frac{1}{p^*})p^* l + (\frac{1}{p} - \frac{1}{p_1})\Big[\lambda p_1 \int_\Omega |x|^{-\beta_1} u^\theta v^\delta dx + \mu p^* \int_\Omega |x|^{-c_1 p^*} u^\alpha v^\gamma dx\Big] \\
&\quad + \mu(\frac{1}{p_1} - \frac{1}{p^*})p^* \int_\Omega |x|^{-c_1 p^*} u_n^\alpha v_n^\gamma dx + O_n(1) \\
&= (\frac{1}{p} - \frac{1}{p^*})p^* l + \lambda p_1(\frac{1}{p} - \frac{1}{p_1})\int_\Omega |x|^{-\beta_1} u^\theta v^\delta dx \\
&\quad + \mu(\frac{1}{p} - \frac{1}{p^*})p^* \int_\Omega |x|^{-c_1 p^*} u_n^\alpha v_n^\gamma dx + O_n(1) \\
&\geq (\frac{1}{p} - \frac{1}{p^*})p^* l + \lambda p_1(\frac{1}{p} - \frac{1}{p_1})\int_\Omega |x|^{-\beta_1} u^\theta v^\delta dx + O_n(1).
\end{aligned}
\tag{4.7}
$$

Using the definition of $\tilde{S}$ we have

$$\Big(\int_\Omega |x|^{-c_1 p^*} u_n^\alpha v_n^\gamma dx\Big)^{p/p^*} \tilde{S} \leq \|u_n\|^p + \|v_n\|^p, \ \forall n.$$

Hence, taking the limit in the above inequality we get

$$(\frac{l}{\mu})^{p/p^*} \tilde{S} \leq (\alpha+\gamma) l = p^* l$$

then

$$l \geq (\mu)^{\frac{-p}{p^*-p}} (p^*)^{\frac{-p^*}{p^*-p}} \tilde{S}^{\frac{p^*}{p^*-p}}. \tag{4.8}$$

Substituting (4.8) in (4.7) and taking the limit, we obtain

$$c \geq (\frac{1}{p} - \frac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}} \tilde{S}^{\frac{p^*}{p^*-p}} + K(\lambda),$$

which contradicts the inequality (4.5). □

Lemma 4.3. *We can choose e in* (2.6) *such that c given by* (2.5) *satisfies*

$$c < (\frac{1}{p} - \frac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}} \tilde{S}^{\frac{p^*}{p^*-p}}. \tag{4.9}$$

Proof. Let us consider $s_0 = s_1(s_1^\alpha t_1^\gamma)^{\frac{-1}{p^*}}$ and $t_0 = t_1(s_1^\alpha t_1^\gamma)^{\frac{-1}{p^*}}$, where $s_1, t_1 > 0$ and $s_1/t_1 = (\alpha/\gamma)^{1/p}$, and u_ϵ the function defined in lemma 2.2. Then, it is suffices to prove that there exists $\epsilon > 0$ such that

$$\sup_{t\geq 0} I(t(s_0 u_\epsilon), t(t_0 u_\epsilon)) < (\frac{1}{p} - \frac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}} \tilde{S}^{\frac{p^*}{p^*-p}}.$$

Due to the geometric conditions of the mountain pass theorem, for each $\epsilon > 0$, there exists $t_\epsilon > 0$ such that

$$0 < \sigma \leq \sup_{t\geq 0} I(t(s_0 u_\epsilon), t(t_0 u_\epsilon)) = I(t_\epsilon(s_0 u_\epsilon), t_\epsilon(t_0 u_\epsilon)).$$

Moreover, supposing by contradiction that there exists a subsequence $\{t_{\epsilon_n}\}$ with $t_{\epsilon_n} \to 0$ as $n \to \infty$, we obtain

$$\begin{aligned}
0 < \sigma &\leq I(t_{\epsilon_n}(s_0 u_{\epsilon_n}), t_{\epsilon_n}(t_0 u_{\epsilon_n})) \\
&\leq \frac{t_{\epsilon_n}^p s_0^p}{p}\|u_{\epsilon_n}\|^p + \frac{t_{\epsilon_n}^p t_0^p}{p}\|u_{\epsilon_n}\|^p \\
&\leq \frac{t_{\epsilon_n}^p}{p}(s_0^p + t_0^p)(\tilde{S}_{a,p,R} + O(\epsilon_n^{(N-d_1 p)/d_1 p})) \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which is an absurd. Then, there exists $l > 0$ with $t_\epsilon \geq l$, for all $\epsilon > 0$. Consequently, by using lemma 2.2 and putting $c_0 = l^{p_1} s_0^\theta t_0^\delta$, we get

$$\sup_{t\geq 0} I(t(s_0 u_\epsilon), t(t_0 u_\epsilon)) \leq \frac{t_\epsilon^p}{p}\Big(\frac{s_1^p + t_1^p}{(s_1^\alpha t_1^\gamma)^{p/p^*}}\|u_\epsilon\|^p\Big) - \lambda c_0 \int_\Omega |x|^{-\beta_1} u_\epsilon^{p_1} dx - \mu t_\epsilon^{p^*}. \quad (4.10)$$

Note that

$$t_{1_\epsilon} = (\mu p^*)^{\frac{-1}{p^*-p}} \Big(\frac{s_1^p + t_1^p}{(s_1^\alpha t_1^\gamma)^{p/p^*}}\Big)^{\frac{1}{p^*-p}} \|u_\epsilon\|^{\frac{p}{p^*-p}} \quad (4.11)$$

is the unique maximum point of $f_\epsilon : (0, \infty) \to \mathbb{R}$, given by

$$f_\epsilon(t) = \frac{(s_1^p + t_1^p)\, t^p}{(s_1^\alpha t_1^\gamma)^{p/p^*} p}\|u_\epsilon\|^p - \mu t^{p^*}.$$

Also we know that

$$(A + B)^k \leq A^k + k(A + B)^{k-1} B, \quad (4.12)$$

for all $A, B \geq 0$ and $k \geq 1$ [26]. Observe that the following identity holds

$$\Big[\frac{s_1^p + t_1^p}{(s_1^\alpha t_1^\gamma)^{p/p^*}}\Big] = \Big[(\alpha/\gamma)^{\gamma/p^*} + (\alpha/\gamma)^{-\alpha/p^*}\Big]. \quad (4.13)$$

By the Caffarelli-Kohn-Nirenberg's inequality, $W_0^{1,p}(\Omega, |x|^{-ap}) \subset W_{a,c_1}^{1,p}(\mathbb{R}^N)$. Then

$$\tilde{S}_{a,p} \leq C_{a,p}^*. \quad (4.14)$$

Substituting (4.11) in (4.10), from (4.12), (4.13), (4.14), and using lemma 2.2, we obtain

$$\begin{aligned}
\sup_{t\geq 0} I(t(s_0u_\epsilon), t(t_0u_\epsilon)) &\leq (\frac{1}{p}-\frac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\Big\{[(\frac{\alpha}{\gamma})^{\gamma/p^*}+(\frac{\alpha}{\gamma})^{-\alpha/p^*}]\tilde{S}_{a,p,R} \\
&\quad + O(\epsilon^{\frac{N-d_1p}{d_1p}})\Big\}^{\frac{p^*}{p^*-p}} - \lambda c_0\int_\Omega |x|^{-\beta_1}u_\epsilon^{p_1}dx \\
&\leq (\frac{1}{p}-\frac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\Big\{[(\frac{\alpha}{\gamma})^{\frac{\gamma}{p^*}}+(\frac{\alpha}{\gamma})^{\frac{-\alpha}{p^*}}]\tilde{S}_{a,p}\Big\}^{\frac{p^*}{p^*-p}} \\
&\quad + O(\epsilon^{\frac{N-d_1p}{d_1p}}) - \lambda c_0\int_\Omega |x|^{-\beta_1}u_\epsilon^{p_1}dx \\
&\leq (\frac{1}{p}-\frac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\Big\{[(\frac{\alpha}{\gamma})^{\frac{\gamma}{p^*}}+(\frac{\alpha}{\gamma})^{\frac{-\alpha}{p^*}}]C^*_{a,p}\Big\}^{\frac{p^*}{p^*-p}} \\
&\quad + O(\epsilon^{\frac{N-d_1p}{d_1p}}) - \lambda c_0\int_\Omega |x|^{-\beta_1}u_\epsilon^{p_1}dx
\end{aligned} \tag{4.15}$$

Now, from lemma 2.1 and (4.15), we get

$$\begin{aligned}
\sup_{t\geq 0} I(t(s_0u_\epsilon), t(t_0u_\epsilon)) &\leq (\frac{1}{p}-\frac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\tilde{S}^{\frac{p^*}{p^*-p}} + O(\epsilon^{\frac{N-d_1p}{d_1p}}) \\
&\quad - \lambda c_0\int_\Omega |x|^{-\beta_1}u_\epsilon^{p_1}dx.
\end{aligned} \tag{4.16}$$

Supposing that $c < \frac{(p_1-p+1)N-(a+1)p_1}{p-1} - \frac{(N-p-ap)(p_1-p)}{p(p-1)}$ we have

$$\frac{(N-p_1+ap_1+c)(p-1)(N-d_1p)}{d_1p(N-p-ap)} - \frac{(N-d_1p)(p-1)p_1}{d_1p^2} < \frac{N-d_1p}{d_1p},$$

then, by lemma 2.2 and by (4.16), we can take a $\epsilon > 0$ small enough such that

$$\begin{aligned}
\sup_{t\geq 0} I(t(s_0u_\epsilon), t(t_0u_\epsilon)) &\leq (\frac{1}{p}-\frac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\tilde{S}^{\frac{p^*}{p^*-p}} + O(\epsilon^{\frac{N-d_1p}{d_1p}}) \\
&\quad - O(\epsilon^{\frac{(N-d_1p)(p-1)(N-p_1-ap_1+c)}{d_1p(N-p-ap)} - \frac{(N-d_1p)(p-1)p_1}{d_1p^2}}) \\
&< (\frac{1}{p}-\frac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\tilde{S}^{\frac{p^*}{p^*-p}}.
\end{aligned}$$

This completes the proof. □

Proof of theorem 1.2. From lemmata 2.4, 2.5, and 4.3, there exists a bounded $(PS)_c$-sequence $\{(u_n, v_n)\}$ in $(W_0^{1,p}(\Omega, |x|^{-ap}))^2$ with $c > 0$ satisfying (4.9) and $u_n, v_n \geq 0$ for a.e. in Ω. Since that $p_1 \in (p, p^*)$, it follows that c verifies (4.5). Thus, we have by lemma 4.2 that there exists $(u, v) \in (W_0^{1,p}(\Omega, |x|^{-ap}))^2$ with $u_n \to u$ and $v_n \to v$ in $W_0^{1,p}(\Omega, |x|^{-ap})$, as $n \to \infty$. Hence, we conclude

$$I(u,v) = c > 0 \quad \text{and} \quad I'(u,v) = 0,$$

that is, (u, v) is a nontrivial and nonnegative weak solution of system (1.2). □

5. Proof of theorem 1.3

The proof follows the steps the proof of theorem 1.2. By lemmata 2.4 and 2.5, there exists a $(PS)_c$-sequence $\{(u_n, v_n)\}$ in $W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$ with $c > 0$ given as in (2.5) and $u_n, v_n \geq 0$ for a.e. in Ω. Moreover, $\{(u_n, v_n)\}$ is bounded uniformly in $\mu > 0$, that is, there exist $M > 0$ such that $\|(u_n, v_n)\| \leq M$ for all $n \in \mathbb{N}$, uniformly in $\mu > 0$. Consequently, we get that $c \leq \overline{M}$ uniformly in $\mu > 0$.

Due to the boundedness of $\{(u_n, v_n)\}$, there exists a subsequence, that we will denote by $\{(u_n, v_n)\}$, and $(u, v) \in W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$ with $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, |x|^{-ap})$ and $v_n \rightharpoonup v$ weakly in $W_0^{1,q}(\Omega, |x|^{-bq})$, as $n \to \infty$. Then, arguing as in lemma 4.1 we obtain that (u, v) is a weak solution of system (1.2) with $u, v \geq 0$ for a.e. in Ω.

Now, we will prove that there exists $\mu_0 > 0$ such that u, v is nontrivial, provided that $0 < \mu < \mu_0$.

Supposing by contradiction that $u(x) \equiv 0$ for a.e. $x \in \Omega$ and proceeding as in the proof of theorem 1.2, we obtain $l > 0$ such that

$$l = \lim_{n\to\infty} \frac{\|u_n\|^p}{\alpha} = \lim_{n\to\infty} \frac{\|v_n\|^q}{\gamma} = \mu \lim_{n\to\infty} \int_\Omega |x|^{-c_1 p^*} u_n^\alpha v_n^\gamma dx$$

and

$$c = \lim_{n\to\infty} I(u_n, v_n) = (\frac{\alpha}{p} + \frac{\gamma}{q} - 1) l > 0. \tag{5.1}$$

On the other hand, by Young's inequality and definitions of $C^*_{a,p}$ and $C^*_{b,q}$, we obtain

$$\begin{aligned} \frac{l}{\mu} &\leq \frac{\alpha^{(p^*+p)/p}}{p^*} (C^*_{a,p})^{-p^*/p} l^{p^*/p} + \frac{\gamma^{(q^*+q)/q}}{q^*} (C^*_{b,q})^{-q^*/q} l^{q^*/q} \\ &\leq \big[\frac{\alpha^{(p^*+p)/p}}{p^*} (C^*_{a,p})^{-p^*/p} + \frac{\gamma^{(q^*+q)/q}}{q^*} (C^*_{b,q})^{-q^*/q}\big] l^\tau, \end{aligned}$$

where $\tau = \max\{p^*/p, q^*/q\}$ if $l > 1$, and $\tau = \min\{p^*/p, q^*/q\}$ if $l \leq 1$. Therefore,

$$l \geq \Big[\mu\Big(\frac{\alpha^{(p^*+p)/p}}{p^*} (C^*_{a,p})^{-p^*/p} + \frac{\gamma^{(q^*+q)/q}}{q^*} (C^*_{b,q})^{-q^*/q}\Big)\Big]^{\frac{-1}{\tau-1}}.$$

Thus substituting the above inequality in (5.1) and taking $\mu_0 > 0$ small enough we conclude

$$c \geq (\frac{\alpha}{p} + \frac{\gamma}{q} - 1)\Big[\mu\Big(\frac{\alpha^{(p^*+p)/p}}{p^*} (C^*_{a,p})^{-p^*/p} + \frac{\gamma^{(q^*+q)/q}}{q^*} (C^*_{b,q})^{-q^*/q}\Big)\Big]^{\frac{-1}{\tau-1}} \geq \overline{M},$$

for all $0 < \mu < \mu_0$, which is an absurd.

6. Appendix

Proof of lemma 2.2. From equation (2.1) we obtain

$$\begin{gathered} \|\nabla y_\epsilon\|^p_{L^p(\mathbb{R}^N, |x|^{-ap})} = (\tilde{S}_{a,p,R})^{p^*/(p^*-p)} = (k_{a,p}(\epsilon))^p \|\nabla U_{a,p,\epsilon}\|^p_{L^p(\mathbb{R}^N, |x|^{-ap})}, \\ \|y_\epsilon\|^{p^*}_{L^{p^*}(\mathbb{R}^N, |x|^{-c_1 p^*})} = (\tilde{S}_{a,p,R})^{p^*/(p^*-p)} = (k_{a,p}(\epsilon))^{p^*} \|U_{a,p,\epsilon}\|^{p^*}_{L^{p^*}(\mathbb{R}^N, |x|^{-c_1 p^*})}. \end{gathered}$$

We observe that

$$\nabla(\psi(x) U_{a,p,\epsilon}(x)) = \begin{cases} \nabla U_{a,p,\epsilon}(x) & \text{if } |x| < R_0 \\ U_{a,p,\epsilon}(x) \nabla\psi(x) + \psi(x) \nabla U_{a,p,\epsilon}(x) & \text{if } R_0 \leq |x| < 2R_0 \\ 0 & \text{if } |x| \geq 2R_0 \end{cases}$$

and

$$\nabla U_{a,p,\epsilon}(x) = -\frac{N-p-ap}{p-1} \cdot \frac{|x|^{[d_1p(N-p-ap)/(p-1)(N-d_1p)]-2}x}{(\epsilon + |x|^{d_1p(N-p-ap)/(p-1)(N-d_1p)})^{N/d_1p}}.$$

Therefore,

$$\begin{aligned}\int_\Omega |x|^{-ap}|\nabla(\psi U_{a,p,\epsilon})(x)|^p dx &= O(1) + \int_{\mathbb{R}^N} |x|^{-ap}|\nabla U_{a,p,\epsilon}(x)|^p dx \\ &= O(1) + (\tilde{S}_{a,p,R})^{\frac{p^*}{p^*-p}}(k_{a,p}(\epsilon))^{-p}\end{aligned}$$

and

$$\begin{aligned}\int_\Omega |x|^{-c_1p^*}|\psi(x)U_{a,p,\epsilon}(x)|^{p^*} dx &= O(1) + \int_{\mathbb{R}^N} |x|^{-c_1p^*}|U_{a,p,\epsilon}(x)|^{p^*} dx \\ &= O(1) + (\tilde{S}_{a,p,R})^{\frac{p^*}{p^*-p}}(k_{a,p}(\epsilon))^{-p^*}.\end{aligned}$$

Consequently,

$$\begin{aligned}\int_\Omega |x|^{-ap}|\nabla u_\epsilon(x)|^p dx &= \frac{O(1) + (\tilde{S}_{a,p,R})^{p^*/(p^*-p)}(k_{a,p}(\epsilon))^{-p}}{[O(1) + (\tilde{S}_{a,p,R})^{p^*/(p^*-p)}(k_{a,p}(\epsilon))^{-p^*}]^{p/p^*}} \\ &= \frac{(k_{a,p}(\epsilon))^p[O(1) + (\tilde{S}_{a,p,R})^{p^*/(p^*-p)}(k_{a,p}(\epsilon))^{-p}]}{[O(k_{a,p}(\epsilon)^{p^*}) + (\tilde{S}_{a,p,R})^{p^*/(p^*-p)}]^{p/p^*}} \\ &\le \tilde{S}_{a,p,R} + O(k_{a,p}(\epsilon)^p) \\ &= \tilde{S}_{a,p,R} + O(\epsilon^{(N-d_1p)/d_1p}).\end{aligned}$$

Now, we prove that $\|u_\epsilon\|^{p_1}_{L^{p_1}(\Omega,|x|^{-\beta_1})}$ is as in (2.2). Considering the changes of variables by the polar coordinates and $s = R_0^{-1}\epsilon^{-1/\alpha}r$ with $\alpha = \frac{d_1p(N-p-ap)}{(p-1)(N-d_1p)}$, we obtain

$$\begin{aligned}&\int_\Omega |x|^{-(a+1)p_1+c}|\psi U_{a,p,\epsilon}|^{p_1} dx \\ &\ge O(1) + \int_{|x|<R_0} |x|^{-(a+1)p_1+c}|U_{a,p,\epsilon}|^{p_1}\,dx \\ &= O(1) + \omega_N \int_0^{R_0} \frac{r^{-(a+1)p_1+c+N-1}}{(\epsilon + r^\alpha)^{(N-d_1p)p_1/d_1p}}\,dr \\ &= O(1) + \omega_N (R_0^\alpha \epsilon)^{\frac{-(N-d_1p)p_1}{d_1p} + \frac{(N-p_1-ap_1+c)(p-1)(N-d_1p)}{d_1p(N-p-ap)}} \\ &\quad \times \int_0^{\epsilon^{-1/\alpha}} \frac{s^{-(a+1)p_1+c+N-1}}{(R_0^{-\alpha} + s^\alpha)^{(N-d_1p)p_1/d_1p}}\,ds.\end{aligned} \tag{6.1}$$

Assuming that $c = [(p_1-p+1)N - (a+1)p_1]/(p-1)$, we see that

$$\frac{-(N-d_1p)p_1}{d_1p} + \frac{(p-1)(N-p_1-ap_1+c)(N-d_1p)}{d_1p(N-p-ap)} = 0,$$

$$-(a+1)p_1 + c + N - \frac{(N-p-ap)p_1}{(p-1)} = 0\cdot$$

Therefore, by (6.1),

$$\int_\Omega |x|^{-(a+1)p_1+c}|\psi U_{a,p,\epsilon}|^{p_1} dx \ge \omega_N \int_1^{\epsilon^{-1/\alpha}} \frac{s^{-(a+1)p_1+c+N-1-\alpha\frac{(N-d_1p)p_1}{d_1p}}}{[(R_0s)^{-\alpha}+1]^{(N-d_1p)p_1/d_1p}}\,ds$$

$$\geq \frac{\omega_N}{\big(R_0^{-\alpha}+1\big)^{(N-d_1p)p_1/d_1p}}|\ln(\epsilon)| = O(|\ln(\epsilon)|).$$

Hence, we obtain

$$\begin{aligned}
\int_\Omega |x|^{-(a+1)p_1+c}|u_\epsilon|^{p_1}\,dx &\geq \frac{O(|\ln(\epsilon)|)}{\big[O(1)+(S_{a,p,R})^{p^*/(p^*-p)}(k_{a,p}(\epsilon))^{-p^*}\big]^{p_1/p^*}} \\
&= \frac{O(|\ln(\epsilon)|)}{(k_{a,p}(\epsilon))^{-p_1}\big[O(k_{a,p}(\epsilon)^{p^*})+(S_{a,p,R})^{p^*/(p^*-p)}\big]^{p_1/p^*}} \\
&\geq O(\epsilon^{(N-d_1p)p_1/d_1p^2}|\ln(\epsilon)|).
\end{aligned}$$

Assuming that $c > [(p_1-p+1)N-(a+1)p_1]/(p-1)$, we have

$$\frac{-(N-d_1p)p_1}{d_1p} + \frac{(N-p_1-ap_1+c)(p-1)(N-d_1p)}{d_1p(N-p-ap)} > 0,$$

$$-(a+1)p_1+c+N-\frac{(N-p-ap)p_1}{(p-1)} > 0.$$

Consequently, by (6.1),

$$\begin{aligned}
&\int_\Omega |x|^{-(a+1)p_1+c}|\psi U_{a,p,\epsilon}|^{p_1}\,dx \\
&\geq O(1)+\omega_N(R_0^\alpha\epsilon)^{\frac{-(N-d_1p)p_1}{d_1p}+\frac{(N-p_1-ap_1+c)(p-1)(N-d_1p)}{d_1p(N-p-ap)}} \\
&\quad\times \frac{1}{(R_0^{-\alpha}+1)^{(N-d_1p)p_1/d_1p}}\int_{1/2}^1 s^{-(a+1)p_1+c+N-1}\,ds \geq O(1).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\int_\Omega |x|^{-(a+1)p_1+c}|u_\epsilon|^{p_1}\,dx &\geq \frac{O(1)}{(k_{a,p}(\epsilon))^{-p_1}[O(k_{a,p}(\epsilon)^{p^*})+(S_{a,p,R})^{p^*/(p^*-p)}]^{p_1/p^*}} \\
&\geq \frac{O(k_{a,p}(\epsilon)^{p_1})}{[O(1)+(S_{a,p,R})^{p^*/(p^*-p)}]^{p_1/p^*}} \\
&\geq O(\epsilon^{(N-d_1p)p_1/d_1p^2}).
\end{aligned}$$

If $c < [(p_1-p+1)N-(a+1)p_1]/(p-1)$, we see that

$$\frac{-(N-d_1p)p_1}{d_1p} + \frac{(N-p_1-ap_1+c)(p-1)(N-d_1p)}{d_1p(N-p-ap)} < 0,$$

$$-(a+1)p_1+c+N-\frac{(N-p-ap)p_1}{(p-1)} < 0.$$

Using (6.1) we obtain

$$\begin{aligned}
&\int_\Omega |x|^{-(a+1)p_1+c}|\psi U_\epsilon|^{p_1}\,dx \\
&\geq O(1)\epsilon^{\frac{-(N-d_1p)p_1}{d_1p}+\frac{(N-p_1-ap_1+c)(p-1)(N-d_1p)}{d_1p(N-p-ap)}}\int_{1/2}^1 s^{-(a+1)p_1+c+N-1}ds \\
&\geq O\Big(\epsilon^{\frac{-(N-d_1p)p_1}{d_1p}+\frac{(N-p_1-ap_1+c)(p-1)(N-d_1p)}{d_1p(N-p-ap)}}\Big).
\end{aligned}$$

Hence, we conclude

$$\begin{aligned}\int_\Omega |x|^{-(a+1)p_1+c}|u_\epsilon|^{p_1}dx &\geq \frac{O\big(\epsilon^{\frac{-(N-d_1p)p_1}{d_1p}+\frac{(N-p_1-ap_1+c)(p-1)(N-d_1p)}{d_1p(N-p-ap)}}\big)}{\big[O(1)+(S_{a,p,R})^{p^*/(p^*-p)}(k_{a,p}(\epsilon))^{-p^*}\big]^{p_1/p^*}}\\ &\geq O\Big(\epsilon^{\frac{(N-p_1-ap_1+c)(p-1)(N-d_1p)}{d_1p(N-p-ap)}-\frac{(N-d_1p)(p-1)p_1}{d_1p^2}}\Big).\end{aligned}$$

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