ON THE EXISTENCE OF WEAK SOLUTIONS FOR $p, q$-LAPLACIAN SYSTEMS WITH WEIGHTS

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Abstract. This paper studies degenerate quasilinear elliptic systems involving $p, q$-superlinear and critical nonlinearities with singularities. Existence results are obtained by using properties of the best Hardy-Sobolev constant together with an approach developed by Brezis and Nirenberg.

1. Introduction

In a well-known paper, Brezis and Nirenberg [11] proved that, under certain conditions, the elliptic problem with Dirichlet boundary condition

$$
-\Delta u = \lambda u^q + u^{2^*-1} \quad \text{in } \Omega,
$$

$$
u > 0 \quad \text{in } \Omega,
$$

$$
u = 0 \quad \text{on } \partial \Omega
$$

possesses at least a solution, for all $\lambda > 0$, where $1 < q < 2^* = 2N/(N-2)$, $N \geq 3$, $2^*$ is said to be the critical Sobolev exponent, and $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded smooth domain. In general, the main difficulty in this type of problem is the lack of compactness of the injection $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$.

We recall that the perturbation $\lambda u^q$ is essential in this kind of the problem. By Pohozaev identity [30], problem (1.1) does not possess any solution when $\lambda \leq 0$.

García and Peral in [19] studied the existence of nontrivial solution for a class of problems involving the p-laplacian operator, namely,

$$
-\Delta_p u \equiv -\text{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{q-2}u + \mu|u|^{p^*-2}u \quad \text{in } \Omega,
$$

$$
u \geq 0 \quad \text{in } \Omega,
$$

$$
u = 0 \quad \text{on } \partial \Omega,
$$
in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ ($N > p$), with $1 < p \leq q < p^* = Np/(N-p)$. When $p < q < p^*$, we say that the above problem is $p$-superlinear. These type of problems, which are related to the Brezis and Nirenberg problem [11] (problem (1.1) with $p = 2$), have been widely treated by several authors and we would like...
to mention some of them, e.g., [14][20][21] for $1 < p < N$ and [26][28][29] for $p = 2$, see also references cited there.

Caffarelli, Kohn and Nirenberg in [12] proved that if $1 < p < N$, $-\infty < a < (N-p)/p$, $a \leq c_1 \leq a + 1$, $d_1 = 1 + a - c_1$, and $p^* = p^*(a, c_1, p) := Np/(N-d_1p)$, there exists $C_{a,p} > 0$ such that the following Hardy-Sobolev type inequality with weights is satisfied

$$\left( \int_{\mathbb{R}^N} |x|^{-c_1p^*} |u|^{p^*} dx \right)^{p/p^*} \leq C_{a,p} \left( \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx \right), \forall u \in C_0^\infty(\mathbb{R}^N).$$

Note that several papers have been appeared on this subject, mainly, the works about the existence of solution for a class of quasilinear elliptic problems of the type

$$-Lu_{ap} = g(x,u) + |x|^{-c_1p^*} |u|^{p^*} - 2u \quad \text{in} \quad \Omega,$$

where $Lu_{ap} = \text{div}(|x|^{-ap} \nabla u^{p-2} \nabla u)$, under certain suppositions on the exponents $1 < p < N$, $-\infty < a < (N-p)/p$, $a \leq c_1 \leq a + 1$, $d_1 = 1 + a - c_1$, and $p^* = Np/(N-dp)$, and on the function $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$. See, for instance, [14][17][20][21][22][23][24][25][26][27] and references therein. The lack of compactness is overcome proving that all the Palais-Smale sequence at the level $c$, ($PS_c$)-sequence, in short, with $c < (d/N)(C^*_{a,p})^{N/dp}$, is relatively compact. $(d/N)(C^*_{a,p})^{N/dp}$ is so called the critical level and $C^*_{a,p}$ is the best Hardy-Sobolev constant and it is characterized by

$$C^*_{a,p} = C^*_{a,p}(\Omega) := \inf \left\{ \frac{\int_\Omega |x|^{-ap} |\nabla u|^p dx}{(\int_\Omega |x|^{-c_1p^*} |u|^{p^*} dx)^{p/p^*}} : u \in W^{1,p}_0(\Omega, |x|^{-ap}) \setminus \{0\} \right\}.$$

Besides the great number of the applications known for the scalar case, for instance, in fluid mechanics, in newtonian fluids, in flow through porous media, reaction-diffusion problems, nonlinear elasticity, petroleum extraction, astronomy, glaciology, etc, see [13], the above systems can involve another phenomena, such as competition model in population dynamics, see [18] and reference therein. For the systems case we would like to mention the papers [2][3][22] and a survey paper [17] as well as in the references therein.

In our work, we will use a version of the well-known mountain pass theorem [6] to establish conditions for the existence of a nontrivial solution for a quasilinear elliptic system involving the above operator and a $p,q$-superlinear nonlinear perturbation

$$-Lu_{ap} = \lambda \theta |x|^{-\beta_1|u|^{\theta-2}}|v|^{\delta} u + \mu \alpha |x|^{-\beta_2|u|^{\alpha-2}}|v|^{\gamma} u \quad \text{in} \quad \Omega,$$

$$-Lv_{bq} = \lambda \delta |x|^{-\beta_1|u|^\theta |v|^{\delta-2}} v + \mu \gamma |x|^{-\beta_2|u|^\alpha |v|^{\gamma-2}} v \quad \text{in} \quad \Omega,$$

$$u = v = 0 \quad \text{on} \quad \partial \Omega,$$

where

$$\Omega$$

is a bounded smooth domain of $\mathbb{R}^N$ with $0 \in \Omega$, the parameters $\lambda, \mu$ are positive real numbers and the exponents satisfy

$$1 < p, \quad q < N, \quad -\infty < a < (N-p)/p, \quad -\infty < b < (N-q)/q,$$

$$a \leq c_1 < a + 1, \quad b \leq c_2 < b + 1, \quad d_1 = 1 + a - c_1, \quad d_2 = 1 + b - c_2,$$

$$p^* = Np/(N-d_1p), \quad q^* = Nq/(N-d_2q), \quad \alpha, \gamma, \theta, \delta > 1, \quad \beta_1, \beta_2 \in \mathbb{R},$$

$$\alpha, \gamma, \theta, \delta > 1.$$
Assume Theorem 1.2. Also, \( \lambda \text{ and nonnegative, for each } \lambda, \mu > 0 \). Then, system (1.2) possesses a weak solution, where each component is nontrivial and nonnegative, for each \( \lambda, \mu > 0 \).

\[
\frac{\theta}{p} + \frac{\delta}{q} \geq \frac{\alpha}{p} + \frac{\gamma}{q} > 1 \quad (p, q\text{-superlinear})
\]

or

\[
\frac{\theta}{p^*} + \frac{\delta}{q^*} < \frac{\alpha}{p^*} + \frac{\gamma}{q^*} < 1 \quad (p, q\text{-subcritical}),
\]

or

\[
\frac{\theta}{p^*} + \frac{\delta}{q^*} < 1 < \frac{\theta}{p} + \frac{\delta}{q} \quad \text{and} \quad \frac{\alpha}{p^*} + \frac{\gamma}{q^*} = 1 \quad p, q\text{-superlinear/critical case})
\]

However, the variational systems behave, in a certain sense, like in the scalar case, there exist some additional difficulties mainly coming from the mutual actions of the variables \( u \) and \( v \), see e. g. [23, 33]. Another difficulty, even in the regular case, are the systems involving \( p\)-laplacian and \( q\)-laplacian operators and their respective critical exponents. In this situation, it is hard to find a well appropriated critical level, mainly, when \( p \neq q \). This open question was pointed out in Adriouch and Hamidi [1]. But, recently Silva and Xavier in [31] were able to prove, in a certain context and in the regular case, the existence of weak solution for a system involving \( p\)-laplacian and \( q\)-laplacian operators with \( p \neq q \). Still in the regular case and \( p = q \), we would like to mention the papers [2, 5, 27, 32, 36], also a survey paper [17]. In particular, Morais and Souto in [27] defined the following critical level number \( S_H/p \), where

\[
S_H = \inf_{W \setminus \{0\}} \left\{ \frac{\int_{\Omega} |\nabla u|^p + |\nabla v|^p}{\int_{\Omega} H(u, v)^{p/p^*}} \right\},
\]

\( W = W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega) \) and \( H \) is homogeneous nonlinearity of degree \( p^* \). In this work, we will improve the critical level by proving that all the Palais Smale sequences at the level \( c \) are relatively compact provided that

\[
c < \left( \frac{1}{p} - \frac{1}{p^*} \right)(\mu p^*)^{rac{p^*}{p^* - p}} \tilde{S}^{rac{p^*}{p^* - p}} + \lambda \left( \frac{1}{p} - \frac{1}{p_1} \right) M,
\]

where \( \tilde{S} \) depends of \( C_{a,p}^* \) and \( M = M(u_n, v_n) \geq 0 \) depends of Palais Smale sequence.

Our first result deals with \( p, q\text{-superlinear and subcritical nonlinear perturbation.} \)

**Theorem 1.1.** In addition to (1.3), (1.4), and (1.5), assume that \( p_i \in (p, p^*) \), \( q_i \in (q, q^*) \), \( i = 1, 2 \), with \( \theta/p_1 + \delta/q_1 = \alpha/p_2 + \gamma/q_2 = 1 \) and

\[
\beta_i < \min \left\{ \left( a + 1 \right) p_i + N \left( 1 - \frac{p_i}{p} \right), \left( b + 1 \right) q_i + N \left( 1 - \frac{q_i}{q} \right) \right\}, \quad i = 1, 2.
\]

Then system (1.2) possesses a weak solution, where each component is nontrivial and nonnegative, for each \( \lambda \geq 0 \) and \( \mu > 0 \).

The next result treats the \( p, q\text{-superlinear and critical case.} \)

**Theorem 1.2.** Assume (1.3), (1.4) and (1.6), with \( p = q \) and \( a = b \geq 0 \). Suppose also \( p_1 = q_1 \in (p, p^*) \), with \( \theta/p_1 + \delta/q_1 = 1 \), \( p^*_2 = q^* \), \( \beta_2 = c_1 p^* \), and \( \beta_1 = (a + 1)p_1 - c \) with

\[
-N \left[ 1 - \left( p_1/p \right) \right] < c < \frac{(p_1 - p + 1)N - (a + 1)p_1}{p - 1} - \frac{(N - p - ap)(p_1 - p)}{p(p - 1)}.
\]

Then, system (1.2) possesses a weak solutions, where each component is nontrivial and nonnegative, for each \( \lambda, \mu > 0 \).
The $p, q$-superlinear and critical case with $p \neq q$ is studied in the following result.

**Theorem 1.3.** In addition to [1.3], [1.4], and [1.6], assume that $p_1 \in (p, p^*)$, $q_1 \in (q, q^*)$, with $\theta/p_1 + \delta/q_1 = 1$, $\beta_2 = c_1 p^* = c_2 q^*$, and $\beta_1$ as in [1.7]. Then there exists $\mu_0$ sufficiently small such that system (1.2) possesses a weak solution, where each component is nontrivial and nonnegative, for each $\lambda > 0$ and $0 < \mu < \mu_0$.

## 2. Preliminaries

We will set some spaces and their norms. If $\alpha \in \mathbb{R}$ and $l \geq 1$, we define $L^l(\Omega, |x|^\alpha)$ as being the subspace of $L^l(\Omega)$ of the Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ satisfying

$$
\|u\|_{L^l(\Omega, |x|^\alpha)} := \left( \int_{\Omega} |x|^\alpha |u|^l \, dx \right)^{1/l} < \infty.
$$

If $1 < p < N$ and $-\infty < a < (N-p)/p$, we define $W^{1,p}_0(\Omega, |x|^{-ap})$ as being the completion of $C_0^\infty(\Omega)$ with respect to the norm $\| \cdot \|$ defined by

$$
\|u\| := \left( \int_{\Omega} |x|^{-ap} |\nabla u|^p \, dx \right)^{1/p}.
$$

First of all, from the Caffarelli, Kohn and Nirenberg inequality (see [12]) and by the boundedness of $\Omega$, it is easy to see that there exists $C > 0$ such that

$$
\left( \int_{\Omega} |x|^{-\delta}|u|^r \, dx \right)^{p/r} \leq C \left( \int_{\Omega} |x|^{-ap} |\nabla u|^p \, dx \right), \quad \forall u \in W^{1,p}_0(\Omega, |x|^{-ap}),
$$

where $1 \leq r \leq Np/(N-p)$ and $\delta \leq (a+1)r + N(1 - (r/p))$.

**Lemma 2.1.** Suppose that $\Omega$ is a bounded smooth domain of $\mathbb{R}^N$ with $0 \in \Omega$, $1 < p < N$, $-\infty < a < (N-p)/p$, $a \leq e_1 < a + 1$, $d_1 = 1 + a - c_1$, $p^* = Np/(N-d_1p)$, and $\alpha + \gamma = p^*$, then

$$
\tilde{S} := \inf_{(u,v) \in \tilde{W}} \left\{ \frac{\int_{\Omega} |x|^{-ap}(|\nabla u|^p + |\nabla v|^p) \, dx}{\left( \int_{\Omega} |x|^{-e_1 p^*} |u|^\alpha |v|^\gamma \, dx \right)^{p/p^*}} \right\},
$$

where

$$
\tilde{W} = \left\{ (u,v) \in (W^{1,p}_0(\Omega, |x|^{-ap}))^2 : |u| \neq 0 \right\},
$$

satisfies

$$
\tilde{S} = [\alpha/\gamma]^{p^*/p^*} + (\alpha/\gamma)^{-\alpha/p^*} C_{a,p}^\alpha.
$$

The proof of the above lemma is similar to the proof of [5] Theorem 5 [see also [27] Lemma 3 for $p \neq 2$].

Let us consider $\Omega$ a smooth domain of $\mathbb{R}^N$ (not necessarily bounded), $0 \in \Omega$, $1 < p < N$, $0 \leq a < (N-p)/p$, $a \leq c_1 < a + 1$, $d_1 = 1 + a - c_1$, and $p^* = Np/(N-d_1p)$. We define the space

$$
W^{1,p}_{a,c_1}(\Omega) = \left\{ u \in L^{p^*}(\Omega, |x|^{-c_1 p^*}) : |\nabla u| \in L^p(\Omega, |x|^{-ap}) \right\},
$$

equipped with the norm

$$
\|u\|_{W^{1,p}_{a,c_1}(\Omega)} = \|u\|_{L^{p^*}(\Omega, |x|^{-c_1 p^*})} + \|\nabla u\|_{L^p(\Omega, |x|^{-ap})}.
$$

We consider the best Hardy-Sobolev constant given by

$$
\tilde{S}_{ap} = \inf_{W^{1,p}_{a,c_1}(\mathbb{R}^N) \setminus \{0\}} \left\{ \frac{\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p \, dx}{\left( \int_{\mathbb{R}^N} |x|^{-c_1 p^*} |u|^{p^*} \, dx \right)^{p/p^*}} \right\}.
$$
Also, we define
\[ R_{a,c}^{1,p}(\Omega) = \{ u \in W_{a,c}^{1,p}(\Omega) : u(x) = u(|x|) \}, \]
euendowed with the norm
\[ \| u \|_{R_{a,c}^{1,p}(\Omega)} = \| u \|_{W_{a,c}^{1,p}(\Omega)}. \]
Actually, Horiuchi in [24] proved that
\[ \inf_{R_{a,c}^{1,p}(\Omega) \setminus \{0\}} \left\{ \frac{\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx}{\left( \int_{\mathbb{R}^N} |x|^{-c_1 p^*} |u|^{p^*} dx \right)^{p/p^*}} \right\} = \tilde{S}_{a,p}, \]
and it is achieved by functions of the form
\[ u_\epsilon(x) := k_{a,p}(\epsilon) U_{a,p,\epsilon}(x), \quad \forall \epsilon > 0, \]
where
\[ k_{a,p}(\epsilon) = c_0 \epsilon^{(N-d_1 p)/d_1 p^2} \quad \text{and} \quad U_{a,p,\epsilon}(x) = \left( \epsilon + |x|^{d_2 p(N-p-\epsilon)} \right)^{-\left(\frac{N-d_1 p}{d_1 p^2}\right)}. \]
Moreover, \( u_\epsilon \) satisfies
\[ \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u_\epsilon|^p dx = \int_{\mathbb{R}^N} |x|^{-c_1 p^*} |u_\epsilon|^{p^*} dx. \quad (2.1) \]
See also Clément, Figueiredo and Mitidieri [19 Proposition 1.4].

The next lemma can be proved arguing as in [11] (see also [35 Lemma 5.1]). For the sake of the completeness we will give the proof in the appendix.

**Lemma 2.2.** In addition to (1.3) and (1.4), assume that \( p_1 = q_1 \in (p, p^*), \theta/p_1 + \delta/q_1 = 1, \beta_2 = c_1 p^* = c_2 q^*, \) and \( \beta_1 = (a+1)p_1 - c \) with
\[ -N \leq (1 - (p_1/p)) \leq c. \]
Let \( R_0 \in (0, 1) \) be such that \( B(0, 2R_0) \subset \Omega \) and \( \psi \in C_0^\infty(B(0, 2R_0)) \) with \( \psi \geq 0 \) in \( B(0, 2R_0) \) and \( \psi \equiv 1 \) in \( B(0, R_0) \), then the function
\[ u_\epsilon(x) = \frac{\psi(x) U_{a,p,\epsilon}(x)}{\| \psi U_{a,p,\epsilon} \|_{L^{p^*}(\Omega, |x|^{-c_1 p^*})}} \]
satisfies
\[ \| u_\epsilon \|_{L^{p^*}(\Omega, |x|^{-c_1 p^*})} = 1, \quad \| \nabla u_\epsilon \|_{L^p(\Omega, |x|^{-\gamma p})} \leq \tilde{S}_{a,p,R} + O(\epsilon^{(N-d_1 p)/d_1 p}), \]
and
\[ \| u_\epsilon \|_{L^{p_1}(\Omega, |x|^{-\beta_1})} \geq \begin{cases} O(\epsilon^{(N-d_1 p)/d_1 p^2}) & \text{if } c > \frac{(p_1-1)(N-(a+1)p_1)}{p-1}, \\ O(\epsilon^{(N-d_1 p)/d_1 p^2} |\ln(\epsilon)|) & \text{if } c = \frac{(p_1-1)(N-(a+1)p_1)}{p-1}, \\ O(\epsilon^{(N-d_1 p)/d_1 p^2} \frac{(N-d_1 p)(p-1)}{p-1}) & \text{if } c < \frac{(p_1-1)(N-(a+1)p_1)}{p-1}. \end{cases} \quad (2.2) \]

The following result, which will be useful in the proof of our results, was proved by Kavian in [28 Lemma 4.8].

**Lemma 2.3.** Let \( \Omega \) be an open subset of \( \mathbb{R}^N \), \( \{ f_n \} \in L^r(\Omega) \), for some \( 1 < r < \infty \), a bounded sequence such that \( f_n(x) \to f(x) \), for a.e. \( x \in \Omega \), as \( n \to \infty \). Then, \( f \in L^r(\Omega) \) and \( f_n \to f \) weakly in \( L^r(\Omega) \) as \( n \to \infty \).
Definition. Let us consider \( \{(u_n, v_n)\} \) in \( W^{1,p}_0(\Omega, |x|^{-ap}) \times W^{1,q}_0(\Omega, |x|^{-bq}) \). We say that the sequence \( \{(u_n, v_n)\} \) is a Palais Smale sequence for operator \( I \) at the level \( c \) (or simply, \( (PS)_c \)-sequence) if
\[
I(u_n, v_n) \to c \quad \text{and} \quad I'(u_n, v_n) \to 0, \quad \text{as} \quad n \to \infty.
\]

Our approach will be to use variational techniques; that is, we have to find the critical points of the Euler-Lagrange functional
\[
I : W^{1,p}_0(\Omega, |x|^{-ap}) \times W^{1,q}_0(\Omega, |x|^{-bq}) \to \mathbb{R}
\]
given by
\[
I(u, v) = \frac{1}{p} \int_\Omega |x|^{-ap} |\nabla u|^p \, dx + \frac{1}{q} \int_\Omega |x|^{-bq} |\nabla v|^q \, dx
- \lambda \int_\Omega |x|^{-\beta_1} u_+^\delta \|v_+\|^\gamma \, dx - \mu \int_\Omega |x|^{-\beta_2} u_+^\gamma \|v_+\|^\delta \, dx,
\]
which is well defined and is of class \( C^1 \), with the Gâteaux derivative
\[
\langle I'(u, v), (w, z) \rangle = \int_\Omega |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla w \, dx + \int_\Omega |x|^{-bq} |\nabla v|^{q-2} \nabla v \nabla z \, dx
- \lambda \theta \int_\Omega |x|^{-\beta_1} u_+^{\theta-1} v_+^\delta \, dx - \lambda \delta \int_\Omega |x|^{-\beta_1} u_+^{\theta} v_+^{\delta-1} \, dx
- \mu \alpha \int_\Omega |x|^{-\beta_2} u_+^{\alpha-1} v_+^\gamma \, dx - \mu \gamma \int_\Omega |x|^{-\beta_2} u_+^{\alpha} v_+^{\gamma-1} \, dx,
\]
where \( u_\pm = \max\{0, \pm u\} \) which is in \( W^{1,p}_0(\Omega, |x|^{-ap}) \) (Similarly \( v_\pm = \max\{0, \pm v\} \) which is in \( W^{1,q}_0(\Omega, |x|^{-bq}) \); see [3]).

First of all, we are going to show the geometric conditions of the mountain pass theorem.

Lemma 2.4. In addition to (1.3) and (1.4), assume that one of the following conditions hold:

(i) the case \( [1.5] \), \( p_i \in (p, p^*) \), \( q_i \in (q, q^*) \), with \( \theta/p_1 + \delta/q_1 = \alpha/p_2 + \gamma/q_2 = 1 \), and \( \beta_i \) as in \( [1.7] \), for \( i = 1, 2 \).

(ii) the case \( [1.6] \), \( p_i \in (p, p^*) \), \( q_i \in (q, q^*) \), with \( \theta/p_1 + \delta/q_1 = 1 \), \( \beta_1 \) as in \( [1.7] \), \( p_2 = p^* \), \( q_2 = q^* \), and \( \beta_2 = c_1 p^* = c_2 q^* \).

Then the Euler-Lagrange functional \( I \) satisfies:

(a) There exist \( \sigma, \rho > 0 \) such that
\[
I(u, v) \geq \sigma \quad \text{if} \quad \|(u, v)\| = \rho. \tag{2.3}
\]

(b) There exists \( e \in W^{1,p}_0(\Omega, |x|^{-ap}) \times W^{1,q}_0(\Omega, |x|^{-bq}) \) such that
\[
I(e) \leq 0, \quad \|e\| \geq R \quad \text{for some} \quad R > \rho.
\]

Proof. Part (a). For \( (u, v) \in W^{1,p}_0(\Omega, |x|^{-ap}) \times W^{1,q}_0(\Omega, |x|^{-bq}) \) with \( \|(u, v)\| \leq 1 \), we have
\[
I(u, v) \geq \left( \frac{1}{p} \|u\|^p - \lambda \frac{\theta C_{p_1}}{p_1} \|u\|^{p_1} - \mu \frac{\alpha C_{p_2}}{p_2} \|u\|^{p_2} \right)
+ \left( \frac{1}{q} \|v\|^q - \lambda \frac{\delta C_{q_1}}{q_1} \|v\|^{q_1} - \mu \frac{\gamma C_{q_2}}{q_2} \|v\|^{q_2} \right)
\]
\[
\begin{align*}
&\geq \frac{1}{p} \|u\|^p - \left(\lambda \frac{\theta C_{p,1/p}}{p_1} + \mu \frac{\alpha C_{p,2/p}}{p_2}\right) \|u\|^\min\{p_1,p_2\} \\
&\quad + \frac{1}{q} \|v\|^q - \left(\lambda \frac{\delta C_{q,1/q}}{q_1} + \mu \frac{\gamma C_{q,2/q}}{q_2}\right) \|v\|^\min\{q_1,q_2\}.
\end{align*}
\]
Hence, as \( p < \min\{p_1,p_2\} \) and \( q < \min\{q_1,q_2\} \), we can choose \( \rho \in (0,1) \) such that
\[
I(u,v) \geq \sigma \quad \text{if} \quad \|(u,v)\| = \rho.
\]

Part (b). The proof follows by taking \((u_0,v_0) \in W_{0}^{1,p}(\Omega, |x|^{-ap}) \times W_{0}^{1,q}(\Omega, |x|^{-bq})\) with \(u_{0_+}, v_{0_+} \neq 0\). Then, defining \((u_t, v_t) = (t^{1/p}u_0, t^{1/q}v_0)\), for \( t > 0 \), we obtain
\[
I(u_t, v_t) \leq \left(\frac{1}{p} \|u_0\|^p + \frac{1}{q} \|v_0\|^q\right) t - \mu t^{\frac{\theta}{\theta_1} + \frac{\delta}{\delta_1}} \int_{\Omega} |x|^{-\beta_1 u_0^\alpha v_0^\gamma} \, dx \to -\infty, \quad (2.4)
\]
as \( t \to \infty \).

From the mountain pass theorem \([6]\) we get a \((PS)_{c}\)-sequence \(\{(u_n,v_n)\}\) in \(W_{0}^{1,p}(\Omega, |x|^{-ap}) \times W_{0}^{1,q}(\Omega, |x|^{-bq})\), where
\[
0 < \sigma \leq c = \inf_{h \in \Gamma} \max_{t \in [0,1]} I(h(t)) \quad (2.5)
\]
and
\[
\Gamma = \{ h \in C([0,1], W_{0}^{1,p}(\Omega, |x|^{-ap}) \times W_{0}^{1,q}(\Omega, |x|^{-bq})) : h(0) = 0, h(1) = c\}, \quad (2.6)
\]
with \( I(c) = I(t_0 u_0, t_0 v_0) < 0 \).

**Lemma 2.5.** In addition to \((1.3)\) and \((1.4)\), assume that one of the two following conditions hold:

(i) the case \((1.5), p_1 \in (p, p^*), q_i \in (q, q^*), \) with \(\theta/p_1 + \delta/q_1 = \alpha/p_2 + \gamma/q_2 = 1, \) and \( \beta_i \) as in \((1.7)\), for \( i = 1, 2. \)

(ii) the case \((1.6), p_1 \in (p, p^*), q_i \in (q, q^*), \) with \(\theta/p_1 + \delta/q_1 = 1, \beta_i \) as in \((1.7)\), \( p_2 = p^*, q_2 = q^*, \) and \( \beta_2 = c_1 p^* = c_2 q^* . \)

Let \(\{(u_n,v_n)\} \subset W_{0}^{1,p}(\Omega, |x|^{-ap}) \times W_{0}^{1,q}(\Omega, |x|^{-bq})\) be a \((PS)_{c}\)-sequence. Then \(\{(u_n,v_n)\}\) is a \((PS)_{c}\)-sequence which is bounded uniformly in \(\mu > 0\).

**Proof.** Let \( \theta_1 = \min\{p_1,p_2\} \) and \( \theta_2 = \min\{q_1,q_2\} \), we have
\[
c + \|(u_n,v_n)\| + O_n(1) \geq I(u_n, v_n) - \langle I'(u_n, v_n), (u_n/\theta_1, v_n/\theta_2) \rangle \\
\quad \geq \left(\frac{1}{p} - \frac{1}{\theta_1}\right) \|u_n\|^p + \left(\frac{1}{q} - \frac{1}{\theta_2}\right) \|v_n\|^q \\
\quad + \lambda \left(\frac{\theta}{\theta_1} + \frac{\delta}{\theta_2} - 1\right) \int_{\Omega} |x|^{-\beta_1 u_0^\alpha v_0^\gamma} \, dx \\
\quad + \mu \left(\frac{\alpha}{\theta_1} + \frac{\gamma}{\theta_2} - 1\right) \int_{\Omega} |x|^{-\beta_2 u_0^\alpha v_0^\gamma} \, dx \\
\quad \geq \left(\frac{1}{p} - \frac{1}{\theta_1}\right) \|u_n\|^p + \left(\frac{1}{q} - \frac{1}{\theta_2}\right) \|v_n\|^q.
\]
Therefore, independently of \(\lambda \geq 0\) and \(\mu > 0\), we conclude that \(\{(u_n,v_n)\}\) is a bounded sequence in \(W_{0}^{1,p}(\Omega, |x|^{-ap}) \times W_{0}^{1,q}(\Omega, |x|^{-bq})\). In particular, we have that \(\{(u_{n_-},v_{n_-})\}\) and \(\{(u_{n_+},v_{n_+})\}\) are bounded sequences in \(W_{0}^{1,p}(\Omega, |x|^{-ap}) \times W_{0}^{1,q}(\Omega, |x|^{-bq})\), then
\[
-\|u_{n_-}\|^p = \langle I'(u_n, v_n), (u_{n_-}, 0) \rangle \to 0 \quad \text{as} \quad n \to \infty \quad (2.7)
\]
and similarly
\[ -\|v_{n-}\|^q = \langle I'(u_n, v_n), (0, v_{n-}) \rangle \to 0 \quad \text{as} \quad n \to \infty. \tag{2.8} \]
Moreover, we get
\[ I(u_{n+}, v_{n+}) = I(u_n, v_n) + \frac{1}{p} \|u_{n-}\|^p + \frac{1}{q} \|v_{n-}\|^q = I(u_n, v_n) + O_n(1). \]

Therefore, from (2.7) and (2.8), we obtain \( I(u_{n+}, v_{n+}) \to c \) as \( n \to \infty \). Similarly, if \( (w, z) \in W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq}) \), we prove that
\[ \langle I'(u_{n+}, v_{n+}), (w, z) \rangle = \langle I'(u_n, v_n), (w, z) \rangle + O_n(1), \]

hence \( I'(u_{n+}, u_{n+}) \to 0 \) as \( n \to \infty \). \( \square \)

3. Proof of theorem 1.1

Lemma 3.1. Suppose that (1.3) and (1.4) hold. Assume that \( p_i \in (p, p^*) \), \( q_i \in (q, q^*) \), \( i = 1, 2 \), with \( \theta/p_1 + \delta/q_1 = \alpha/p_2 + \gamma/q_2 = 1 \), and \( \beta_i \), \( i = 1, 2 \), as in (1.7). Then, every \( (PS)_\epsilon \)-sequence \( \{ (u_n, v_n) \} \) with \( u_n, v_n \geq 0 \), for a.e. in \( \Omega \), is precompact.

Proof. From lemma 2.5, the sequence \( \{ (u_n, v_n) \} \) is bounded in \( W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq}) \). We can assume, passing to a subsequence if necessary, that there exists \( (u, v) \in W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq}) \) satisfying \( u_n \rightharpoonup u \) and \( v_n \rightharpoonup v \) weakly, as \( n \to \infty \). From the compact embedding theorem [35, Theorem 2.1], we obtain
\[ u_n \to u \quad \text{in} \quad L^{p_1}(\Omega, |x|^{-\beta_1}) \cap L^{p_2}(\Omega, |x|^{-\beta_2}) \quad \text{as} \quad n \to \infty, \]
\[ v_n \to v \quad \text{in} \quad L^{q_1}(\Omega, |x|^{-\beta_1}) \cap L^{q_2}(\Omega, |x|^{-\beta_2}) \quad \text{as} \quad n \to \infty. \]

Since there exist \( f \in L^{p_1}(\Omega, |x|^{-\beta_1}) \) and \( g \in L^{q_1}(\Omega, |x|^{-\beta_1}) \) such that \( |u_n|(x) \leq f(x) \) and \( |v_n|(x) \leq g(x) \), for a.e. \( x \in \Omega \) and all \( n \in \mathbb{N} \), applying the Lebesgue’s dominated convergence theorem we infer that
\[ \lim_{n \to \infty} \int_{\Omega} |x|^{-\beta_1} u_n^{\theta-1} v_n^{\delta}(u_n - u) dx = 0, \tag{3.1} \]
and similarly
\[ \lim_{n \to \infty} \int_{\Omega} |x|^{-\beta_2} u_n^{\alpha-1} v_n^{\gamma}(u_n - u) dx = 0. \tag{3.2} \]

Now, taking the upper limit in the equation
\[ \int_{\Omega} |x|^{-ap} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \nabla (u_n - u) dx \]
\[ = \langle I'(u_n, v_n), (u_n - u, 0) \rangle - \int_{\Omega} |x|^{-ap} |\nabla u|^{p-2}\nabla u \nabla (u_n - u) dx \]
\[ + \lambda \theta \int_{\Omega} |x|^{-\beta_1} u_n^{\theta-1} v_n^{\delta}(u_n - u) dx + \mu \alpha \int_{\Omega} |x|^{-\beta_2} u_n^{\alpha-1} v_n^{\gamma}(u_n - u) dx. \]

Using the definition of \((PS)_\epsilon\)-sequence, the weak convergence, (3.1), and (3.2), we obtain
\[ \limsup_{n \to \infty} \int_{\Omega} |x|^{-ap} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \nabla (u_n - u) dx = 0. \]
Consequently, by a well known lemma (see e.g. [20, lemma 4.1]) we achieve, up to a subsequence, that $u_n \to u$ strongly in $W_0^{1,p}(\Omega, |x|^{-ap})$ as $n \to \infty$. Analogously, we get $v_n \to v$ strongly in $W_0^{1,q}(\Omega, |x|^{-bq})$ as $n \to \infty$. □

Proof of theorem 4.1. By combining lemmata 2.4 and 2.5 there exists a $(PS)_c$-sequence $\{u_n, v_n\} \subset (W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$ with $u_n, v_n \geq 0$, for a.e. in $\Omega$. Moreover, from lemma 3.1 there exist $(u, v) \in W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$ and a subsequence of $\{(u_n, v_n)\}$, that we will denote by $\{(u_n, v_n)\}$, such that $u_n \to u$ strongly in $W_0^{1,p}(\Omega, |x|^{-ap})$ and $v_n \to v$ strongly in $W_0^{1,q}(\Omega, |x|^{-bq})$, as $n \to \infty$. Then, we conclude that

$$I(u, v) = c > 0 \quad \text{and} \quad I'(u, v) = 0,$$

that is, $(u, v)$ is a nonnegative weak solution of system (1.2). Moreover, it is easy to check that $u, v \neq 0$. □

4. PROOF OF THEOREM 4.2

First of all, notice that by lemma 2.4 the geometric conditions of the mountain pass theorem for the functional $I$ are satisfied.

The next three lemmata are crucial in the proof of this theorem.

Lemma 4.1. Let $\{(u_n, v_n)\} \subset (W_0^{1,p}(\Omega, |x|^{-ap}))^2$ be a bounded $(PS)_c$-sequence such that $u_n, v_n \geq 0$, for a.e. in $\Omega$, and there exists $(u, v) \in (W_0^{1,p}(\Omega, |x|^{-ap}))^2$ satisfying $u_n \to u$ and $v_n \to v$ in $W_0^{1,p}(\Omega, |x|^{-ap})$, as $n \to \infty$. Then, $(u, v)$ is a weak solution of system (1.2) and $u, v \geq 0$ for a.e. in $\Omega$.

Proof. Arguing as in the proof of lemma 3.1, by combining the compact embedding theorem 3.5 Theorem 2.1 with the Lebesgue’s dominated convergence theorem, we obtain that $u, v \geq 0$ for a.e. in $\Omega$,

$$\lim_{n \to \infty} \int_{\Omega} |x|^{-\beta_1} u_n^{\beta_1-1} v_n^{\beta_2} w dx = \int_{\Omega} |x|^{-\beta_1} u^{\beta_1-1} v^{\beta_2} w dx, \quad \forall w \in W_0^{1,p}(\Omega, |x|^{-ap}), \quad (4.1)$$

and

$$\lim_{n \to \infty} \int_{\Omega} |x|^{-\delta_1} u_n^{\delta_1-1} v_n^{\delta_2} z dx = \int_{\Omega} |x|^{-\delta_1} u^{\delta_1-1} v^{\delta_2} z dx, \quad \forall z \in W_0^{1,p}(\Omega, |x|^{-ap}). \quad (4.2)$$

Notice that $\nabla u_n(x) \to \nabla u(x)$ and $\nabla v_n(x) \to \nabla v(x)$, for a.e. $x \in \Omega$, as $n \to \infty$. These facts can be proved arguing as in [9] (see also [8, 20, 22]).

Since $\{(u_n, v_n)\}$ is bounded in $W_0^{1,p}(\Omega, |x|^{-ap})^2$, we have $|\nabla u_n|^p - 2\nabla u_n$ and $|\nabla v_n|^{p-2}\nabla v_n$ are bounded in $(L^{\frac{p}{p-1}}(\Omega, |x|^{-ap}))^N$. On the other hand, since $\alpha + \gamma = p^*$, by the Hölder’s inequality, we infer that $\{u_n^{\alpha-1} v_n^\gamma\}$ and $\{u_n^{\alpha} v_n^{\gamma-1}\}$ are bounded in $L^{\frac{p^*}{p^*-1}}(\Omega, |x|^{-c_1p^*})$. Therefore, by lemma 2.3 we get

$$\nabla u_n \to \nabla u \quad \text{and} \quad \nabla v_n \to \nabla v \quad \text{weakly in} \quad (L^{\frac{p}{p-1}}(\Omega, |x|^{-ap}))^N \quad (4.3)$$

and

$$u_n^{\alpha-1} v_n^\gamma \to u^{\alpha-1} v^\gamma, \quad u_n^{\alpha} v_n^{\gamma-1} \to u^{\alpha} v^{\gamma-1} \quad \text{weakly in} \quad L^{\frac{p^*}{p^*-1}}(\Omega, |x|^{-c_1p^*}), \quad (4.4)$$

as $n \to \infty$. Consequently, using (4.1) - (4.4) we obtain

$$\langle I'(u, v), (w, z) \rangle = \lim_{n \to \infty} \langle I'(u_n, v_n), (w, z) \rangle = 0, \quad \forall (w, z) \in (W_0^{1,p}(\Omega, |x|^{-ap}))^2,$$

that is, $(u, v)$ is a weak solution of system (1.2). □
Lemma 4.2. In addition to (1.3), (1.4), and (1.6), assume that \( p = q, 0 \leq a = b, \)
\( p_1 = q_1 \in (p, p^*), \) with \( \theta/p_1 + \delta/q_1 = 1, \)
\( p^* = q^*, \) and \( \beta_2 = 0 < p^* . \) Then, all the
Palais Smale sequences \( \{ (u_n, v_n) \} \subset (W_0^{1, p}(\Omega, |x|^{-ap}))^2 \)
for the operator \( I \) at the level \( c, \) with \( u_n, v_n \geq 0 \) for a.e. in \( \Omega, \)
are precompact provided that
\[
c < \left( 1 - \frac{1}{p} \right) (\mu p^*)^{\frac{1}{p-1}} S_\frac{\theta}{p-\theta} + K(\lambda), \tag{4.5}
\]
where
\[
K(\lambda) = \lambda p_1 \left( 1 - \frac{1}{p_1} \right) \lim_{n \to \infty} \int_\Omega |x|^{-\beta_1} u_n^\alpha v_n^\delta dx.
\]

Proof. By Lemma 2.5 the sequence \( \{ (u_n, v_n) \} \) is bounded in \( (W_0^{1, p}(\Omega, |x|^{-ap}))^2; \)
consequently, there exists \( (u, v) \in (W_0^{1, p}(\Omega, |x|^{-ap}))^2 \) such that \( u_n \rightharpoonup u \) and \( v_n \rightharpoonup v \)
weakly in \( W_0^{1, p}(\Omega, |x|^{-ap}), \) as \( n \to \infty. \) Then, by combining the compact embedding
theorem [35, Theorem 2.1] with the Lebesgue’s dominated convergence theorem,
we infer that \( u_n(x) \rightharpoonup u(x), v_n(x) \rightharpoonup v(x), \) for a.e. in \( \Omega, \)
as \( n \to \infty, \) and
\[
\lim_{n \to \infty} \int_\Omega |x|^{-\beta_1} u_n^\alpha v_n^\delta dx = \int_\Omega |x|^{-\beta_1} u^\alpha v^\delta dx. \tag{4.6}
\]
Moreover, as in Lemma 4.1 we can suppose that \( \nabla u_n(x) \rightharpoonup \nabla u(x) \)
and \( \nabla v_n(x) \rightharpoonup \nabla v(x), \) for a.e. \( x \in \Omega, \) as \( n \to \infty. \)

Define \( \tilde{u}_n = u_n - u \) and \( \tilde{v}_n = v_n - v. \) By Brezis and Lieb [10, Theorem 1]
we have
\[
\begin{align*}
\text{(i)} & \quad \|u_n\|^p = \|\tilde{u}_n\|^p + \|u\|^p + O_n(1), \text{ as } n \to \infty. \\
\text{(ii)} & \quad \|v_n\|^p = \|\tilde{v}_n\|^p + \|v\|^p + O_n(1), \text{ as } n \to \infty. \\
\text{(iii)} & \quad \int_\Omega |x|^{-\epsilon_1 p^*} |u_n|^\alpha |v_n|^\gamma dx - \int_\Omega |x|^{-\epsilon_1 p^*} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\gamma dx \\
& \quad = \int_\Omega |x|^{-\epsilon_1 p^*} |u|^\alpha |v|^\gamma dx + O_n(1), \text{ as } n \to \infty.
\end{align*}
\]
We recall that the proof of identity (iii) follows arguing as in [27, Lemma 8].

By Lemma 4.1 we have that \( (u, v) \) is a weak solution of system (1.2), that is,
\( \langle I'(u, v), (w, z) \rangle = 0 \) for all \( (w, z) \in (W_0^{1, p}(\mathbb{R}^N, |x|^{-ap}))^2. \) By using (4.6) and (i)–(iii), we get
\[
\begin{align*}
\|\tilde{u}_n\|^p - \mu \alpha & \int_\Omega |x|^{-\epsilon_1 p^*} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\gamma dx \\
& = \|u_n\|^p - \|u\|^p - \mu \alpha \left( \int_\Omega |x|^{-\epsilon_1 p^*} u_n^\alpha v_n^\gamma dx - \int_\Omega |x|^{-\epsilon_1 p^*} u^\alpha v^\gamma dx \right) + O_n(1) \\
& = \langle I'(u_n, v_n), (u_n, 0) \rangle - \langle I'(u, v), (u, 0) \rangle + O_n(1) \\
& = O_n(1), \text{ as } n \to \infty.
\end{align*}
\]
Analogously, we obtain
\[
\|\tilde{v}_n\|^p - \mu \gamma \int_\Omega |x|^{-\epsilon_1 p^*} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\gamma dx = O_n(1).
\]
Thus, we can take \( l \geq 0 \) such that
\[
l = \lim_{n \to \infty} \frac{\|\tilde{u}_n\|^p}{\alpha} = \lim_{n \to \infty} \frac{\|\tilde{v}_n\|^p}{\gamma} = \mu \lim_{n \to \infty} \int_\Omega |x|^{-\epsilon_1 p^*} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\gamma dx.
\]
If \( l = 0 \) the result is proved. Suppose by contradiction that \( l > 0 \). By the definition of \((PS)_{e}\)-sequence we get

\[
c + O_n(1)
\]

\[
= I(u_n, v_n) - \frac{1}{p_1}(I'(u_n, v_n), (u_n, v_n))
\]

\[
= \left(\frac{1}{p} - \frac{1}{p_1}\right)(\|u_n\|^p + \|v_n\|^p) + \mu(\frac{\alpha + \gamma}{p} - 1) \int |x|^{-c_1} p^* u_n^\alpha v_n^\gamma dx
\]

\[
= \left(\frac{1}{p} - \frac{1}{p_1}\right)(\|\tilde{u}_n\|^p + \|\tilde{v}_n\|^p) + \mu(\frac{p^*}{p_1} - 1) \int |x|^{-c_1} p^* \tilde{u}_n^\alpha \tilde{v}_n^\gamma dx + \int |x|^{-c_1} p^* u_n^\alpha v_n^\gamma dx
\]

\[
+ O_n(1)
\]

\[
= \left(\frac{1}{p} - \frac{1}{p_1}\right)p^* l + \left(\frac{1}{p} - \frac{1}{p_1}\right)(\|u\|^p + \|v\|^p)
\]

\[
+ \frac{1}{p_1} - \frac{1}{p} p^* \left[l + \mu \int |x|^{-c_1} p^* u_n^\alpha v_n^\gamma dx\right] + O_n(1) \quad (4.7)
\]

\[
= \left(\frac{1}{p} - \frac{1}{p^*}\right)p^* l + \left(\frac{1}{p} - \frac{1}{p_1}\right) \left[\lambda p_1 \int |x|^{-\beta} u^\theta v^\delta dx + \mu p^* \int |x|^{-c_1} p^* u^\alpha v^\gamma dx\right]
\]

\[
+ \mu(\frac{1}{p_1} - \frac{1}{p^*}) p^* \int |x|^{-c_1} p^* u_n^\alpha v_n^\gamma dx + O_n(1)
\]

\[
\ge \left(\frac{1}{p} - \frac{1}{p^*}\right)p^* l + \lambda p_1 \left(\frac{1}{p} - \frac{1}{p_1}\right) \int |x|^{-\beta} u^\theta v^\delta dx
\]

\[
+ \mu(\frac{1}{p_1} - \frac{1}{p^*}) p^* \int |x|^{-c_1} p^* u_n^\alpha v_n^\gamma dx + O_n(1).
\]

Using the definition of \( \tilde{S} \) we have

\[
\left(\int |x|^{-c_1} p^* u_n^\alpha v_n^\gamma dx\right)^{p/p^*} \tilde{S} \leq \|u_n\|^p + \|v_n\|^p, \forall n.
\]

Hence, taking the limit in the above inequality we get

\[
\left(\frac{1}{\mu}\right)^{p/p^*} \tilde{S} \leq (\alpha + \gamma) l = p^* l
\]

then

\[
l \geq (\mu)^{\frac{p}{p^*}} (\mu p^*) \frac{p}{p^*} \tilde{S} \frac{p^*}{p^*}.
\]

Substituting \((4.8)\) in \((4.7)\) and taking the limit, we obtain

\[
c \geq \left(\frac{1}{p} - \frac{1}{p^*}\right)(\mu p^*) \frac{p}{p^*} \tilde{S} \frac{p^*}{p^*} + K(\lambda),
\]

which contradicts the inequality \((4.5)\). \( \square \)

**Lemma 4.3.** We can choose \( e \) in \((2.6)\) such that \( c \) given by \((2.5)\) satisfies

\[
c < \left(\frac{1}{p} - \frac{1}{p^*}\right)(\mu p^*) \frac{p}{p^*} \tilde{S} \frac{p^*}{p^*}. \quad (4.9)
\]
\textbf{Proof.} Let us consider \( s_0 = s_1 (s_1 t_1^2)^{\frac{1}{p}} \) and \( t_0 = t_1 (s_1 t_1^2)^{\frac{1}{p}} \), where \( s_1, t_1 > 0 \) and \( s_1/t_1 = (\alpha/\gamma)^{1/p} \), and \( u_\epsilon \) the function defined in lemma 2.2. Then, it is suffices to prove that there exists \( \epsilon > 0 \) such that

\[
\sup_{t \geq 0} I(t(s_0 u_\epsilon), t(t_0 u_\epsilon)) < \left( \frac{1}{p} - \frac{1}{p^*} \right) (\mu p^*) \frac{p}{p - p^*} S^{p^*}.
\]

Due to the geometric conditions of the mountain pass theorem, for each \( \epsilon > 0 \), there exists \( t_\epsilon > 0 \) such that

\[
0 < \sigma \leq \sup_{t \geq 0} I(t(s_0 u_\epsilon), t(t_0 u_\epsilon)) = I(t_\epsilon(s_0 u_\epsilon), t_\epsilon(t_0 u_\epsilon)).
\]

Moreover, supposing by contradiction that there exists a subsequence \( \{t_{\epsilon_n}\} \) with \( t_{\epsilon_n} \to 0 \) as \( n \to \infty \), we obtain

\[
0 < \sigma \leq I(t_{\epsilon_n}(s_0 u_{\epsilon_n}), t_{\epsilon_n}(t_0 u_{\epsilon_n})) \leq \frac{t_\epsilon p}{p} \frac{p}{s_{\epsilon_n}} \left( s_0^{p} + t_0^{p} \right) \|u_{\epsilon_n}\|^{p} + \frac{t_\epsilon p}{p} \|u_{\epsilon_n}\|^{p}
\]

\[
\leq \frac{t_\epsilon p}{p} \left( s_0^{p} + t_0^{p} \right) (\tilde{S}_{a,p,R} + O(c_n^{(N-d_1)/d_1})) \to 0 \text{ as } n \to \infty,
\]

which is an absurd. Then, there exists \( l > 0 \) with \( t_{\epsilon} \geq l \), for all \( \epsilon > 0 \). Consequently, by using lemma 2.2 and putting \( c_0 = l \tilde{S}_{a,p,R}^{1/2} \), we get

\[
\sup_{t \geq 0} I(t(s_0 u_\epsilon), t(t_0 u_\epsilon)) \leq \frac{t_\epsilon p}{p} \left( \frac{s_0^{p} + t_0^{p}}{(s_1 t_1^{p})^{p/p'}} \|u_\epsilon\|^{p} \right) - \lambda c_0 \int_{\Omega} |x|^{-\beta_1} u_\epsilon^{p_1} dx - \mu t_\epsilon^{p^*}. \quad (4.10)
\]

Note that

\[
t_\epsilon = (\mu p^*) \frac{p}{p - p^*} \left( \frac{s_0^{p} + t_0^{p}}{(s_1 t_1^{p})^{p/p'}} \|u_\epsilon\|^{p} \right) \frac{1}{p} \|u_\epsilon\|^{p - p^*}.
\]

(4.11)

is the unique maximum point of \( f_\epsilon : (0, \infty) \to \mathbb{R} \), given by

\[
f_\epsilon(t) = \left( \frac{s_0^{p} + t_0^{p}}{(s_1 t_1^{p})^{p/p'}} \|u_\epsilon\|^{p} \right) - \mu t_\epsilon^{p^*}.
\]

Also we know that

\[
(A + B)^k \leq A^k + k(A + B)^{k-1}B, \quad (4.12)
\]

for all \( A, B \geq 0 \) and \( k \geq 1 \). Observe that the following identity holds

\[
\left[ \frac{s_0^{p} + t_0^{p}}{(s_1 t_1^{p})^{p/p'}} \right]^k = \left[ (\alpha/\gamma)^{1/p^*} + (\alpha/\gamma)^{-1/p^*} \right].
\]

(4.13)

By the Caffarelli-Kohn-Nirenberg’s inequality, \( W^{1,p}_0(\Omega, |x|^{-ap}) \subset W^{1,p}_{a,c_1}(\mathbb{R}^N) \). Then

\[
\tilde{S}_{a,p} \leq C_{a,p}^*.
\]

(4.14)
Substituting (4.11) in (4.10), from (4.12), (4.13), (4.14), and using lemma 2.2, we obtain

\[
\sup_{t \geq 0} I(t(s_0 u_e), t(t_0 u_e)) \leq \left( \frac{1}{p} - \frac{1}{p^*} \right)(\mu p^*) \frac{1}{p^* - p} \left\{ \left[ \frac{\alpha}{\gamma} \right]^2 + \left( \frac{\alpha}{\gamma} \right)^{-} \right\} \mathcal{S}_{a,p,R}^* \left[ \mathcal{S} \mathcal{S} \right] + O\left( \varepsilon^{\frac{N - d_1 p}{p^* - p}} \right) \int_{\Omega} |x|^{-\beta_1 + \varepsilon} u_{p^1} dx \\
\leq \left( \frac{1}{p} - \frac{1}{p^*} \right)(\mu p^*) \frac{1}{p^* - p} \left\{ \left[ \frac{\alpha}{\gamma} \right]^2 + \left( \frac{\alpha}{\gamma} \right)^{-} \right\} \mathcal{S}_{a,p,R}^* \left[ \mathcal{S} \mathcal{S} \right] + O\left( \varepsilon^{\frac{N - d_1 p}{p^* - p}} \right) - \lambda_0 \int_{\Omega} |x|^{-\beta_1 + \varepsilon} u_{p^1} dx \\
\leq \left( \frac{1}{p} - \frac{1}{p^*} \right)(\mu p^*) \frac{1}{p^* - p} \left\{ \left[ \frac{\alpha}{\gamma} \right]^2 + \left( \frac{\alpha}{\gamma} \right)^{-} \right\} \mathcal{S}_{a,p,R}^* \left[ \mathcal{S} \mathcal{S} \right] + O\left( \varepsilon^{\frac{N - d_1 p}{p^* - p}} \right) - \lambda_0 \int_{\Omega} |x|^{-\beta_1 + \varepsilon} u_{p^1} dx.
\]

(4.15)

Now, from lemma 2.1 and (4.15), we get

\[
\sup_{t \geq 0} I(t(s_0 u_e), t(t_0 u_e)) \leq \left( \frac{1}{p} - \frac{1}{p^*} \right)(\mu p^*) \frac{1}{p^* - p} \left\{ \left[ \frac{\alpha}{\gamma} \right]^2 + \left( \frac{\alpha}{\gamma} \right)^{-} \right\} \mathcal{S}_{a,p,R}^* \left[ \mathcal{S} \mathcal{S} \right] + O\left( \varepsilon^{\frac{N - d_1 p}{p^* - p}} \right) - \lambda_0 \int_{\Omega} |x|^{-\beta_1 + \varepsilon} u_{p^1} dx.
\]

(4.16)

Supposing that \( c < \frac{(p_1 - p + 1)N - (a + 1)p_1}{p - 1} - \frac{(N - p - ap)(p_1 - p)}{p(p - 1)} \) we have

\[
\frac{(N - p_1 + ap_1 + c)(p - 1)(N - d_1 p)}{d_1 p(N - p - ap)} - \frac{(N - d_1 p)(p - 1)p_1}{d_1 p^2} < \frac{N - d_1 p}{d_1 p},
\]

then, by lemma 2.2 and by (4.16), we can take a \( \varepsilon > 0 \) small enough such that

\[
\sup_{t \geq 0} I(t(s_0 u_e), t(t_0 u_e)) \leq \left( \frac{1}{p} - \frac{1}{p^*} \right)(\mu p^*) \frac{1}{p^* - p} \left\{ \left[ \frac{\alpha}{\gamma} \right]^2 + \left( \frac{\alpha}{\gamma} \right)^{-} \right\} \mathcal{S}_{a,p,R}^* \left[ \mathcal{S} \mathcal{S} \right] + O\left( \varepsilon^{\frac{N - d_1 p}{p^* - p}} \right) - \lambda_0 \int_{\Omega} |x|^{-\beta_1 + \varepsilon} u_{p^1} dx \\
< \left( \frac{1}{p} - \frac{1}{p^*} \right)(\mu p^*) \frac{1}{p^* - p} \left\{ \left[ \frac{\alpha}{\gamma} \right]^2 + \left( \frac{\alpha}{\gamma} \right)^{-} \right\} \mathcal{S}_{a,p,R}^* \left[ \mathcal{S} \mathcal{S} \right] + O\left( \varepsilon^{\frac{N - d_1 p}{p^* - p}} \right).
\]

This completes the proof. \( \square \)

Proof of theorem 1.2 From lemmata 2.4, 2.5, and 4.3 there exists a bounded (PS)\(_c\)-sequence \( \{(u_n, v_n)\} \) in \( (W^1_p(\Omega, |x|^{-ap}))^2 \) with \( c > 0 \) satisfying (4.9) and \( u_n, v_n \geq 0 \) for a.e. in \( \Omega \). Since that \( p_1 \in (p, p^*) \), it follows that \( c \) verifies (4.5). Thus, we have by lemma 4.2 that there exists \( (u, v) \in (W^1_p(\Omega, |x|^{-ap}))^2 \) with \( u_n \to u \) and \( v_n \to v \) in \( W^1_p(\Omega, |x|^{-ap}) \), as \( n \to \infty \). Hence, we conclude

\[
I(u, v) = c > 0 \quad \text{and} \quad I'(u, v) = 0,
\]

that is, \( (u, v) \) is a nontrivial and nonnegative weak solution of system (1.2). \( \square \)
5. Proof of theorem 1.2

The proof follows the steps the proof of theorem 1.2: By lemmata 2.4 and 2.5 there exists a \( (PS)_c \)-sequence \( \{(u_n, v_n)\} \) in \( W_0^{1,p}(\Omega, |x|^{-\alpha p}) \times W_0^{1,q}(\Omega, |x|^{-\beta q}) \) with \( c > 0 \) given as in \( \text{(2.5)} \) and \( u_n, v_n \geq 0 \) for a.e. in \( \Omega \). Moreover, \( \{(u_n, v_n)\} \) is bounded uniformly in \( \mu > 0 \), that is, there exist \( M > 0 \) such that \( \| (u_n, v_n) \| \leq M \) for all \( n \in \mathbb{N} \), uniformly in \( \mu > 0 \). Consequently, we get that \( c \leq M \) uniformly in \( \mu > 0 \).

Due to the boundedness of \( \{(u_n, v_n)\} \), there exists a subsequence, that we will denote by \( \{(u_n, v_n)\} \), and \( (u, v) \in W_0^{1,p}(\Omega, |x|^{-\alpha p}) \times W_0^{1,q}(\Omega, |x|^{-\beta q}) \) with \( u_n \rightharpoonup u \) weakly in \( W_0^{1,p}(\Omega, |x|^{-\alpha p}) \) and \( v_n \rightharpoonup v \) weakly in \( W_0^{1,q}(\Omega, |x|^{-\beta q}) \), as \( n \to \infty \). Then, arguing as in lemma 4.1 we obtain that \( (u, v) \) is a weak solution of system \( \text{(1.2)} \) with \( u, v \geq 0 \) for a.e. in \( \Omega \).

Now, we will prove that there exists \( \mu_0 > 0 \) such that \( u, v \) is nontrivial, provided that \( 0 < \mu < \mu_0 \).

Supposing by contradiction that \( u(x) \equiv 0 \) for a.e. \( x \in \Omega \) and proceeding as in the proof of theorem 1.2 we obtain \( l > 0 \) such that

\[
l = \lim_{n \to \infty} \frac{\|u_n\|^p}{\alpha} = \lim_{n \to \infty} \frac{\|v_n\|^q}{\gamma} = \mu \lim_{n \to \infty} \int_{\Omega} |x|^{-c_1} u_n^\alpha v_n^\gamma dx
\]

and

\[
c = \lim_{n \to \infty} I(u_n, v_n) = \left( \frac{\alpha}{p} + \frac{\gamma}{q} - 1 \right) l > 0.
\]

On the other hand, by Young’s inequality and definitions of \( C_{a,p}^* \) and \( C_{b,q}^* \), we obtain

\[
l \leq \frac{\alpha(p^* + p)/p}{\mu} (C_{a,p}^*)^{-p^*/p} + \frac{\gamma(q^* + q)/q}{q^*} (C_{b,q}^*)^{-q^*/q}\]

where \( \tau = \max\{p^*/p, q^*/q\} \) if \( l > 1 \), and \( \tau = \min\{p^*/p, q^*/q\} \) if \( l \leq 1 \). Therefore,

\[
l \geq \mu \left( \left( \frac{\alpha(p^* + p)/p}{\mu} (C_{a,p}^*)^{-p^*/p} + \frac{\gamma(q^* + q)/q}{q^*} (C_{b,q}^*)^{-q^*/q} \right) \right)^{\frac{1}{\tau}}
\]

Thus substituting the above inequality in \( \text{(5.1)} \) and taking \( \mu_0 > 0 \) small enough we conclude

\[
c \geq \left( \frac{\alpha}{p} + \frac{\gamma}{q} - 1 \right) \left( \left( \frac{\alpha(p^* + p)/p}{\mu} (C_{a,p}^*)^{-p^*/p} + \frac{\gamma(q^* + q)/q}{q^*} (C_{b,q}^*)^{-q^*/q} \right) \right)^{\frac{1}{\tau}} \geq M,
\]

for all \( 0 < \mu < \mu_0 \), which is an absurd.

6. Appendix

Proof of lemma 2.2

From equation \( \text{(2.1)} \) we obtain

\[
\| \nabla y_\epsilon \|^p_{L^p([-R_0, R_0], |x|^{-\alpha p})} = (\tilde{S}_{a,p,R})^{p^*/(p^* - p)} = (k_{a,p,R})^p \| \nabla U_{a,p,R} \|^p_{L^p([-R_0, R_0], |x|^{-\alpha p})},
\]

\[
\| y_\epsilon \|^p_{L^p([-R_0, R_0], |x|^{-\alpha p})} = (\tilde{S}_{a,p,R})^{p^*/(p^* - p)} = (k_{a,p,R})^p \| U_{a,p,R} \|^p_{L^p([-R_0, R_0], |x|^{-\alpha p})}.
\]

We observe that

\[
\nabla (\psi(x) U_{a,p,R}(x)) = \begin{cases} 
\nabla U_{a,p,R}(x) & \text{if } |x| < R_0 \\
U_{a,p,R}(x) \nabla \psi(x) + \psi(x) \nabla U_{a,p,R}(x) & \text{if } R_0 \leq |x| < 2R_0 \\
0 & \text{if } |x| \geq 2R_0
\end{cases}
\]
and
\[ \nabla U_{a,p,\epsilon}(x) = -\frac{N - p - ap}{p - 1} \cdot \frac{|x|^{d_4 p(N - p - ap)/(p - 1)(N - d_4 p) - 2} \cdot \epsilon}{|x|^{d_4 p(N - p - ap)/(p - 1)(N - d_4 p) - 2} \cdot \epsilon} \cdot |x|^{d_4 p(N - p - ap)/(p - 1)(N - d_4 p) - 2} \cdot \epsilon. \]

Therefore,
\[
\int_{\Omega} |x|^{-ap} |\nabla (\psi U_{a,p,\epsilon}(x))|^p dx = O(1) + \int_{\mathbb{R}^N} |x|^{-ap} |\nabla U_{a,p,\epsilon}(x)|^p dx
\]
\[ = O(1) + (\hat{S}_{a,p,R})^{p/(p'-p)} (k_{a,p}(\epsilon))^\sigma \]
and
\[
\int_{\Omega} |x|^{-c_1 p} |\psi(x) U_{a,p,\epsilon}(x)|^p dx = O(1) + \int_{\mathbb{R}^N} |x|^{-c_1 p} |U_{a,p,\epsilon}(x)|^p dx
\]
\[ = O(1) + (\hat{S}_{a,p,R})^{p/(p'-p)} (k_{a,p}(\epsilon))^\sigma. \]

Consequently,
\[
\int_{\Omega} |x|^{-ap} |\nabla u_{\epsilon}(x)|^p dx = \frac{O(1) + (\hat{S}_{a,p,R})^{p/(p'-p)} (k_{a,p}(\epsilon))^\sigma}{O(1) + (\hat{S}_{a,p,R})^{p/(p'-p)} (k_{a,p}(\epsilon))^\sigma} \frac{(k_{a,p}(\epsilon))^{p/(p'-p)} (k_{a,p}(\epsilon))^\sigma}{(k_{a,p}(\epsilon))^{p/(p'-p)} (k_{a,p}(\epsilon))^\sigma}
\]
\[ \leq \hat{S}_{a,p,R} + O(k_{a,p}(\epsilon)^p)
\]
\[ = \hat{S}_{a,p,R} + O(\epsilon^{(N-d_4 p)/(d_1 p)}). \]

Now, we prove that \(\|u_{\epsilon}\|_{L^p(\Omega,|x|^{-\beta_1})}^p \) is as in (2.2). Considering the changes of variables by the polar coordinates and \( s = R_0^{-1} \epsilon^{-1/\alpha} \) with \( \alpha = \frac{d_4 p(N - p - ap)}{(p - 1)(N - d_4 p)} \), we obtain
\[
\int_{\Omega} |x|^{-(a+1)p_1 + c} |\psi U_{a,p,\epsilon}|^p dx
\]
\[ \geq O(1) + \int_{|x| < R_0} |x|^{-(a+1)p_1 + c} |U_{a,p,\epsilon}|^p dx
\]
\[ = O(1) + \omega_N \int_{0}^{R_0}(\epsilon + s)(N-d_4 p)^{p_1} \frac{d_1 p}{d_1 p} dr
\]
\[
= O(1) + \omega_N (R_0^\alpha) \int_{0}^{(N-d_4 p)^{p_1} d_1 p} \frac{d_1 p(N - p - ap)}{d_1 p (N - d_4 p)}
\]
\[ \times \int_{0}^{(N-d_4 p)^{p_1} d_1 p} ds.
\]

Assuming that \( \epsilon = [(p_1 - p + 1)N - (a + 1)p_1]/(p - 1) \), we see that
\[
-\frac{(N-d_4 p)p_1}{d_1 p} + \frac{(p - 1)(N - p - ap_1 + c)(N - d_4 p)}{d_1 p (N - p - ap)} = 0,
\]
\[
-(a + 1)p_1 + c + N - \frac{(N - p - ap_1)}{p - 1} = 0.
\]

Therefore, by (6.1),
\[
\int_{\Omega} |x|^{-(a+1)p_1 + c} |\psi U_{a,p,\epsilon}|^p dx \geq \omega_N \int_{1}^{\epsilon^{1/\alpha}} s^{-(a+1)p_1 + c + N - 1} \frac{(N-d_4 p)p_1}{R_0^\alpha + s^{N-d_4 p)}^{p_1} d_1 p ds
\]
Using (6.1) we obtain

\[ \begin{align*}
\int_{\Omega} |x|^{-(a+1)p_1+c} |u_e|^{p_1} dx & \geq \frac{\omega_N}{R_0^{-\alpha} + 1} |(N-d_1p)p_1/d_1p| \ln(\epsilon) = O(|\ln(\epsilon)|).
\end{align*} \]

Hence, we obtain

\[ \begin{align*}
\int_{\Omega} |x|^{-(a+1)p_1+c} |u_e|^{p_1} dx & \geq O(|\ln(\epsilon)|) \\
& \geq O(1) + (S_{a,p,R})^{p_1/(p^* - p)}(k_{a,p}(\epsilon) - p^*)^{p_1/p^*} \\
& \geq O(\epsilon(N-d_1p)p_1/d_1p^2 |\ln(\epsilon)|).
\end{align*} \]

Assuming that \( c > [(p_1 - p + 1)N - (a + 1)p_1]/(p - 1) \), we have

\[ -\frac{(N - d_1p)p_1}{d_1p} + \frac{(N - p_1 - ap_1 + c)(p - 1)(N - d_1p)}{d_1p(N - p - ap)} > 0, \]

\[ -(a + 1)p_1 + c + N - \frac{(N - p - ap)p_1}{(p - 1)} > 0. \]

Consequently, by (6.1),

\[ \begin{align*}
\int_{\Omega} |x|^{-(a+1)p_1+c} |\psi U_{a,p,e}|^{p_1} dx & \geq O(1) + \omega_N(R_0^a \epsilon)^{-(N-d_1p)p_1} + \frac{(N - p_1 - ap_1 + c)(p - 1)(N - d_1p)}{d_1p(N - p - ap)} \\
& \times \frac{1}{(R_0^{-\alpha} + 1)(N-d_1p)p_1/d_1p} \int_{1/2}^1 s^{-(a+1)p_1+c+N-1} ds \geq O(1).
\end{align*} \]

Hence, we get

\[ \begin{align*}
\int_{\Omega} |x|^{-(a+1)p_1+c} |u_e|^{p_1} dx & \geq O(1) \\
& \geq \frac{(k_{a,p}(\epsilon) - p^*)^{p_1} [O(k_{a,p}(\epsilon) - p^*) + (S_{a,p,R})^{p_1/(p^* - p)}]^{p_1/p^*}}{O(1) + (S_{a,p,R})^{p_1/(p^* - p)}p_1/p^*} \\
& \geq O(\epsilon(N-d_1p)p_1/d_1p^2).
\end{align*} \]

If \( c < [(p_1 - p + 1)N - (a + 1)p_1]/(p - 1) \), we see that

\[ -\frac{(N - d_1p)p_1}{d_1p} + \frac{(N - p_1 - ap_1 + c)(p - 1)(N - d_1p)}{d_1p(N - p - ap)} < 0, \]

\[ -(a + 1)p_1 + c + N - \frac{(N - p - ap)p_1}{(p - 1)} < 0. \]

Using (6.1) we obtain

\[ \begin{align*}
\int_{\Omega} |x|^{-(a+1)p_1+c} |\psi U_e|^{p_1} dx & \geq O(1) \\
& \geq O(1) + \frac{(N - p_1 - ap_1 + c)(p - 1)(N - d_1p)}{d_1p(N - p - ap)} \\
& \times \frac{1}{(R_0^{-\alpha} + 1)(N-d_1p)p_1/d_1p} \int_{1/2}^1 s^{-(a+1)p_1+c+N-1} ds \\
& \geq O(\epsilon \frac{(N-d_1p)p_1}{d_1p} + \frac{(N - p_1 - ap_1 + c)(p - 1)(N - d_1p)}{d_1p(N - p - ap)}).
\end{align*} \]
Hence, we conclude

$$\int_{\Omega} |x|^{-(\alpha+1)p_1+c} |u_{\epsilon}|^{p_1} dx \geq O\left( \frac{\epsilon^{-(N-d_1)p_1} + \frac{(N-p_1-1)(N-d_1)}{2}(N-p-1)(N-d_1)}{a_1 p_1} \right) \frac{O(1) + \left( \frac{S_{a,p,R}}{p^{*}} \right) (k_a(p))(e) - p^{*}}{P / p^{*}}$$

$$\geq O \left( \frac{\epsilon^{-(N-d_1)p_1} + \frac{(N-p_1-1)(N-d_1)}{2}(N-p-1)(N-d_1)}{a_1 p_1} \right).$$

**References**


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