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OSCILLATION AND ASYMPTOTIC BEHAVIOUR OF A HIGHER ORDER NEUTRAL DIFFERENTIAL EQUATION WITH POSITIVE AND NEGATIVE COEFFICIENTS

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Abstract. In this paper, we obtain necessary and sufficient conditions so that every solution of

$$\big(y(t) - p(t)y(r(t))\big)^{(n)} + q(t)G(y(g(t))) - u(t)H(y(h(t))) = f(t)$$

oscillates or tends to zero as $t \to \infty$, where n is an integer $n \geq 2$, $q > 0$, $u \geq 0$. Both bounded and unbounded solutions are considered in this paper. The results hold also when $u \equiv 0$, $f(t) \equiv 0$, and $G(u) \equiv u$. This paper extends and generalizes some recent results.

1. Introduction

In this article, we obtain necessary and sufficient conditions for every solution of the higher-order neutral functional differential equation

$$\big(y(t) - p(t)y(r(t))\big)^{(n)} + q(t)G(y(g(t))) - u(t)H(y(h(t))) = f(t) \tag{1.1}$$

to oscillate or to tend to zero as t tends to infinity, where, n is an integer $n \geq 2$, $p, f \in C([0,\infty),\mathbb{R})$, $q, u \in C([0,\infty),[0,\infty))$, and $G, H \in C(\mathbb{R},\mathbb{R})$. The functional delays $r(t), g(t)$ and $h(t)$ are continuous, strictly increasing and unbounded functions for $t \geq t_0$ such that $r(t) \leq t, g(t) \leq t$, and $h(t) \leq t$. Some of the following assumptions will be used in this article.

- (H0) G is non-decreasing, $xG(x) > 0$ for $x \neq 0$.
- (H1) $\liminf_{t\to\infty} r(t)/t > 0$.
- (H2) H is bounded.
- (H3) $\liminf_{|u|\to\infty} G(u)/u \geq \delta$ where $\delta > 0$.
- (H4) $\int_{t_0}^{\infty} t^{n-2}q(t)\,dt = \infty$, $n \geq 2$.
- (H5) $\int_{t_0}^{\infty} t^{n-1}u(t)\,dt < \infty$.
- (H6) $\int_{t_0}^{\infty} t^{n-1}q(t)\,dt = \infty$.
- (H7) There exists a bounded function $F \in C^{(n)}([0,\infty),\mathbb{R})$ such that $F^{(n)}(t) = f(t)$ and $\lim_{t\to\infty} F(t) = 0$.

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(H8) There exists a bounded function $F \in C^{(n)}([0,\infty),\mathbb{R})$ such that $F^{(n)}(t) = f(t)$.

(H9) $\liminf_{t\to\infty} g(t)/t > 0$.

(H10) $\int^{\infty} q(t)\,dt = \infty$

Note that, we do not need the condition "$xH(x) > 0$ for $x \neq 0$" in the proofs of our results. However, one may assume them for technical reasons; i.e, to make (1.1) a neutral equation with positive and negative coefficients. Further, we note that, for $n \geq 2$, condition (H10) implies (H4) and furthermore, (H4) implies (H6). For τ, σ, α positive constants, we put $r(t) = t - \tau$, $g(t) = t - \sigma$ and $h(t) = t - \alpha$. Then (1.1) reduces to

$$\big(y(t) - p(t)y(t-\tau)\big)^{(n)} + q(t)G(y(t-\sigma)) - u(t)H(y(t-\alpha)) = f(t)\,. \tag{1.2}$$

If $n = 1$ then (1.2) reduces to

$$\big(y(t) - p(t)y(t-\tau)\big)' + q(t)G(y(t-\sigma)) - u(t)G(y(t-\alpha)) = f(t), \tag{1.3}$$

which was studied in [23]. Our objective is to generalize the results in [23] to the higher-order equation (1.1). Further, if $u = 0$, then (1.1) takes the form

$$\big(y(t) - p(t)y(r(t))\big)^{(n)} + q(t)G(y(g(t))) = f(t). \tag{1.4}$$

Hence (1.2)–(1.4) are particular cases of (1.1).

The authors in [25, p. 195], suggested the study of unbounded solutions for (1.4), particularly when $0 \leq p(t) \leq p < 1$; this paper accomplishes that task. The motivation of this work came from the fact that almost no work is done on oscillatory behaviour of unbounded solutions of neutral differential equations (1.2) of order $n > 2$. For the case when n is odd, the authors in [26], have presented a result for the linear equation

$$\big(y(t) - p(t)y(t-\tau)\big)^{(n)} + q(t)y(t-\sigma) - u(t)y(t-\alpha) = 0, \tag{1.5}$$

with the assumptions

(AD1) $q(t) > u(t - \sigma + \alpha)$ and

(AD2) $\sigma > \alpha$ or $\alpha > \sigma$.

In [14], the author obtained sufficient conditions for the oscillation of solutions of the linear homogeneous equation

$$\big(y(t) - p(t)y(t-\tau)\big)' + q(t)y(t-\sigma) - u(t)y(t-\alpha) = 0, \tag{1.6}$$

with the assumptions (AD1) and (AD2). In [22] the authors obtained sufficient conditions for oscillation of the equation (1.3) and other results under the conditions (AD1), (AD2), and

(AD3) $\liminf_{|u|\to\infty} G(u)/u < \beta$, for some $\beta > 0$.

In [13] the authors studied a second order neutral equation with several delay terms of the form

$$\big(y(t) - p(t)y(t-\tau)\big)'' + \sum_{i=1}^{k} q_i(t)y(t-\sigma_i) - \sum_{i=1}^{m} u_i(t)y(t-\alpha_i) = f(t). \tag{1.7}$$

When $k = m = 1$, the above equation takes the form

$$\big(y(t) - p(t)y(t-\tau)\big)'' + q(t)y(t-\sigma) - u(t)y(t-\alpha) = f(t). \tag{1.8}$$

Here also, the authors require the conditions (AD1), (AD2),

(AD4) $u(t) < u(t-\alpha)$,
(AD5) $q(t) \geq u(t-\alpha) \geq k > 0$.

In this paper, an attempt is made to relax the conditions (AD1)–(AD5) and study the oscillation and non-oscillation of (1.1). Our results hold also when $u \equiv 0, f(t) \equiv 0$, and $G(u) \equiv u$. As a consequence, this paper extends and generalizes some of the recent results in [13, 22, 23, 25]. Appropriate examples are included to illustrate our results.

Let $t_0 \geq 0$ and $t_{-1} := \min\{r(t_0), g(t_0), h(t_0)\}$. By a *solution* of (1.1), we mean a function $y \in C([t_{-1}, \infty), \mathbb{R})$ such that $y(t) - p(t)y(r(t))$ is n times continuously differentiable on $[t_0, \infty)$ and the neutral equation (1.1) is satisfied by $y(t)$ for all $t \geq t_0$. It is known that (1.1) has a unique solution provided that an initial function $\phi \in C([t_{-1}, t_0], \mathbb{R})$ is given to satisfy $y(t) = \phi(t)$ for all $t \in [t_{-1}, t_0]$. Such a solution is said to be *non-oscillatory* if it is eventually positive or eventually negative for large t, otherwise it is called *oscillatory*.

In this work we assume the existence of solutions and study only their qualitative behaviour. For existence and uniqueness of solutions, the reader is referred to [8]. In the sequel, unless otherwise specified, when we write a functional inequality, it will be assumed to hold for all sufficiently large values of t.

2. Main results

We assume that $p(t)$ satisfies one of the following conditions in this work.

(A1) $0 \leq p(t) \leq p < 1$,
(A2) $-1 < -p \leq p(t) \leq 0$,
(A3) $0 \leq p(t) \leq p_1$,
(A4) $-p_2 \leq p(t) \leq -p_1 < -1$,
(A5) $1 < p_1 \leq p(t) \leq p_2$,
(A6) $-p_1 \leq p(t) \leq 0$,

where p, p_1 and p_2 are real numbers.

Before we present our main results, we state some Lemmas.

Lemma 2.1. [11, p.193] *Let $y \in C^n([0,\infty), \mathbb{R})$ be of constant sign and not identically zero on any interval $[T, \infty)$, $T \geq 0$, and $y^{(n)}(t)y(t) \leq 0$. Then there exists a number $t_0 \geq 0$ such that the functions $y^{(j)}(t)$, $j = 1, 2, \dots, n-1$, are of constant sign on $[t_0, \infty)$ and there exists a number $m \in \{1, 3, \dots, n-1\}$ when n is even or $m \in \{0, 2, 4, \dots, n-1\}$ when n is odd such that*

$$y(t)y^{(j)}(t) > 0, \quad \text{for } j = 0, 1, 2, \dots, m,\ t \geq t_0,$$
$$(-1)^{n+j-1}y(t)y^{(j)}(t) > 0 \quad \text{for } j = m+1, m+2, \dots, n-1,\ t \geq t_0.$$

Lemma 2.2. [25, Lemma 1] *Let $u, v, p : [0, \infty) \to \mathbb{R}$ be such that $u(t) = v(t) - p(t)v(r(t))$, $t \geq T_0$, where $r(t)$ is a continuous, monotonic increasing and unbounded function such that $r(t) \leq t$. Suppose that $p(t)$ satisfies one of the conditions* (A2), (A3), (A4). *If $v(t) > 0$ for $t \geq 0$ and $\liminf_{t\to\infty} v(t) = 0$ and $\lim_{t\to\infty} u(t) = L$ exists, then $L = 0$.*

Lemma 2.3. [21, Lema 2.1] *If $\int_0^\infty t^{n-1}|f(t)|\,dt < \infty$, then* (H7) *holds.*

Proof. Let

$$F(t) = \frac{(-1)^n}{(n-1)!}\int_t^\infty (s-t)^{n-1}f(s)\,ds, t \geq 0.$$

Then $F^{(n)}(t) = f(t)$ and $\lim_{t\to\infty} F(t) = 0$. Thus (H7) holds. $\square$

Theorem 2.4. *Suppose that $n \geq 2$ and $p(t)$ satisfies any one of the two conditions* (A1), (A2). *Then under assumptions* (H0), (H2)–(H5), (H8), (H9), *every unbounded solution of* (1.1) *oscillates.*

Proof. For the sake of contradiction, suppose that $y(t)$ is a non-oscillatory and unbounded solution of (1.1) for large t. Assume that $y(t) > 0$, eventually. Then there exists $t_0 > T_1$ such that $y(t) > 0$, $y(r(t)) > 0$, $y(g(t))$ and $y(h(t)) > 0$ for $t \geq t_0$. For simplicity of notation, define for $t \geq t_0$,

$$z(t) = y(t) - p(t)y(r(t))\,. \tag{2.1}$$

Further, due to the assumption (H2) and (H5), we define for $t \geq t_0$

$$k(t) = \frac{(-1)^{n-1}}{(n-1)!}\int_t^\infty (s-t)^{n-1}u(s)H(y(h(s)))\,ds. \tag{2.2}$$

Then

$$k^{(n)}(t) = -u(t)H(y(h(t))). \tag{2.3}$$

Set

$$w(t) = z(t) + k(t) - F(t). \tag{2.4}$$

Then using (2.1)–(2.4) in (1.1), we obtain

$$w^{(n)}(t) = -q(t)G(y(g(t))) \leq 0. \tag{2.5}$$

Hence $w, w', \dots w^{(n-1)}$ are monotonic and single sign for $t \geq t_1 \geq t_0$. Then $\lim_{t\to\infty} w(t) = \lambda$, where $-\infty \leq \lambda \leq +\infty$. From (2.2), it follows, due to (H2) and (H5) that

$$k(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{2.6}$$

Since $y(t)$ is unbounded, there exists a sequence $\{a_n\}$ such that

$$a_n \to \infty, \quad y(a_n) \to \infty, \quad \text{as } n \to \infty,$$

and

$$y(a_n) = \max\{y(s) : t_1 \leq s \leq a_n\}. \tag{2.7}$$

We may choose n large enough so that for $n \geq N_0$, $\min\{r(a_n), g(a_n), h(a_n)\} > t_1$. Then from (2.6) and (H8), it follows that, for $0 < \epsilon$, we can find a positive integer N_1 and a real number η such that, $|k(a_n)| < \epsilon$ and $|F(a_n)| < \eta$ for $n \geq N_1 \geq N_0$. Hence for $n \geq N_1$, if (A1) holds, then we have

$$w(a_n) \geq y(a_n)(1-p) - \epsilon - \eta.$$

If (A2) holds, then for $n \geq N_1$,we have

$$w(a_n) \geq y(a_n) - \epsilon - \eta.$$

Taking $n \to \infty$, we find $\lim_{t\to\infty} w(t) = \infty$, because of the monotonic nature of $w(t)$. Hence $w > 0, w' > 0$ for $t \geq t_2 \geq t_1$,. Since $w^{(n)}(t) \not\equiv 0$ and is non positive, it follows from Lemma 2.1 that there exists a positive integer m such that $n-m$ is odd and for $t \geq t_3 \geq t_2$, we have $w^{(j)}(t) > 0$ for $j = 0, 1, \dots, m$ and $w^{(j)}(t)w^{(j+1)}(t) < 0$ for $j = m, m+1, \dots, n-2$. Then $\lim_{t\to\infty} w^{(m)}(t) = l$ exists (as a finite number). Hence $m \geq 1$. Integrating (2.5), $n-m$ times from t to ∞, we obtain for $t \geq t_3$

$$w^{(m)}(t) = l + \frac{(-1)^{n-m-1}}{(n-m-1)!}\int_t^\infty (s-t)^{n-m-1}q(s)G(y(g(s)))\,ds. \tag{2.8}$$

This implies

$$\int_t^\infty (s-t)^{n-m-1} q(s) G(y(g(s)))\, ds < \infty, \quad \text{for } t \geq t_3. \tag{2.9}$$

From (H4) and the above inequality we obtain $\liminf_{t\to\infty} G(y(g(t)))/t^{m-1} = 0$. By (H0) and (H3), we get $\liminf_{t\to\infty} y(g(t))/t^{m-1} = 0$. From (H9), it follows that, we can find $b > 0$ such that $g(t)/t \geq b > 0$ for large t and since $\lim_{t\to\infty} g(t) = \infty$ then we have

$$\liminf_{t\to\infty} \frac{y(t)}{t^{m-1}} = 0.$$

Since $m \geq 1$,we can choose $M_0 > 0$ such that $w(t) > M_0 t^{m-1}$ for $t \geq t_4 \geq t_3$. Thus

$$\liminf_{t\to\infty} \frac{y(t)}{w(t)} = 0. \tag{2.10}$$

Set, for $t \geq t_4$,

$$p^*(t) = p(t)\frac{w(r(t))}{w(t)}.$$

From (H8), (2.6) and $\lim_{t\to\infty} w(t) = \infty$, we obtain

$$\lim_{t\to\infty} \frac{(F(t) - k(t))}{w(t)} = 0.$$

Then we have

$$\begin{aligned}
1 &= \lim_{t\to\infty} \big[\frac{w(t)}{w(t)}\big] \\
&= \lim_{t\to\infty} \big[\frac{y(t) - p(t)y(r(t)) - (F(t) - k(t))}{w(t)}\big] \\
&= \lim_{t\to\infty} \big[\frac{y(t)}{w(t)} - \frac{p^*(t)y(r(t))}{w(r(t))} - \frac{(F(t) - k(t))}{w(t)}\big] \\
&= \lim_{t\to\infty} \big[\frac{y(t)}{w(t)} - \frac{p^*(t)y(r(t))}{w(r(t))}\big].
\end{aligned} \tag{2.11}$$

Since $w(t)$ is a increasing function, $w(r(t))/w(t) < 1$. If $p(t)$ satisfies (A1) then $0 \leq p^*(t) < p(t) \leq p < 1$. However, if $p(t)$ satisfies (A2), then $0 \geq p^*(t) \geq p(t) \geq -p > -1$. Hence it is clear that if $p(t)$ satisfies (A1) or (A2) then $p^*(t)$ also lies in the ranges (A1) or (A2) accordingly. Hence use of Lemma 2.2 yields, due to (2.10), that

$$\lim_{t\to\infty} \big[\frac{y(t)}{w(t)} - \frac{p^*(t)y(r(t))}{w(r(t))}\big] = 0,$$

a contradiction to (2.11). Hence the unbounded solution $y(t)$ cannot be eventually positive.

Next, if $y(t)$ is an eventually negative solution of (1.1) for large t, then we set $x(t) = -y(t)$ to obtain $x(t) > 0$ and then (1.1) reduces to

$$\big(x(t) - p(t)x(\tau(t))\big) + q(t)\tilde{G}(x(g(t))) - u(t)\tilde{H}(x(h(t))) = \tilde{f}(t) \tag{2.12}$$

where

$$\tilde{f}(t) = -f(t), \quad \tilde{G}(v) = -G(-v), \quad \tilde{H}(v) = -H(-v). \tag{2.13}$$

Further, $\tilde{F}(t) = -F(t)$ implies $\tilde{F}^n(t) = \tilde{f}(t)$.

In view of the above facts, it can be easily verified that the following conditions hold:

(H0') $\tilde{G}$ is non-decreasing and $x\tilde{G}(x) > 0$ for $x \neq 0$,
(H2') $\tilde{H}$ is bounded,
(H3') $\liminf_{|v|\to\infty} \tilde{G}(v)/v \geq \delta > 0$,
(H8') There exists a bounded function $\tilde{F}(t)$ such that $\tilde{F}^n(t) = \tilde{f}(t)$.

Proceeding as in the proof for the case $y(t) > 0$, we obtain a contradiction. Hence $y(t)$ is oscillatory and the proof is complete. ☐

Remark 2.5. The above theorem answers the open problem in [25, p. 195]; i.e, to study oscillatory behaviour of unbounded solutions of (1.1), when $p(t)$ satisfies (A1).

The following example illustrates Theorem 2.4.

Example 2.6. Consider the neutral equation

$$\begin{aligned}&(y(t) - \alpha y(t-2\pi))'' + 2e^{-3\pi/2}(e^{2\pi} - \alpha)y(t-\pi/2) - e^{-4\pi}t^{-2}H(y(t-4\pi))\\ &= \frac{-e^t\cos(t)}{t^2(e^{8\pi} + e^{2t}\cos^2(t))},\end{aligned}$$

where $0 \leq \alpha < 1$ or $-1 < \alpha \leq 0$, $H(u) = u/(u^2+1)$. Clearly, the above equation satisfies all the conditions of the Theorem 2.4, hence it admits an oscillatory solution. In this example, $y(t) = e^t\cos(t)$. Note that, by Lemma 2.3, (H8) is satisfied with

$$F(t) = \int_t^\infty \frac{(t-s)e^s\cos(s)}{s^2(e^{8\pi} + e^{2s}\cos^2(s))}\,ds\,.$$

In Theorem2.4, if we put $p(t) = 0$, then we get the following result about higher order delay differential equation with positive and negative coefficients.

Corollary 2.7. *Suppose $n \geq 2$. If the conditions* (H0), (H2)–(H5), (H8), (H9) *are met, then every unbounded solution of the equation*

$$y^{(n)}(t) + q(t)G(y(g(t))) - u(t)H(y(h(t))) = f(t)$$

oscillates.

Remark 2.8. We note that (H7) implies (H8), but not conversely. Note that (H7) is equivalent to the condition:

> There exists a function $F(t) \in C^n([0,\infty),\mathbb{R})$ such that $F^{(n)}(t) = f(t)$ and $\lim_{t\to\infty} F(t) = \gamma$.

Clearly (H7) implies the above condition. Conversely, suppose that the above condition holds. If $\lim_{t\to\infty} F(t) = \gamma \neq 0$, then we may take $L(t) = F(t) - \gamma$. Consequently, $L(t)$ satisfies (H7). Hence without any loss of generality, we may assume (H7) in the subsequent results of this section.

Remark 2.9. In Theorem 2.4, if we assume $y(t)$ is bounded, then we have the theorem with the condition (H6), which is weaker than (H4) and thus, we would be able to completely relax the conditions (H2), (H3) and (H9) in the following result, which extends and generalizes [22, Theorem 2.7] and [15, Theorem 2.4].

Theorem 2.10. *If $p(t)$ satisfies one of the conditions* (A1), (A2), (A4), (A5)*, then under the assumptions* (H0), (H5)–(H7)*, every bounded solution of* (1.1) *oscillates or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be an eventually positive solution of (1.1), which is bounded. Then set $z(t), k(t)$, and $w(t)$ as in (2.1), (2.2) and (2.4) respectively and obtain (2.5). Note that, here $k(t)$ is well defined, bounded, and satisfies (2.6) due to boundedness of $y(t)$ and (H5). From the facts: $w(t)$ is monotonic, $y(t)$ is bounded,(H7) and (2.6) hold, we obtain $\lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t) = \lambda$, which exists (as a finite number). From Lemma 2.1, it follows that $(-1)^{n+k}w^{(k)}(t) < 0$ and $\lim_{t\to\infty} w^{(k)}(t) = 0$, for $k = 1, 2, \dots, n-1$. Then integrating (2.5), n-times from t to ∞, we obtain for $t \ge t_1$,

$$w(t) = \lambda + \frac{(-1)^{n-1}}{(n-1)!}\int_t^\infty (s-t)^{n-1}q(s)G(y(g(s)))\,ds. \tag{2.14}$$

Thus,

$$\frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}q(s)G(y(g(s)))\,ds < \infty, \quad t \ge t_1. \tag{2.15}$$

Then from (H6) and (2.15), it follows that $\liminf_{t\to\infty} G(y(t)) = 0$. Then by (H0) we get $\liminf_{t\to\infty} y(t) = 0$. Application of Lemma2.2 yields $\lim_{t\to\infty} z(t) = 0$. If $p(t)$ is in (A1) then

$$\begin{aligned}
0 = \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty}(y(t) - p(t)y(r(t)))\\
&\ge \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(-p(t)y(r(t)))\\
&\ge (1-p)\limsup_{t\to\infty} y(t).
\end{aligned}$$

This implies $\limsup_{t\to\infty} y(t) = 0$. Hence $y(t) \to 0$ as $t \to \infty$. If $p(t)$ satisfies (A2) or (A4), then, since $y(t) \le z(t)$, it follows that $y(t) \to 0$ as $t \to \infty$. If $p(t)$ satisfies (A5), then $z(t) \le y(t) - p_1 y(r(t))$. Hence, it follows that

$$\begin{aligned}
0 &= \liminf_{t\to\infty} z(t)\\
&\le \liminf_{t\to\infty}[y(t) - p_1 y(r(t))]\\
&\le \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}[-p_1 y(r(t))]\\
&= (1-p_1)\limsup_{t\to\infty} y(t).
\end{aligned}$$

Then $\limsup_{t\to\infty} y(t) = 0$. Thus $\lim_{t\to\infty} y(t) = 0$. The proof for the case when $y(t) < 0$ eventually, is similar. Thus the proof is complete. □

The following example justifies the assumption of boundedness on $y(t)$ in the above Theorem when $p(t)$ satisfies (A5).

Example 2.11. The neutral difference equation

$$(y(t) - 2ey(t-1))^{(n)} + e^2 y(t-2) - e^{-t+2}\frac{y(t-2)}{y^2(t-2)+1} = -(e^{t-2}+1)^{-1},$$

satisfies all the conditions of Theorem 2.4, with the exception that $p(t)$ satisfies the condition (A5). But this neutral equation has an unbounded solution e^t, which tends to ∞ as $t \to \infty$.

The following examples illustrates Theorem 2.10.

Example 2.12. Consider the neutral equation

$$[y(t) + e^{-t/2}y(t/2)]^{(n)} + q(t)y^3(t/3) - u(t)y^5(t/7) = 0 \quad \text{for } t \ge 1.$$

Here n is any positive integer, may be odd or even. Further, in the above equation take

$$q(t) = \begin{cases} 3, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

Moreover, assume

$$u(t) = \begin{cases} e^{-2t/7}, & \text{if } n \text{ odd,} \\ 3e^{-2t/7}, & \text{if } n \text{ is even.} \end{cases}$$

Then $G(u) = u^3$, $H(u) = u^5$, $r(t) = t/2$, $g(t) = t/3$, $h(t) = t/7$. $p(t) = -e^{-t/2}$ which satisfies (A2). This equation satisfies all the conditions of Theorem 2.10 with(A2). Hence all non-oscillatory solutions tend to zero as $t \to \infty$, which is the case of the solution $y(t) = e^{-t}$. Most of the results in the reference fail to apply to this equation because of the functional delays.

Example 2.13. The neutral differential equation

$$(y(t) - e^{-1}y(t-1))^{(n)} + y^3(t-2) - e^{-t}y(t-2)e^{-y^2(t-2)} = e^{6-3t} - e^{-(2t-2+e^{4-2t})},$$

satisfies all the conditions of Theorem 2.10, for (A1). Hence by the theorem, all bounded solutions are either oscillatory or tend to zero as $t \to \infty$, which is the case of the solution $y(t) = e^{-t}$. On the other hand, if we compare the above equation to (1.2), then we observe the condition (AD2) is not satisfied,because $\sigma = \alpha = 2$. Hence the results of the papers [13, 14, 17, 22, 26] can not be applied to this equation.

We combine Theorem 2.4 with Theorem 2.10 (restricted to (A1) or (A2)) to get the following result.

Theorem 2.14. *Suppose that $n \geq 2$ and $p(t)$ satisfies any one of the ranges* (A1), (A2). *Then under assumptions* (H0), (H2)–(H5), (H7), (H9), *every solution of* (1.1) *oscillates or tends to zero as $t \to \infty$.*

Remark 2.15. The above theorem extends and generalizes [13, Theorems 2,4], [22, theorems 2.2 and 2.4], and [18, Theorem 2.2]. Further, note that (H4) implies (H6), for $n \geq 2$, but not conversely. While dealing with unbounded solutions in Theorem2.14, we need the stronger assumption (H4), where as, dealing with bounded solutions in Theorem 2.10, we needed the weaker one; i.e, (H6). This is justified from the following example.

Example 2.16. Consider the equation

$$\big(y(t) - py(t-1)\big)'' + \Big(\frac{1}{t^2 \log(t-1)} - \frac{p}{(t-1)^2 \log(t-1)}\Big)y(t-1) = 0, \tag{2.16}$$

for $t > \max\{3, 1/(1-\sqrt{p})\}$, where $0 < p < 1$. Here, $p(t)$ is in (A1), $u(t) \equiv 0$, and $f(t) \equiv 0$. This equation satisfies all the conditions of Theorem 2.14 except (H4), but satisfies (H6). It admits a non-oscillatory solution $y(t) = \log(t) \to \infty$ as $t \to \infty$.

Remark 2.17. The assumption (H3) implies that G is linear or super linear. To deal with the unbounded solutions, we used (H3) in Theorem 2.14, though we do not require this, while dealing with bounded solutions in Theorem 2.10. Hence to justify our assumption (H3) in Theorem 2.14, we give the following example.

Example 2.18. Consider the neutral equation

$$\Big(y(t)-\frac{1}{16}y(t-1)\Big)'''+\frac{3}{8(t-1)^{1/2}}\Big[\frac{1}{t^{3/2}}-\frac{1}{16(t-1)^{3/2}}\Big]y^{1/3}(t-1)=0, \quad (2.17)$$

for $t \ge t_0 > 2$. It is obvious that (H4) holds; i.e, $\int_{t_0}^\infty tq(t)\,dt = \infty$, and all the conditions of Theorem 2.14 (with (A1)) are satisfied, except for (H3). Hence we do not get the conclusion of the Theorem 2.14 from (2.17). That is why, (2.17) admits a solution $y(t) = t^{3/2}$, which tends to ∞ as $t \to \infty$.

For our next result, we need to state the following Lemma.

Lemma 2.19. [12, p. 193] *Let f and g be two positive functions in $[a,t]$ with $\lim_{t\to\infty} f(t)/g(t) = l$, where l is non-zero real number. Then $\int_a^\infty f(t)\,dt$ and $\int_a^\infty g(t)\,dt$ converge or diverge together. Also, if $f/g \to 0$ and $\int_a^\infty g(t)\,dt$ converges, then $\int_a^\infty f(t)\,dt$ converges and if $f/g \to \infty$ and $\int_a^\infty g(t)\,dt$ diverges, then $\int_a^\infty f(t)\,dt$ diverges.*

Our next result, where $p(t)$ satisfies (A6), improves and generalizes [20, Theorem 2.1], [13, Theorem 3], and [22, Theorem 2.11].

Theorem 2.20. *Assume* (A6)*, that $n \ge 2$, that $\liminf_{t\to\infty} r'(t) > 0$, and that $r(g(t)) = g(r(t))$. Also assume* (H0)–(H3), (H5), (H7), (H9),

- (H11) *$\int_{t_0}^\infty t^{n-2}q^*(t)\,dt = \infty$, where $q^*(t) = \min[q(t), q(r(t))]$;*
- (H12) *$G(-u) = -G(u)$;*
- (H13) *For $u > 0$, $v > 0$, there exists a constant $\beta > 0$ such that, $G(u)G(v) \ge G(uv)$ and $G(u) + G(v) \ge \beta G(u + v)$.*

Then every solution of (1.1) *oscillates or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be an eventually positive solution of (1.1) for $t \ge t_0 \ge T_1$. Then set $z(t)$, $k(t)$, and $w(t)$ as in (2.1), (2.2) and (2.4) respectively to get (2.5) for $t > t_1 \ge t_0$. Then (2.6) holds. Hence $w(t), w'(t), w''(t), \dots, w^{(n-1)}(t)$ are monotonic in $[t_1,\infty)$. Consequently, From (2.6) and (H7) it follows that

$$\lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t) = \lambda, \quad \text{where } -\infty \le \lambda \le \infty. \quad (2.18)$$

If $\lambda < 0$, then $z(t) < 0$, for large t, a contradiction. If $\lambda = 0$, then $y(t) \le z(t)$, implies $\lim_{t\to\infty} y(t) = 0$. If $\lambda > 0$, then $w(t) > 0$ for large $t \ge t_2 \ge t_1$. Then from Lemma 2.1, it follows that, there exists an integer m, $0 \le m \le n-1$ such that $n-m$ is odd, and for $t \ge t_3 \ge t_2$, we have $w^{(j)}(t) > 0$ for $j = 0, 1, \dots, m$ and $(-1)^{n+j-1}w^{(j)}(t) > 0$ for $j = m+1, m+2, \dots, n-1$. Hence $\lim_{t\to\infty} w^{(m)}(t) = l$ exists and $\lim_{t\to\infty} w^{(i)}(t) = 0$ for $i = m+1, m+2, \dots, n-1$. Note that $0 < \lambda < \infty$, implies $m = 0$, but $\lambda = \infty$ implies $m > 0$ such that $n-m$ is odd. Integrating (2.5), $(n-m)$ times from t to ∞, we obtain (2.8) and (2.9). Hence from Lemma 2.19 and (2.9) we obtain for $\rho \ge t_3$,

$$\int_\rho^\infty t^{n-m-1}q(t)G(y(g(t)))\,dt < \infty. \quad (2.19)$$

Note that, since $r(t)$ is monotonic increasing, its inverse function, $r^{-1}(t)$ exists, such that $r^{-1}(r(t)) = t$. Since $q(t) > q^*(r^{-1}(t))$, it follows that

$$\int_\rho^\infty t^{n-m-1}q^*(r^{-1}(t))G(y(g(t)))\,dt < \infty.$$

Then replacing t by $r(t)$ in the above inequality, multiplying by the scalar $G(p_1)$, we obtain

$$G(p_1)\int_{T_2}^{\infty}(r(t))^{n-m-1}q^*(t)G(y(g(r(t))))r'(t)\,dt<\infty,$$

where $T_2\geq r^{-1}(\rho)$. Since $\liminf_{t\to\infty} r'(t)>0$ then, $r'(t)>c>0$ for $t\geq T_3\geq T_2$. Then we have

$$cG(p_1)\int_{T_3}^{\infty}(r(t))^{n-m-1}q^*(t)G(y(g(r(t))))\,dt<\infty.$$

(H1) implies there exists a scalar a such that $r(t)/t>a>0$ for $t\geq T_4\geq T_3$, and $p(t)\geq -p_1$. Using this, and (H0), we obtain

$$\int_{T_4}^{\infty}t^{n-m-1}q^*(t)G(-p(g(t)))G(y(g(r(t))))\,dt<\infty.$$

This with the use of (H13) yields

$$\int_{T_4}^{\infty}t^{n-m-1}q^*(t)G(-p(g(t))(y(g(r(t)))))\,dt<\infty.$$

Since $g(r(t))=r(g(t))$, then the above inequality yields

$$\int_{T_4}^{\infty}t^{n-m-1}q^*(t)G(-p(g(t))(y(r(g(t)))))\,dt<\infty. \tag{2.20}$$

From (2.19) and the fact that $q(t)\geq q^*(t)$, we obtain

$$\int_{T_4}^{\infty}t^{n-m-1}q^*(t)G(y(g(t)))\,dt<\infty. \tag{2.21}$$

Further, using (H13), (2.20) and (2.21), one may get

$$\int_{T_4}^{\infty}t^{n-m-1}q^*(t)G(z(g(t)))\,dt<\infty. \tag{2.22}$$

If $m=0$ then (H11) and (2.22) implies $\liminf_{t\to\infty} tG(z(g(t)))=0$, which with application of (H0) and the assumption $\lim_{t\to\infty} g(t)=\infty$ yields $\lim_{t\to\infty} z(t)=0$, a contradiction. If $m>0$ then there exists $M_0>0$ such that $w(t)>M_0t^{m-1}$ and by (H7) and (2.6), we can find $0<M_1<M_0$ such that

$$z(t)>M_1t^{m-1}\quad\text{for } t\geq T_5\geq T_4. \tag{2.23}$$

Due to (H9), we can find $b>0$ such that $g(t)/t>b>0$ for large t. Then further use of (2.23) and (H3) yields

$$\begin{aligned}\int_{T_5}^{\infty}t^{n-m-1}q^*(t)G(z(g(t)))\,dt &\geq \int_{T_5}^{\infty}t^{n-m-1}q^*(t)G(M_1(g(t))^{m-1})\,dt\\ &\geq \delta M_1b^{m-1}\int_{T_5}^{\infty}q^*(t)t^{n-2}\,dt=\infty,\end{aligned}$$

by (H11), a contradiction due to (2.22). Hence the proof for the case $y(t)>0$ is complete. If $y(t)<0$ for large t then, proceeding as above and using (H12), we complete the proof. $\square$

Remark 2.21. Clearly, (H11) implies (H4). To justify, the stronger assumption (H11) for the above Theorem, an example is given in [20]. We claim,if $q(t)$ is monotonic and (H1) holds then (H11) is equivalent to (H4). Indeed, if $q(t)$ is decreasing then $q^*(t) = q(t)$, hence the equivalence of (H4) and (H11)is immediate. On the other hand, if $q(t)$ is increasing, then suppose that (H4) holds. Since $q(t) > 0$, we find T_1 and $\eta > 0$ such that $q(r(t)) > \eta$ for $t \geq T_1$. Now, $q^*(t) = q(r(t))$ and $n \geq 2$ implies

$$\int_{T_1}^{\infty} t^{n-2} q^*(t)\,dt = \int_{T_1}^{\infty} t^{n-2} q(r(t))\,dt \geq \eta \int_{T_1}^{\infty} t^{n-2}\,dt = \infty.$$

Thus, (H11) holds. Hence, the equivalence of (H4) and (H11) is established. In [18, theorem 2.5], the authors assume (H10) and that $q(t)$ is monotonic. This implies (H11) is a weaker condition that we have used in our theorem. For [13, Theorem 3], where $-p \leq p(t) \leq 0$, the authors use the condition $\liminf_{t\to\infty} q(t) > 0$, which implies $\int_{t_0}^{\infty} q^*(t) = \infty$, and this further implies (H11) for $n \geq 2$. Hence the above result; i.e, Theorem2.20 extends and generalizes [13, Theorem 3].

Remark 2.22. The prototype function G satisfying (H0), (H3), (H12) and (H13) is $G(u) = (\beta + |u|^{\mu})|u|^{\lambda} \operatorname{sgn} u$, where $\lambda > 0$, $\mu > 0$, $\lambda + \mu \geq 1$, $\beta \geq 1$. For verifying it we may use the well known inequality (see[9, p. 292])

$$u^p + v^p \geq \begin{cases} (u+v)^p, & 0 \leq p < 1, \\ 2^{1-p}(u+v)^p, & p \geq 1. \end{cases}$$

3. Necessary Conditions

In this section we prove that if every solution of the neutral equation (1.1) oscillates or tends to zero as $t \to \infty$, then (H6) holds.

Theorem 3.1. *Suppose that $p(t)$ satisfies* (A1) *or* (A2). *If* (H5) *and* (H8) *hold and every solution of* (1.1) *oscillates or tends to zero as $t \to \infty$, then* (H6) *holds.*

Proof. Suppose that $p(t)$ satisfies (A1). Assume for the sake of contradiction, that (H6) does not hold. Hence

$$\int_{t_0}^{\infty} s^{n-1} q(s)\,ds < \infty. \tag{3.1}$$

Hence, all we need to show is the existence of a bounded solution $y(t)$ of (1.1) with $\liminf y(t) > 0$. From (H8), we find a constant k and a real number t_1 such that $t \geq t_1$ implies

$$|F(t)| < k \quad \text{for } t \geq t_1. \tag{3.2}$$

Choose two positive constants L and c such that $L \geq 7k/(1-p)$ and $c \leq k$. Since $G, H \in C(\mathbb{R}, \mathbb{R})$, we let

$$\eta = \max\{|G(x)| : c \leq x \leq L\}, \tag{3.3}$$

$$\gamma = \max\{|H(x)| : c \leq x \leq L\}. \tag{3.4}$$

Let $\mu = \max\{\eta, \gamma\}$. From (H5), we find $t_2 > t_1$ such that $t > t_2$ implies

$$\frac{\mu}{(n-1)!} \int_{t}^{\infty} (s-t)^{n-1} u(s)\,ds < \epsilon. \tag{3.5}$$

From (3.1) we find $t_3 > t_2$ such that $t \geq t_3$ implies

$$\frac{\mu}{(n-1)!}\int_t^\infty (s-t)^{n-1}q(s)\,ds < \epsilon. \tag{3.6}$$

In this case take $\epsilon \leq k$, and choose $T \geq t_3$ such that $T_0 = \min\{r(T), g(T), h(T)\} \geq t_3$. Then (3.2), (3.5) and (3.6) hold, for $t \geq T_0$. Let $X = C([T_0,\infty),\mathbb{R})$ be the set of all continuous functions with norm $\|x\| = \sup_{t\geq T_0}|x(t)| < \infty$. Clearly X is a Banach space. Let

$$S = \{u \in BC([T_0,\infty),\mathbb{R}) : c \leq u(t) \leq L\}$$

with the supremum norm $\|u\| = \sup\{|u(t)| : t \geq T_0\}$. Clearly S is a closed, bounded and convex subset of $C([T_0,\infty),\mathbb{R})$. Define two maps A and $B : S \to X$ as follows. For $x \in S$,

$$Ax(t) = \begin{cases} Ax(T), & t \in [T_0,T],\\ p(t)x(r(t)) + F(t) + \lambda, & t \geq T, \end{cases} \tag{3.7}$$

where $\lambda = 4k$, and

$$Bx(t) = \begin{cases} Bx(T), & t \in [T_0,T]\\ \frac{(-1)^{n-1}}{(n-1)!}\int_t^\infty (s-t)^{n-1}q(s)G(x(g(s)))\,ds & \\ +\frac{(-1)^n}{(n-1)!}\int_t^\infty (s-t)^{n-1}u(s)H(x(h(s)))\,ds, & t \geq T. \end{cases} \tag{3.8}$$

First we show that if $x, y \in S$ then $Ax + By \in S$. In fact, for every $x, y \in S$ and $t \geq T$, we get

$$\begin{aligned}
&(Ax)(t) + (By)(t)\\
&\leq p(t)x(r(t)) + 4k + |F(t)| + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}q(s)|G(y(g(s)))|\,ds\\
&\quad + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}u(s)|H(y(h(s)))|\,ds\\
&\leq pL + 4k + k + k + k \leq L.
\end{aligned}$$

On the other hand, for $t \geq T$,

$$\begin{aligned}
&(Ax)(t) + (By)(t)\\
&\geq 4k - |F(t)| - \frac{\mu}{(n-1)!}\int_t^\infty (s-t)^{n-1}q(s)\,ds - \frac{\mu}{(n-1)!}\int_t^\infty (s-t)^{n-1}u(s)\,ds\\
&\geq 4k - k - k - k \geq c.
\end{aligned}$$

Hence

$$c \leq (Ax)(t) + (By)(t) \leq L \quad \text{for } t \geq T.$$

So that $Ax + By \in S$ for all $x, y \in S$.

Next we show that A is a contraction in S and B is completely continuous, by following the arguments given in [24, Theorem 2.2]. Then by Lemma [7, Krasnoselskiis Fixed point Theorem], there is an $x_0 \in S$ such that $Ax_0 + Bx_0 = x_0$. It is easy to see that $x_0(t)$ is the required non oscillatory solution of the equation (1.1), which is bounded below by the positive constant c. If $p(t)$ satisfies (A2) then the proof is similar, only thing we have to do is to suitably fix c, L, ϵ and λ. In this regard, first decrement p if necessary, so that $p < 1/7$. Then select L, c and λ such that $7k \leq L < k/p$, $0 < c \leq k - pL$, and $\lambda = 4k$. The mappings A and B

are defined similarly. Then proceeding as above we complete the proof, when $p(t)$ satisfies (A2). □

Remark 3.2. The above theorem generalizes and extends [25, Theorem 1], [22, Theorem 2.5], and the necessity part of [15, Theorem 2.2], because we do not require (H0) or the condition that G be Lipschitzian in intervals of the form $[a, b]$. Further, the above theorem holds, even if $q(t)$ changes sign. In that case we have to replace $q(t)$ by $|q(t)|$ in (H6).

Theorem 3.3. *Suppose that $p(t)$ satisfies* (A4) *or* (A5). *Let* (H5) *and* (H8) *hold. If every solution of* (1.1) *oscillates or tends to zero as $t \to \infty$ then* (H6) *holds.*

Proof. Suppose that $p(t)$ satisfies (A5). The proof is similar to the proof of the above theorem with the following changes in the parameters c, L, λ and ϵ. Choose $L = k(3p_1 + 4p_2)/p_1(p_1 - 1)$, and $c = k/p_1$. Then $L > c > 0$ and assume $\epsilon = k$. We define the mappings A and B as

$$Ax(t) = \begin{cases} Ax(T), & \text{if } t \in [T_0, T] \\ \frac{x(r^{-1}(t))}{p(r^{-1}(t))} + \frac{\lambda}{p(r^{-1}(t))} + \frac{F(r^{-1}(t))}{p(r^{-1}(t))}, & \text{if } t \geq T, \end{cases}$$

where $\lambda = 4p_2k/p_1$, and

$$Bx(t) = \begin{cases} Bx(T), & \text{if } t \in [T_0, T] \\ \frac{(-1)^n}{(n-1)!p(r^{-1}(t))} \int_{r^{-1}(t)}^{\infty} (s - r^{-1}(t))^{n-1} q(s) G(x(g(s))) ds \\ + \frac{(-1)^{n-1}}{(n-1)!\, p(r^{-1}(t))} \int_{r^{-1}(t)}^{\infty} (s - r^{-1}(t))^{n-1} u(s) H(x(h(s))) ds, & \text{if } t \geq T. \end{cases}$$

The function r^{-1} used in the definition of the operators A and B, is the inverse function of $r(t)$, which exists because $r(t)$ is monotonic. Further, note that, $r^{-1}(r(t)) = t$. Proceeding as in the proof of the Theorem 3.1, we find a positive bounded solution with limit infimum $\geq c > 0$. If $p(t)$ satisfies (A4) then, the proof is similar to the proof for the case (A5), hence we let the reader find the values of c, L, ϵ, λ and complete the proof. □

In view of the Theorems 2.10, 3.1 and 3.3, we have the following result.

Corollary 3.4. *Suppose that $p(t)$ satisfies one of conditions* (A1), (A2), (A4), (A5). *Under the assumptions* (H0), (H1), (H5), (H7), *every bounded solution of* (1.1) *oscillates or tends to zero as $t \to \infty$ if and only if* (H6) *holds.*

The above corollary extends and generalizes of [22, corollary 2.9].

Open Problems. Before we close, we state two problems for further research.

- If $p(t) \geq 1$, can we find sufficient condition for the oscillation of (1.1) under one of the conditions (H6), (H4) or (H10)?
- In Theorem 2.20 can we replace (H11) by a weaker condition?

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