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STABILITY OF QUASI-LINEAR DIFFERENTIAL EQUATIONS WITH TRANSITION CONDITIONS

WEIZHEN FENG, YUBIN LIU

Abstract. This paper investigates the stability of quasi-linear differential equations on certain time scales with transition condition (DETC). We establish Sufficient conditions for stability and illustrate our results with examples.

1. Introduction

The study of quasi-linear systems is valuable because it is an important transition from linear systems to nonlinear systems. B. Liu [3] introduced the stability of a class of quasi-linear impulsive hybrid systems by using the methods of Lyapunov functions and the linearization technique. Y. Liu [4] gained sufficient conditions of the exponential stability of a class of quasi-linear switched systems by using the Cauchy matrices of its corresponding linear systems.

Differential equations on certain time scales with transition conditions (DETS) are in some sense more general than dynamic equations on time scales. Akhmet [1] investigated DETS on the basic of reduction to the impulsive differential equations (IDE). A special transformation (ψ-substitution) was used in [1] which allowed the reduced IDE to inherit all similar properties of the corresponding DETC.

In this paper, we make an attempt to investigate the stability of the DETC

$$\begin{gathered} y' = A(t)y + f(t,y) + A_iY(t_{2i+1}), \quad t \in [t_{2i+1}, t_{2i+2}],\ t \geq t^0, \\ y(t_{2i+1}) = B_i y(t_{2i}) + J_i(y(t_{2i})) + y(t_{2i}), \quad t_{2i} \geq t^0, \\ y(t^0) = y_0, \quad t^0 \in T_0, \end{gathered} \tag{1.1}$$

where the time scale $T_0 = \bigcup_{i=0}^{+\infty}[t_{2i-1}, t_{2i}], t_i < t_{i+1},\ t_{-1} < 0 < t_0, \lim_{i\to+\infty} t_i = +\infty$, and the derivative is one sided at the boundary points of T_0,

$$Y(t_{2i+1}) = \begin{cases} y_0, & t_{2i+1} < t^0 < t_{2i+2}, \\ y(t_{2i+1}), & t_{2i+1} \geq t^0\,. \end{cases}$$

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Note that (1.1) is more general than

$$y^{\Delta} = A(t)y + f(t,y), \quad t \in T_0, \tag{1.2}$$

since this equation can be written as

$$\begin{gathered} y' = A(t)y + f(t,y), \quad t \in [t_{2i+1}, t_{2i+2}],\ t \geq t^0, \\ y(t_{2i+1}) = [A(t_{2i})y(t_{2i}) + f(t_{2i}, y(t_{2i}))](t_{2i+1} - t_{2i}) + y(t_{2i}), \\ y(t^0) = y_0, \quad t^0 \in T_0. \end{gathered} \tag{1.3}$$

2. Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional real space and $\|A\|$ the norm of an $n \times n$ matrix A induced by the Euclidean vector norm, i.e. $\|A\| = [\lambda_{\max}(A^T A)]^{\frac{1}{2}}$. Let $\mathbb{R}^+ = [0, +\infty)$, $N_+ = \{1, 2, \dots\}$ and $N = \{0\} \cup N_+$.

In the following discussion, we suppose that:

(A1) $A(t) \in C(T_0, \mathbb{R}^{n\times n})$; $B_i, A_i \in \mathbb{R}^{n\times n}$; $f(t,y) \in C(T_0 \times \mathbb{R}^n, \mathbb{R}^n)$; $J_i(y) \in C(\mathbb{R}^n, \mathbb{R}^n)$; $f(t,0) \equiv 0$ and $J_i(0) \equiv 0$ for $t \in T_0$ and $i \in N$.

The following conditions will be used in some theorems:

(H2) There exist $\delta^* > 0$, $\sigma_i > 0$ ($i \in N$) such that $\|B_i y + J_i(y) + y\| \leq \sigma_i \|y\|$ whenever $\|y\| \leq \delta^*$;

(H3) $0 < \lambda_1 \leq \inf\{t_{2i} - t_{2i-1}\} \leq \sup_{i\in N}\{t_{2i} - t_{2i-1}\} = \lambda < +\infty$,
$$0 < \lambda_3 \leq \inf\{t_{2i+1} - t_{2i}\} \leq \sup_{i\in N}\{t_{2i+1} - t_{2i}\} = \lambda_2 < +\infty;$$

(H4) $\|f(t,y)\| \leq \|F(t)\| \cdot \|y\|$, $(t,y) \in T_0 \times \mathbb{R}^n$;

(H5) for all $\varepsilon \geq 0$ there exists $\delta(\varepsilon) > 0$, such that $\|f(t,y)\| \leq \varepsilon\|y\|$ for all $t \in T_0$ whenever $\|y\| \leq \delta$.

Without loss of generality, we assume $t^0 \in [t_{2m-1}, t_{2m}]$ and $t^0 \geq 0$. Denote $d(T_0[t^0, t]) = t - t_{2p+1} + \sum_{i=m+1}^{p}(t_{2i} - t_{2i-1}) + (t_{2m} - t^0)$ for $t \in [t_{2p+1}, t_{2p+2}]$, and $T_0[a, +\infty) = T_0 \cap [a, +\infty)$.

A function $y : [t^0, +\infty) \cap T_0 \to \mathbb{R}^n$ is said to be a solution of (1.1) if

(i) $y(t^0) = y_0$;
(ii) $y' = A(t)y + f(t,y) + A_i Y(t_{2i+1})$ if $t \in [t_{2i+1}, t_{2i+2}]$ and $t \geq t^0, i \in N$;
(iii) $y(t_{2i+1}) = B_i y(t_{2i}) + J_i(y(t_{2i})) + y(t_{2i})$, $t_{2i} \geq t^0$, $i \in N$.

Denote the solution of (1.1) as $y(t, t^0, y_0)$.

Definition 2.1 ([1])**.** The ψ-substitution on the set $T_0' = T_0 \backslash \bigcup_{i=0}^{+\infty}\{t_{2i-1}\}$ is defined as

$$\psi(t) = t - \sum_{0 \leq t_{2k} < t} \delta_k, \quad t \in T_0',\ t \geq 0, \tag{2.1}$$

where $\delta_k = t_{2k+1} - t_{2k}$.

Definition 2.2. (I) The zero solution of (1.1) is said to be uniformly stable, if for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that for any $t^0 \in T_0, \|y_0\| < \delta$ implies $\|y(t, t^0, y_0)\| < \varepsilon$ for all $t \in T_0[t^0, +\infty)$;

(II) The zero solution of (1.1) is said to be uniformly attractive, if there exists $\delta_0 > 0$, such for all $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that for any $t^0 \in T_0, \|y_0\| < \delta_0$ implies $\|y(t, t^0, y_0)\| < \varepsilon$ for all $t \in T_0[t^0 + T, +\infty)$;

(III) The zero solution of (1.1) is said to be uniformly asymptotically stable if the zero solution of (1.1) is uniformly stable and uniformly attractive;

(IV) The zero solution of (1.1) is said to be exponentially stable, if there exists $\alpha > 0$ such that for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that for any $t^0 \in T_0$, $\|y_0\| < \delta$ implies $\|y(t, t^0, y_0)\| < \varepsilon e^{-\alpha(t-t^0)}$ for all $t \in T_0[t^0, +\infty)$.

Let $x(s^+) = \lim_{h\to 0^+} x(s+h), x(s) = x(s^-) = \lim_{h\to 0^-} x(s+h)$.

Lemma 2.3 ([1]). *$\psi(t)$ is a one-to-one map, $\psi(0) = 0$, $\psi(T_0') = R$. The inverse transformation is*

$$\psi^{-1}(s) = s + \sum_{0\le s_k<s} \delta_k, \quad s \ge 0,\ s_i = \psi(t_{2i}),\ i \in N. \tag{2.2}$$

Lemma 2.4 ([1]). *$\psi'(t) = 1$ if $t \in T_0'$. $\frac{d}{ds}(\psi^{-1}(s)) = 1$ if $s \ne s_i$, $i \in N$.*

Lemma 2.5. *If $y(t)$ is the solution of (1.1), then $x(s) = y(\psi^{-1}(s))$ is the solution of*

$$\begin{gathered} x' = \widetilde{A}(s)x + \widetilde{f}(s, x) + A_i\widetilde{X}(s_i^+), \quad s \in (s_i, s_{i+1}], s \ge s_0,\\ x(s_i^+) = B_i x(s_i) + J_i(x(s_i)) + x(s_i), \quad s_i \ge s_0,\\ x(s^0) = y_0. \end{gathered} \tag{2.3}$$

and vice versa, where $\widetilde{A}(s) = A(\psi^{-1}(s))$, $\widetilde{f}(s, x) = f(\psi^{-1}(s), x)$, $s^0 = \psi(t_0)$,

$$\widetilde{X}(s_i^+) = \begin{cases} y_0, & s_i < s^0 < s_{i+1},\\ x(s_i^+), & s_i \ge s^0, \end{cases}$$

$x(s_i^+) = y(t_{2i+1})$.

Lemma 2.6. *The zero solution of (1.1) is uniformly stable if and only if the zero solution of (2.3) is uniformly stable.*

Lemma 2.7. *Suppose that (H3) holds, then the zero solution of (1.1) is uniformly asymptotically stable if the zero solution of (2.3) is uniformly asymptotically stable.*

Proof. If the zero solution of (2.3) is uniformly attractive, then there exists a $\delta_0 > 0$ such that for all $\varepsilon > 0$, there exists $T_1(\varepsilon) > 0$, such that $\|y_0\| < \delta, t^0 \in T_0$ implies

$$\|x(s, s^0, y_0)\| < \varepsilon \quad \text{for all } s \ge s^0 + T_1. \tag{2.4}$$

Hence, for the ε above, there exists a $T = T_1 + \frac{T_1}{\lambda_1}\lambda_2 > 0$, such that $t \in T_0[t^0 + T, +\infty)$ implies $s = \psi(t) \ge s^0 + T_1$; that is,

$$\|y(t, t^0, y_0)\| = \|x(s, s^0, y_0)\| < \varepsilon \quad \text{for all } t \in T_0[t^0 + T, +\infty) \setminus \{t_{2i+1}\}, s = \psi(t),$$

and

$$\|y(t_{2i+1}, t^0, y_0)\| = \|x(s_i^+, s^0, y_0)\| = \lim_{h\to 0^+} \|x(s_i + h, s^0, y_0)\| \le \varepsilon. \tag{2.5}$$

Therefore, the zero solution of (1.1) is uniformly attractive.

By the above conclusion and Lemma 2.6, we can get that the zero solution of (1.1) is uniformly asymptotically stable. □

Lemma 2.8. *Suppose that (H3) holds, then the zero solution of (1.1) is exponentially stable if the zero solution of (2.3) is exponentially stable.*

Proof. If the zero solution of (2.3) is exponentially stable, then there exists $\alpha > 0$ such that for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, such that $\|y_0\| < \delta$ implies

$$\|x(s, s^0, y_0)\| \le \varepsilon e^{-\alpha(s-s^0)} \quad \text{for } s \ge s^0. \tag{2.6}$$

For $t \ge t^0, t \in [t_{2(m+p)-1}, t_{2(m+p)}]$, by (H3), we have

$$d(T_0[t^0, t]) \ge p\lambda_1 = \frac{p\lambda_1}{(p+1)\lambda + p\lambda_2}[(p+1)\lambda + p\lambda_2] \ge l(t - t^0), \tag{2.7}$$

where $l = \frac{p\lambda_1}{(p+1)\lambda+p\lambda_2}$. Hence, for the above ε, and $\|y_0\| < \delta$, we have

$$\|y(t, t^0, y_0)\| \le \varepsilon e^{-\alpha d(T_0[t^0,t])} \le \varepsilon e^{-\alpha l(t-t^0)},\ t \in T_0[t^0, +\infty); \tag{2.8}$$

that is, the zero solution of (1.1) is exponentially stable. □

Denote $Y(t, \tau)$, $X(s, \gamma)$ as the Cauchy matrices of $y' = A(t)y$ and $x' = \widetilde{A}(s)x$ respectively. It is easy to verify that $Y(t, \tau) = X(s, \gamma)$ where $t = \psi^{-1}(s), \tau = \psi^{-1}(\gamma)$ when $\gamma \ne s_i, t = \psi^{-1}(s)$ and $\tau = t_{2i+1}$ when $\gamma = s_i$. Consider

$$\begin{gathered} y' = A(t)y, \quad t \in [t_{2i+1}, t_{2i+2}],\ t \ge t^0, \\ y(t_{2i+1}) = B_i y(t_{2i}) + y(t_{2i}), \quad t_{2i} \ge t^0, \\ y(t^0) = y_0 \end{gathered} \tag{2.9}$$

and

$$\begin{gathered} x' = \widetilde{A}(s)x, \quad s \ne s_i,\ s \ge s^0, \\ x(s_i^+) = (B_i + I)x(s_i), \quad s_i \ge s^0, \\ x(s^0) = y_0. \end{gathered} \tag{2.10}$$

It is easy to verify that the solution of (2.9), for $t \in [t_{2p+1}, t_{2p+2}]$, is

$$y(t) = Y(t, t_{2p+1})(B_p + I)[\prod_{k=m+1}^{p} Y(t_{2k}, t_{2k-1})(B_{k-1} + I)]Y(t_{2m}, t^0)y_0, \tag{2.11}$$

and the solution of (2.10), for $s \in (s_p, s_{p+1}]$, is

$$x(s) = X(s, s_p^+)(B_p + I)[\prod_{k=m+1}^{p} X(s_k, s_{k-1})(B_{k-1} + I)]X(s_m, s^0)y_0. \tag{2.12}$$

Denote

$$E(t, t^0) = Y(t, t_{2p+1})(B_p + I)[\prod_{k=m+1}^{p} Y(t_{2k}, t_{2k-1})(B_{k-1} + I)]Y(t_{2m}, t^0),$$

$$E_1(s, s^0) = X(s, s_p^+)(B_p + I)[\prod_{k=m+1}^{p} X(s_k, s_{k-1})(B_{k-1} + I)]X(s_m, s^0).$$

3. Stability

Lemma 3.1. *The zero solution of* (2.9) *is uniformly stable if and only if* $E(t, t^0)$ *is uniformly bounded on* T_0 *with respect to* t^0.

The above lemma follows from (2.11); so we omit the proof.

Theorem 3.2. *Assume that*

(i) *the zero solution of* (2.9) *is uniformly stable;*

(ii) *(H4) is satisfied and there exists a* $\beta \geq 0$ *such that for all* $\tau \in T_0$, $\int_{T_0[\tau,+\infty)} \|F(t)\| dt \leq \beta$;
(iii) $A_i = 0, J_i(y) \equiv 0$ *for all* $y \in \mathbb{R}^n$, $i \in N$.

Then the zero solution of (1.1) *is uniformly stable.*

Proof. By (i), Lemma 2.5 and Lemma 3.1, we can get that there exists an $M > 0$ such that $\|E_1(t,\tau)\| \leq M$ for $t, \tau \in T_0$, $t \geq \tau$.

We are going to prove that the zero solution of (2.3) is uniformly stable. It is important that under (iii), (2.3) can be re-written

$$\begin{gathered} x' = \widetilde{A}(s)x + \widetilde{f}(s, x(s)), \quad s \in (s_i, s_{i+1}],\ s \geq s^0, \\ x(s_i^+) = B_i x(s_i) + x(s_i), \quad s_i \geq s^0, \\ x(s^0) = y_0. \end{gathered} \tag{3.1}$$

For $s \in (s^0, s_m]$, the solution of (3.1) is

$$\begin{gathered} x(s) = X(s, s^{0+})y_0 + \int_{s^0}^{s} X(s,\tau)\widetilde{f}(\tau, x(\tau))d\tau, \\ x(s_m^+) = (B_m + I)X(s_m, s^{0+})y_0 + \int_{s^0}^{s_m} (B_m + I)X(s_m,\tau)\widetilde{f}(\tau, x(\tau))d\tau\,. \end{gathered}$$

For $s \in (s_m, s_{m+1}]$,

$$\begin{aligned} x(s) &= X(s, s_m^+)x(s_m^+) + \int_{s_m}^{s} X(s,\tau)\widetilde{f}(\tau, x(\tau))d\tau \\ &= X(s, s_m^+)(B_m + I)X(s_m, s^{0+})y_0 \\ &\quad + \int_{s^0}^{s_m} X(s, s_m^+)(B_m + I)X(s_m,\tau)\widetilde{f}(\tau, x(\tau))d\tau + \int_{s_m}^{s} X(s_m,\tau)\widetilde{f}(\tau, x(\tau))d\tau \\ &= E_1(s, s^{0+})y_0 + \int_{s^0}^{s_m} E_1(s,\tau)\widetilde{f}(\tau, x(\tau))d\tau + \int_{s_m}^{s} E_1(s,\tau)\widetilde{f}(\tau, x(\tau))d\tau \\ &= E_1(s, s^{0+})y_0 + \int_{s^0}^{s} E_1(s,\tau)\widetilde{f}(\tau, x(\tau))d\tau\,. \end{aligned}$$

By (3.1), the above equality, and mathematical induction, we conclude easily that

$$x(s) = E_1(s, s^{0+})y_0 + \int_{s^0}^{s} E_1(s,\tau)\widetilde{f}(\tau, x(\tau))d\tau, \quad s \geq s^0. \tag{3.2}$$

Therefore,

$$\|x(s)\| \leq M\|y_0\| + M\int_{s^0}^{s} \|F(\tau)\| \cdot \|x(\tau)\| d\tau. \tag{3.3}$$

By the Bellman inequality,

$$\|x(s)\| \leq M\|y_0\| \exp(\int_{s^0}^{s} \|F(\tau)\| d\tau) \leq M\|y_0\|e^{\beta}, \tag{3.4}$$

which leads to that the solution $y(t)$ of (1.1) satisfies

$$\|y(t)\| \leq Me^{\beta}\|y_0\|.$$

which yields that the zero solution of (1.1) is uniformly stable. The proof is complete. □

Theorem 3.3. *Assume that* (H2), (H3), (H5) *hold and*

(i) *there exist* $M, \alpha > 0$ *such that* $\sup_{t\in[t_{2i-1},t_{2i}]} \|Y(t, t_{2i-1})\| \le M$, $\sup_{t\in[t_{2i-1},t_{2i}]} \|A(t)\| \le \alpha$, $i \in N$;
(ii) *there exist* $\mu, \sigma > 0$ *such that* $\|A_i\| \le \mu$, $\sigma_i \le \sigma$ *for* $i \in N$ *and* $M(1+\lambda\mu)\sigma \le q < 1$.

Then the zero solution of (1.1) *is uniformly asymptotically stable.*

Proof. From (H5) there exists $\delta_1 > 0$ such that

$$\|\widetilde{f}(t, x)\| \le \|x\|, \quad \text{whenever } \|x\| < \delta_1. \tag{3.5}$$

We claim that there exists $\delta_2 > 0$ such that the solution of (2.3) satisfies

$$\|x(s)\| < \delta_1, s \in (s^0, s_m] \quad \text{if } \|x(s^{0+})\| < \delta_2, \tag{3.6}$$

and

$$\|x(s)\| < \delta_1, s \in (s_k, s_{k+1}] \quad \text{whenever } \|x(s_k^+)\| < \delta_2, k \ge m, \tag{3.7}$$

where $\delta_2(1 + \lambda\mu)\exp[(1 + \alpha)\lambda] \le \delta_1$. Note that $x(s^{0+}) = x(s^0)$ if $s^0 \in (s_i, s_{i+1})$, and $x(s^{0+}) = (B_i + I)x(s^0)$ if $s^0 = s_i, i \in N$.

Otherwise, if (3.6) is not true, there must exists $\tau_0 \in (s^0, s_m]$ such that $\|x(\tau_0)\| = \delta_1$ while $\|x(s)\| < \delta_1$ for all $s \in (s^0, \tau_0]$. Then, $\|\widetilde{f}(s, x(s))\| \le \|x(s)\|$ for all $s \in (s^0, \tau_0]$.

For any $s \in (s^0, \tau_0]$, it is true that

$$x(s) = x(s^{0+}) + \int_{s^0}^{s} [\widetilde{A}(r)x(r) + \widetilde{f}(r, x(r)) + A_{m-1}x(s^{0+})]dr,$$

which implies

$$\|x(s)\| \le \|x(s^{0+})\|(1 + \lambda\mu) + \int_{s^0}^{s} (1 + \alpha)\|x(r)\|dr. \tag{3.8}$$

By the Bellman inequality, for $s \in (s^0, \tau_0]$,

$$\|x(s)\| \le \|x(s^{0+})\|(1 + \lambda\mu)\exp[(1 + \alpha)\lambda]; \tag{3.9}$$

that is,

$$\|x(s)\| \le \delta_2(1 + \lambda\mu)\exp[(1 + \alpha)\lambda] < \delta_1,$$

which contradicts the fact that $\|x(\tau_0)\| = \delta_1$. Hence, (3.6) is true. Similarly, we can get that (3.7) is true. Thus, if $\|x(s_k^+)\| < \delta_2$, then

$$\|x(s)\| \le \|x(s_k^+)\|(1 + \lambda\mu)\exp[(1 + \alpha)\lambda], \quad s \in (s_k, s_{k+1}],\ k \ge m. \tag{3.10}$$

For a given ε_0: $0 < \varepsilon_0 < \min\{1 - q, \frac{1-q}{\sigma}\}$, we choose ε_1: $0 < \varepsilon_1 < 1$ such that $\varepsilon_1 M\lambda(1 + \lambda\mu)\exp[(1 + \alpha)\lambda] \le \varepsilon_0$. By (H5) there exists a δ_3: $0 < \delta_3 < \min\{\delta_1, \delta^*\}$ such that $\|\widetilde{f}(s, x)\| \le \varepsilon_1\|x\|$ whenever $\|x\| < \delta_3$.

By a proof similar to the above discussion, we can obtain $0 < \delta_4 < \delta_2$ such that $\|x(s)\| \le \delta_3, s \in (s^0, s_m]$, and (3.9) holds for $s \in (s^0, s_m]$ if $\|x(s^{0+})\| \le \delta_4$. We can also get that, for $k \ge m$, $\|x(s)\| \le \delta_3, s \in (s_k, s_{k+1}]$ and (3.10) holds for $s \in (s_k, s_{k+1}]$ whenever $\|x(s_k^+)\| \le \delta_4$.

Denote $\delta_5 :\ 0 < \delta_5 < \min\{\delta_4, \frac{\delta_4}{\sigma}\}$. Let $\|y_0\| < \delta_5$. Then $\|x(s^{0+})\| < \delta_4$. From (2.3) we get that for $s \in (s^0, s_m]$,

$$\begin{aligned} x(s) &= X(s, s^{0+})x(s^{0+}) + \int_{s^0}^{s} X(s, \tau)[\widetilde{f}(\tau, x(\tau)) + A_{m-1}x(s^{0+})]d\tau \\ &= [X(s, s^{0+}) + \int_{s^0}^{s} X(s, \tau)d\tau \cdot A_{m-1}]x(s^{0+}) + \Delta_s, \end{aligned}$$

where $\Delta_s = \int_{s^0}^{s} X(s,\tau)\widetilde{f}(\tau, x(\tau))d\tau$.

That $\|y_0\| < \delta_5$ and that (3.9) holds for $s \in (s^0, s_m]$ leads us to

$$\Delta_s \le M\varepsilon_1 \int_{s^0}^{s} \|x(\tau)\|d\tau \le \|x(s^{0+})\| \cdot M\varepsilon_1(1+\mu\lambda)\exp[(1+\alpha)\lambda]\lambda \le \varepsilon_0\|x(s^{0+})\|$$

for $s \in (s^0, s_m]$. Hence, for $s \in (s^0, s_m]$,

$$\|x(s)\| \le (M + \varepsilon_0 + M\mu\lambda)\|x(s^{0+})\|. \tag{3.11}$$

Therefore,

$$\begin{aligned} \|x(s_m^+)\| &= \|B_m x(s_m) + J_m(x(s_m)) + x(s_m)\| \\ &\le \sigma(M + \varepsilon_0 + M\mu\lambda)\|x(s^{0+})\| \\ &\le (q + \sigma\varepsilon_0)\|x(s^{0+})\| < \delta_4. \end{aligned} \tag{3.12}$$

From (2.3) we obtain that for $s \in (s_m, s_{m+1}]$,

$$x(s) = [X(s, s_m^+) + \int_{s_m}^{s} X(s,\tau)d\tau \cdot A_m]x(s_m^+) + \Delta_s, \tag{3.13}$$

$\Delta_s = \int_{s_m}^{s} X(s,\tau)\widetilde{f}(\tau, x(\tau))d\tau$, $s \in (s_m, s_{m+1}]$.

As for (3.11), we can get that, for $s \in (s_m, s_{m+1}]$,

$$\|x(s)\| \le (M + \varepsilon_0 + M\mu\lambda)\|x(s_m^+)\|, \tag{3.14}$$

$$\|x(s_{m+1}^+)\| \le (q + \sigma\varepsilon_0)\|x(s_m^+)\| < \delta_4. \tag{3.15}$$

By mathematical induction, for $s \in (s_k, s_{k+1}]$, we obtain

$$\|x(s_k^+)\| < \delta_4, \quad \|x(s)\| \le (M + \varepsilon_0 + M\mu\lambda)\|x(s_k^+)\|, \tag{3.16}$$

$$\|x(s_{k+1}^+)\| \le (q + \sigma\varepsilon_0)\|x(s_k^+)\| < \delta_4, k \ge m. \tag{3.17}$$

Hence,

$$\|x(s_k^+)\| \le (q + \sigma\varepsilon_0)^{k-m+1}\|x(s^{0+})\|, k \ge m. \tag{3.18}$$

Since for each $\varepsilon > 0$, there exists $N_1 \in N_+$ such that $n \ge N_1$ implies $(q+\sigma\varepsilon_0)^n < \varepsilon$, there exists $T_1 = N_1\lambda$ such that for any $\|y_0\| < \delta_5$, $s^0 \in T_0, s_k \ge s^0 + T_1$,

$$\|x(s_k^+)\| < \delta_5\varepsilon, \quad \text{and} \quad \|x(s)\| \le (M + \varepsilon_0 + M\mu\lambda)\delta_5\varepsilon, \quad s \in (s_k, s_{k+1}];$$

that is, the zero solution of (2.3) is uniformly attractive.

Form (3.12),(3.18), we conclude that

$$\|x(s_k^+)\| \le \|x(s^{0+})\|. \tag{3.19}$$

Therefore, for each $\varepsilon > 0$, there exists $0 < \delta < \min\{\delta_5, \varepsilon\}$, such that $\|y_0\| < \delta$ implies

$$\|x(s_k^+)\| \le (1 + \sigma)\varepsilon, k \ge m,$$

and

$$\|x(s)\| \le (1 + \sigma)(M + \varepsilon_0 + M\mu\lambda)\varepsilon, \quad s \in [s^0, +\infty) \cup T_0; \tag{3.20}$$

that is, the zero solution of (2.3) is uniformly stable.

Summing up the above discussion, we can get that the zero solution of (2.3) is uniformly asymptotically stable. Hence, by Lemma 2.7, we get that the zero solution of (1.1) is uniformly asymptotically stable. □

Theorem 3.4. *Assume that* (H2), (H3), (H5) *hold and for* $k \in N$*,*

(i) *There exist* $M, \alpha > 0$ *such that* $\|Y(t,s)\| \le Me^{-\alpha(t-s)}$, $t_{2k-1} \le s \le t \le t_{2k}$*;*

(ii) *there exist* $\sigma, \beta > 0$ *such that* $\sigma_k \le \sigma$, $\beta < \alpha$, *and* $\sigma_k M(1+\frac{1}{\alpha})e^{-(\alpha-\beta)(t_{2k}-t_{2k-1})} \le 1$;

(iii) $\|A_k\| \le e^{-\alpha\lambda}$.

Then the zero solution of (1.1) *is exponentially stable.*

Proof. At first, we investigate the solution of (2.3),

$$x(s) = \begin{cases} X(s,s_p^+)x(s_p^+) + \int_{s_p}^s X(s,\tau)\widetilde{f}(\tau,x(\tau))d\tau \\ +x(s_p^+)\int_{s_p}^s X(s,\tau)d\tau\cdot A_p, & s\in(s_p,s_{p+1}],\ p\ge m, \\ X(s,s^{0+})x(s^{0+}) + \int_{s^0}^s X(s,\tau)\widetilde{f}(\tau,x(\tau))d\tau \\ +x(s^{0+})\int_{s^0}^s X(s,\tau)d\tau\cdot A_{m-1}, & s\in(s^0,s_m]\,. \end{cases} \tag{3.21}$$

Case I: $M > 1$**.** Let $M' = \max\{1,\sigma\}, \varepsilon_0 > 0$ such that $\varepsilon_0 M'M^2(1+\frac{1}{\alpha}) < \beta$. (H5) yields that there exists a $\delta(\varepsilon_0) > 0$ which ensure $\|\widetilde{f}(s,x)\| \le \varepsilon_0\|x\|$ whenever $\|x\| < \delta$ and $s\in T_0$.

For $\|y_0\| < \frac{\delta}{2MM'^2(1+\frac{1}{\alpha})}$ (which ensure $\|x(s^{0+})\| < \frac{\delta}{2MM'(1+\frac{1}{\alpha})}$), there exists T_1 : $s^0+T_1 < s_m$, such that $\|x(s)\| < \delta$ for $s\in(s^0,s^0+T_1]$. Hence, for $s\in(s^0,s^0+T_1]$,

$$\begin{aligned} \|x(s)\| &\le \|X(s,s^0)\|\cdot\|x(s^{0+})\| + \int_{s^0}^s \|X(s,\tau)\|\cdot\|\widetilde{f}(\tau,x(\tau))\|d\tau \\ &\quad + \|A_{m-1}\|\cdot\|x(s^{0+})\|\int_{s^0}^s \|X(s,\tau)\|d\tau \\ &\le Me^{-\alpha(s-s^0)}\|x(s^{0+})\| + \int_{s^0}^s Me^{-\alpha(s-\tau)}\varepsilon_0\|x(\tau)\|d\tau \\ &\quad + e^{-\alpha\lambda}\|x(s^{0+})\|\int_{s^0}^s Me^{-\alpha(s-\tau)}d\tau \\ &\le M(1+\frac{1-e^{-\alpha\lambda}}{\alpha})e^{-\alpha(s-s^0)}\|x(s^{0+})\| + M\varepsilon_0\int_{s^0}^s e^{-\alpha(s-\tau)}\|x(\tau)\|d\tau \\ &\le M(1+\frac{1}{\alpha})e^{-\beta(s-s^0)}\|x(s^{0+})\| + M'M^2\varepsilon_0(1+\frac{1}{\alpha})\int_{s^0}^s e^{-\beta(s-\tau)}\|x(\tau)\|d\tau\,; \end{aligned} \tag{3.22}$$

that is,

$$\|x(s)\|e^{\beta s} \le M(1+\frac{1}{\alpha})e^{\beta s^0}\|x(s^{0+})\| + M'M^2\varepsilon_0(1+\frac{1}{\alpha})\int_{s^0}^s e^{\beta\tau}\|x(\tau)\|d\tau. \tag{3.23}$$

By the Bellman inequality, for $s\in(s^0,s^0+T_1]$,

$$\|x(s)\| \le M(1+\frac{1}{\alpha})e^{-[\beta-\varepsilon_0 M'M^2(1+\frac{1}{\alpha})](s-s^0)}\|x(s^{0+})\|, \tag{3.24}$$

which leads to

$$\|x(s)\| < \frac{\delta}{2} \quad \text{for } s\in(s^0,s^0+T_1]. \tag{3.25}$$

We claim that

$$\|x(s)\| < \delta \quad \text{for } s\in(s^0,s_m]. \tag{3.26}$$

In fact, if (3.26) is not true, there exists $s'\in(s^0+T_1,s_m]$, such that $\|x(s')\| = \delta$ and $\|x(s)\| < \delta$ for $s\in(s^0,s')$. As above, we can prove that (3.24) holds for $s\in(s^0,s']$; that is, $\|x(s)\| < \frac{\delta}{2}$ for $s\in(s^0,s')$. Hence, $\|x(s')\| = \lim_{s\to s'-}\|x(s)\| \le \frac{\delta}{2} < \delta$, which is a contradiction to that $\|x(s')\| = \delta$. (3.26) leads us to that (3.22) is true for $s \in (s^0, s_m]$.

Also (3.26) yields that (3.24) is true for $s \in (s^0, s_m]$. Therefore, $\|x(s_m)\| \le \frac{\delta}{2M'}$, and

$$\|x(s_m^+)\| = \|B_m x(s_m) + J_m(x(s_m)) + x(s_m)\| \le \sigma_m \|x(s_m)\| \le \frac{\delta}{2}. \tag{3.27}$$

So, there exists $T_2 : s_m + T_2 < s_{m+1}$, such that $\|x(s)\| < \delta$ for $s \in (s_m, s_m + T_2]$. Since (3.22) holds for $s \in (s^0, s_m]$, for $s \in (s_m, s_m + T_2]$,

$$\begin{aligned}
\|x(s)\| &\le \|X(s, s_m)\| \cdot \|x(s_m^+)\| + \int_{s_m}^{s} \|X(s,\tau)\| \cdot \|\widetilde{f}(\tau, x(\tau))\| d\tau \\
&\quad + \|A_m\| \cdot \|x(s_m^+)\| \int_{s_m}^{s} \|X(s,\tau)\| d\tau \\
&\le [M e^{-\alpha(s-s_m)} + e^{-\alpha\lambda} M \int_{s_m}^{s} e^{-\alpha(s-\tau)} d\tau] \|x(s_m^+)\| \\
&\quad + \varepsilon_0 M \int_{s_m}^{s} e^{-\alpha(s-\tau)} \|x(\tau)\| d\tau \\
&\le \sigma_m M (1 + \frac{1}{\alpha}) e^{-\alpha(s-s_m)} [M(1 + \frac{1}{\alpha}) e^{-\alpha(s_m - s^0)} \|x(s^{0+})\| \\
&\quad + \varepsilon_0 M \int_{s^0}^{s_m} e^{-\alpha(s_m-\tau)} \|x(\tau)\| d\tau] + M\varepsilon_0 \int_{s_m}^{s} e^{-\alpha(s-\tau)} \|x(\tau)\| d\tau \\
&= \sigma_m M^2 (1 + \frac{1}{\alpha})^2 e^{-\alpha(s-s^0)} \|x(s^{0+})\| \\
&\quad + \sigma_m M^2 (1 + \frac{1}{\alpha}) \varepsilon_0 \int_{s^0}^{s_m} e^{-\alpha(s-\tau)} \|x(\tau)\| d\tau + M\varepsilon_0 \int_{s_m}^{s} e^{-\alpha(s-\tau)} \|x(\tau)\| d\tau.
\end{aligned}$$

Hence, for $s \in (s_m, s_m + T_2]$,

$$\|x(s)\| \le M(1 + \frac{1}{\alpha}) e^{-\beta(s-s^0)} \|x(s^{0+})\| + M' M^2 \varepsilon_0 (1 + \frac{1}{\alpha}) \int_{s^0}^{s} e^{-\beta(s-\tau)} \|x(\tau)\| d\tau. \tag{3.28}$$

Hence, for $s \in (s_m, s_m + T_2]$, (3.24) holds. So, $\|x(s)\| < \delta$ for $s \in (s_m, s_m + T_2]$. Similarly, we can prove that $\|x(s)\| < \delta$ for $s \in (s_m, s_{m+1}]$ and (3.24) holds for $s \in (s_m, s_{m+1}]$.

Suppose that for $\|y_0\| < \frac{\delta}{2MM'^2(1+\frac{1}{\alpha})}$, (3.24) holds for $s \in (s^0, s_k]$, $k \ge m$. Then by (3.24), we have $\|x(s_k)\| < \frac{\delta}{2M'}$. Hence, $\|x(s_k^+)\| \le \sigma_k \|x(s_k)\| < \delta/2$. There must be an $H_1 > 0 : s_k + H_1 < s_{k+1}$, such that $\|x(s)\| < \delta$ for $s \in (s_k, s_k + H_1]$. Therefore, for $s \in (s_k, s_k + H_1]$,

$$\begin{aligned}
&\|x(s)\| \\
&\le M(1 + \frac{1}{\alpha}) e^{-\alpha(s-s_k)} \|x(s_k^+)\| + M\varepsilon_0 \int_{s_k}^{s} e^{-\alpha(s-\tau)} \|x(\tau)\| d\tau \\
&\le M(1 + \frac{1}{\alpha}) e^{-\alpha(s-s_k)} \sigma_k [M(1 + \frac{1}{\alpha}) e^{-\alpha(s_k - s_{k-1})} \|x(s_{k-1}^+)\| \\
&\quad + M\varepsilon_0 \int_{s_{k-1}}^{s_k} e^{-\alpha(s_k-\tau)} \|x(\tau)\| d\tau] + M\varepsilon_0 \int_{s_k}^{s} e^{-\alpha(s-\tau)} \|x(\tau)\| d\tau \\
&\le \sigma_k M^2 (1 + \frac{1}{\alpha})^2 e^{-\alpha(s-s_{k-1})} \|x(s_{k-1}^+)\|
\end{aligned}$$

$$
\begin{aligned}
&+\sigma_k M^2(1+\frac{1}{\alpha})\varepsilon_0\int_{s_{k-1}}^{s_k} e^{-\alpha(s-\tau)}\|x(\tau)\|d\tau + M\varepsilon_0\int_{s_k}^{s} e^{-\alpha(s-\tau)}\|x(\tau)\|d\tau\\
&\le M(1+\frac{1}{\alpha})e^{-\beta(s-s_{k-1})}\|x(s_{k-1}^+)\| + M'M^2(1+\frac{1}{\alpha})\varepsilon_0\int_{s_{k-1}}^{s} e^{-\beta(s-\tau)}\|x(\tau)\|d\tau\\
&\le M(1+\frac{1}{\alpha})e^{-\beta(s-s_{k-1})}[M(1+\frac{1}{\alpha})e^{-\alpha(s_{k-1}-s_{k-2})}\|x(s_{k-2}^+)\|\\
&+M\varepsilon_0\int_{s_{k-2}}^{s_{k-1}} e^{-\alpha(s_{k-1}-\tau)}\|x(\tau)\|d\tau] + M'M^2(1+\frac{1}{\alpha})\varepsilon_0\int_{s_{k-1}}^{s} e^{-\beta(s-\tau)}\|x(\tau)\|d\tau\\
&\le \sigma_{k-1}M^2(1+\frac{1}{\alpha})^2 e^{-\beta(s-s_{k-1})-\alpha(s_{k-1}-s_{k-2})}\|x(s_{k-2}^+)\|\\
&+M'M^2(1+\frac{1}{\alpha})\varepsilon_0\int_{s_{k-2}}^{s_{k-1}} e^{-\beta(s-\tau)}\|x(\tau)\|d\tau\\
&+M'M^2(1+\frac{1}{\alpha})\varepsilon_0\int_{s_{k-1}}^{s} e^{-\beta(s-\tau)}\|x(\tau)\|d\tau\\
&\le M(1+\frac{1}{\alpha})e^{(\alpha-\beta)(s_{k-1}-s_{k-2})-\beta(s-s_{k-1})-\alpha(s_{k-1}-s_{k-2})}\|x(s_{k-2}^+)\|\\
&+M'M^2(1+\frac{1}{\alpha})\varepsilon_0\int_{s_{k-2}}^{s} e^{-\beta(s-\tau)}\|x(\tau)\|d\tau\\
&\le M(1+\frac{1}{\alpha})e^{-\beta(s-s_{k-2})}\|x(s_{k-2}^+)\| + M'M^2(1+\frac{1}{\alpha})\varepsilon_0\int_{s_{k-2}}^{s} e^{-\beta(s-\tau)}\|x(\tau)\|d\tau\\
&\le \dots \le M(1+\frac{1}{\alpha})e^{-\beta(s-s^0)}\|x(s^{0+})\| + M'M^2(1+\frac{1}{\alpha})\varepsilon_0\int_{s^0}^{s} e^{-\beta(s-\tau)}\|x(\tau)\|d\tau
\end{aligned}
$$

So, for $s\in(s_k,s_k+H_1]$, (3.24) holds, which implies that $\|x(s)\|<\delta$ for $s\in(s^0,s_k+H_1]$. Similarly, we can prove that $\|x(s)\|<\delta$ for $s\in(s^0,s_{k+1}]$ and (3.28) holds for $s\in(s^0,s_{k+1}]$. Therefore (3.24) holds for $s\in(s^0,s_{k+1}]$. By mathematical induction, we can conclude that $\|y_0\|<\frac{\delta}{2MM'^2(1+\frac{1}{\alpha})}$ leads to (3.24) holds for $s\in(s^0,+\infty)$.

Case II: $M\le 1$**.** Let $0<\varepsilon_0<\frac{\beta}{M'(1+\frac{1}{\alpha})}$. We can get

$$
\|x(s)\|\le(1+\frac{1}{\alpha})e^{-(\beta-\varepsilon_0 M'(1+\frac{1}{\alpha}))(s-s^0)}\|x(s^{0+})\|.
$$

Summing up the above discussion, we can conclude that no matter if $M>1$ or $M\le 1$, there exists a $\gamma(M,M',\alpha)>0$, such that

$$
\|x(s)\|\le M'(1+\frac{1}{\alpha})e^{-\gamma(s-s^0)}\|y_0\|,\ s\ge s^0.
$$

Therefore, the zero solution of (2.3) is exponentially stable. By Lemma 2.8, we can conclude that the zero solution of (1.1) is exponentially stable. □

4. Examples

As a first example consider the equation

$$
\begin{gathered}
y' = Ay + f(t,y), \quad t\in[2k+1,2k+2]\\
y(2k+1) = By(2k) + y(2k), \quad k\in N
\end{gathered}
\tag{4.1}
$$

where $A,B\in\mathbb{R}^{n\times n}$, $\|f(t,y)\|\le F(t)\cdot\|y\|$, $\int_0^{+\infty}F(t)dt<+\infty$.

(i) If $e^{\|A\|}\cdot\|B+I\|\le 1$, then the zero solution of (4.1) is uniformly stable.

(ii) If

$$\min\{\mathrm{Re}(\lambda_j) : \lambda_j \text{ is an eigenvalue of } A\} = -\alpha < 0 \tag{4.2}$$

and $e^{-\alpha}\|B + I\| \le 1$, then the zero solution of (4.1) is uniformly stable.

As a second example consider the equation

$$\begin{gathered} y' = Ay + f(t, y) + CY(2k+1), \quad t \in [2k+1, 2k+2] \\ y(2k+1) = By(2k) + y(2k), \quad k \in N \end{gathered} \tag{4.3}$$

where $A, B \in \mathbb{R}^{n\times n}$, $f(t, y)$ satisfies (H5) (for example, $f(t, y) = (y_1^2, \dots, y_n^2)^T$).

(i) If $e^{\|A\|} \cdot (1 + \|C\|)\|B + I\| \le q < 1$, then the zero solution of (4.3) is uniformly stable.

(ii) If (4.2) holds, $e^{-\alpha}(1 + \|C\|)\|B + I\| \le q < 1$, then the zero solution of (4.3) is uniformly stable.

(iii) If (4.2) holds, $e^{-\alpha}(1 + \frac{1}{\alpha})\|B + I\| \le q < 1$ and $\|C\| \le e^{-\alpha}$, then the zero solution of (4.3) is exponentially stable.

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Weizhen Feng

Department of Mathematics, South China Normal University, Guangzhou 510631, China

E-mail address: `fengweizhen1@yahoo.cn`

Yubin Liu

Department of Mathematics, Huizhou University, Huizhou 516007, China

E-mail address: `gdliuyubin@163.com`