*Electronic Journal of Differential Equations*, Vol. 2008(2008), No. 109, pp. 1–22.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

# ENERGY DECAY FOR WAVE EQUATIONS OF $\phi$-LAPLACIAN TYPE WITH WEAKLY NONLINEAR DISSIPATION

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Abstract. In this paper, first we prove the existence of global solutions in Sobolev spaces for the initial boundary value problem of the wave equation of $\phi$-Laplacian with a general dissipation of the form

$$(|u'|^{l-2}u')' - \Delta_\phi u + \sigma(t)g(u') = 0 \quad \text{in } \Omega \times \mathbb{R}_+,$$

where $\Delta_\phi = \sum_{i=1}^n \partial_{x_i}\big(\phi(|\partial_{x_i}|^2)\partial_{x_i}\big)$. Then we prove general stability estimates using multiplier method and general weighted integral inequalities proved by the second author in [18]. Without imposing any growth condition at the origin on $g$ and $\phi$, we show that the energy of the system is bounded above by a quantity, depending on $\phi$, $\sigma$ and $g$, which tends to zero (as time approaches infinity). These estimates allows us to consider large class of functions $g$ and $\phi$ with general growth at the origin. We give some examples to illustrate how to derive from our general estimates the polynomial, exponential or logarithmic decay. The results of this paper improve and generalize many existing results in the literature, and generate some interesting open problems.

## 1. Introduction

In this paper we investigate the existence of global solutions and their decay properties for the initial boundary value problem of the wave equation with weak dissipation

$$\begin{gathered}(|u'|^{l-2}u')' - \Delta_\phi u + \sigma(t)g(u') = 0 \quad \text{in } \Omega \times \mathbb{R}_+ \\ u = 0 \quad \text{on } \Gamma \times \mathbb{R}_+ \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{on } \Omega\end{gathered} \tag{1.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega = \Gamma$, $l \geq 2$, $\phi$, $\sigma$ and $g$ are given functions, and $\Delta_\phi = \sum_{i=1}^n \partial_{x_i}\big(\phi(|\partial_{x_i}|^2)\partial_{x_i}\big)$. The functions $(u_0, u_1)$ are the given initial data.

Concrete examples of (1.1) include the dissipative wave equation

$$\begin{gathered}u'' - \Delta_x u + g(u') = 0 \quad \text{in } \Omega \times \mathbb{R}_+ \\ u = 0 \quad \text{on } \Gamma \times \mathbb{R}_+ \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{on } \Omega\end{gathered} \tag{1.2}$$

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2000 *Mathematics Subject Classification.* 35B40, 35L70.
*Key words and phrases.* Wave equation; global existence; general dissipative term; rate of decay; multiplier method; integral inequalities.

Submitted September 19, 2007. Published August 11, 2008.

where $l = 2$, $\phi \equiv 1$ and $\sigma \equiv$const. The degenerate Laplace operator

$$\begin{gathered} u'' - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + g(u') = 0 \quad \text{in } \Omega \times \mathbb{R}_+ \\ u = 0 \quad \text{on } \Gamma \times \mathbb{R}_+ \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{on } \Omega \end{gathered} \tag{1.3}$$

where $l = 2$, $\phi = s^{\frac{p-2}{2}}$ with $p \geq 2$ and $\sigma \equiv$const. And the quasilinear wave equation

$$\begin{gathered} u'' - \operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) + g(u') = 0 \quad \text{in } \Omega \times \mathbb{R}_+ \\ u = 0 \quad \text{on } \Gamma \times \mathbb{R}_+ \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{on } \Omega \end{gathered} \tag{1.4}$$

when $l = 2$, $\phi = 1/\sqrt{1+s}$ and $\sigma \equiv$const. Problem (1.4), with $-\Delta u'$ instead of $g(u')$, describe the motion of fixed membrane with strong viscosity. This problem with $n = 1$ was proposed by Greenberg [16] and Greenberg-MacCamy-Mizel [17] as a model of quasilinear wave equation which admits a global solution for large data. Quite recently, Kobayashi-Pecher-Shibata [25] have treated such nonlinearity and proved the global existence of smooth solutions. Subsequently, Nakao [31] derived a decay estimate of the solutions under the assumption that the mean curvature of $\partial\Omega$ is non positive.

Our purpose in this paper is: firstly to give an existence and uniqueness theorem for global solutions in Orlitz-Sobolev spaces to the problem (1.1).

Secondly (for the stabilization problem), the aim of this paper is to obtain an explicit and general decay rate, depending on $\sigma$, $g$ and $\phi$, for the energy of solutions of (1.1) without any growth assumption on $g$ and $\phi$ at the origin, and on $\sigma$ at infinity. More precisely, we intend to obtain a general relation between the decay rate for the energy (when $t$ goes to infinity), the functions $\sigma$, $\phi$ and $g$. The proof is based on some general weighted integral inequalities proved by the second author in [18] and some properties of convex functions, in particular, the dual function of convex function to use the general Young's inequality and Jensen's inequality (instead of Hölder inequality widely used in the classical case of linear or polynomial growth of $g$ at the origin) in objective to prove our general decay estimate under a general growth of $g$ at the origin. These arguments of convexity were used for the first time (in our knowledge) by Liu and Zuazua [28], and then by Eller, Lagnese and Nicaise [15] and Alabau-Boussouira [5].

In particular, we can consider the cases where $g$ and $\phi$ degenerate near the origin polynomially, between polynomially and exponentially, exponentially or faster than exponentially. This kind of growth was considered by Liu and Zuazua [28] and Alabau-Boussouira [5] for the wave equation, and Eller, Lagnese and Nicaise [15] for Maxwell system. So we complement the results obtained in [8] and [9].

In this paper, the functions considered are all real valued. We omit the space and time variables $x$ and $t$ of $u(t,x)$, $u_t(t,x)$ and simply denote $u(t,x)$, $u_t(t,x)$ by $u$, $u'$, respectively, when no confusion arises. Let $p$ be a number with $2 \leq p \leq +\infty$. We denote by $\| \, . \, \|_p$ the $L^p$ norm over $\Omega$. In particular, $L^2$ norm is denoted $\| \, . \, \|_2$. $( \, . \, )$ denotes the usual $L^2$ inner product. We use familiar function spaces $W_0^{1,2}$.

The paper is organized as follow: in section 2, we give some hypotheses and we announce the main results of this paper. In section 3 and section 4, we prove all the announced results. In section 5, we give some applications. Finally, we conclude and give some comments and open questions in section 6.

## 2. Preliminaries and main results

We use the following hypotheses:

(H1) $\sigma : \mathbb{R}_+ \to ]0, +\infty[$ is a non increasing function of class $C^1(\mathbb{R}_+)$ satisfying
$$\int_0^{+\infty} \sigma(\tau)\, d\tau = +\infty. \tag{2.1}$$

(H2) $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $C^1(]0, +\infty[) \cap C([0, +\infty[)$ satisfying: $\phi(s) > 0$ on $]0, +\infty[$ and $\phi$ is non decreasing.

(H3) $g : \mathbb{R} \to \mathbb{R}$ is a non decreasing function of class $C(\mathbb{R})$ such that there exist $\epsilon_1$, $c_1$, $c_2 > 0$, $l - 1 \le r$, $(n-2)r \le n+2$ and a convex and increasing function $G : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ satisfying $G(0) = 0$, and $G'(0) = 0$ or $G$ is linear on $[0, \epsilon_1]$ such that
$$c_1|s|^{l-1} \le |g(s)| \le c_2|s|^r \quad \text{if } |s| \ge \epsilon_1, \tag{2.2}$$
$$|s|^l + g^2(s) \le G^{-1}(sg(s)) \quad \text{if } |s| \le \epsilon_1. \tag{2.3}$$

**Remark 2.1.** 1. We have $\int_0^{+\infty} \phi(\tau) d\tau = +\infty$, and $s \mapsto \int_0^s \phi(\tau) d\tau$ is a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$.

2. The function $\tilde{\phi}(s) = \frac{1}{2} \int_0^s \phi(\tau)\, d\tau$ is a convex function. Indeed, let $x_1 \ne 0$ and $x_1 \ne 0$ such that $x_1 < x_2$, as $\phi$ is of class $C^1([x_1, x_2])$ and a non decreasing function, then $\tilde{\phi}$ is a convex function. Now if $x_1 = 0$, we have for all $0 \le \lambda \le 1$
$$\tilde{\phi}(\lambda x_2) = \frac{1}{2} \int_0^{\lambda x_2} \phi(s)\, ds = \frac{1}{2} \lambda \int_0^{x_2} \phi(\lambda z)\, dz$$
where we have make the change of variable $s = \lambda z$. As $\phi$ is a non decreasing function and $\lambda x_2 \le x_2$ for all $\lambda \in [0, 1]$, then
$$\tilde{\phi}(\lambda x_2) \le \lambda \tilde{\phi}(x_2).$$

3. If $g$ satisfies
$$H(|s|) \le |g(s)| \le H^{-1}(|s|) \quad \text{if } |s| \le \epsilon_1$$
for a function $H : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $H'(0) = 0$ or $H$ being linear on $[0, \sqrt{\frac{\epsilon_1}{\delta}}]$ where $\delta = 2 \max\{1, \epsilon_1^{l-2}\}$ such that the function $s \mapsto \sqrt{s} H(\sqrt{s})$ is convex and increasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[$, then the condition (2.3) is satisfied for
$$G(s) = \sqrt{\frac{s}{\delta}} H(\sqrt{\frac{s}{\delta}}).$$
In the other hand, $g$ satisfies (H3) for any $\epsilon_1' \in ]0, \epsilon_1]$ (with some $c_1'$, $c_2' > 0$ instead of $c_1$, $c_2$, respectively).

Now we define (as before) $\tilde{\phi}(s) = \frac{1}{2} \int_0^s \phi(\tau) d\tau$ and the energy associated to the solution of the system (1.1) by the following formula:
$$E(t) = \frac{l-1}{l} \int_\Omega |u'|^l dx + \int_\Omega \sum_{i=1}^n \tilde{\phi}(|\partial_{x_i} u|^2) dx. \tag{2.4}$$

By a simple computation, we have

$$E'(t) = -\sigma(t)\int_\Omega u' g(u')dx, \tag{2.5}$$

so $E$ is non negative and non increasing function. We first state two lemmas which will be needed later.

**Lemma 2.1** (Sobolev-Poincaré's inequality). *Let $p > 1$ and $q > 1$ with $(n-p)q \leq np$, then there is a constant $c_* = c_*(\Omega, p, q)$ such that*

$$\|u\|_q \leq c_* \|\nabla u\|_p \quad \text{for} \quad u \in W_0^{1,p}(\Omega).$$

*The case $p = q = 2$ gives the known Poincaré's inequality.*

**Lemma 2.2** (Guesmia [18]). *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ differentiable function, $\lambda \in \mathbb{R}_+$ and $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ convex and increasing function such that $\Psi(0) = 0$. Assume that*

$$\int_s^{+\infty} \Psi(E(t))\,dt \leq E(s), \quad \forall s \geq 0$$
$$E'(t) \leq \lambda E(t), \quad \forall t \geq 0.$$

*Then $E$ satisfies the estimate*

$$E(t) \leq e^{\tau_0 \lambda T_0} d^{-1}\Big(e^{\lambda(t-h(t))}\Psi\Big(\psi^{-1}\Big(h(t) + \psi(E(0))\Big)\Big)\Big), \quad \forall t \geq 0$$

*where*

$$\psi(t) = \int_t^1 \frac{1}{\Psi(s)}\,ds, \quad \forall t > 0,$$
$$d(t) = \begin{cases} \Psi(t) & \text{if } \lambda = 0, \\ \int_0^t \frac{\Psi(s)}{s}\,ds & \text{if } \lambda > 0, \end{cases} \quad \forall t \geq 0,$$
$$h(t) = \begin{cases} K^{-1}(D(t)), & \text{if } t > T_0, \\ 0 & \text{if } t \in [0, T_0] \end{cases}$$
$$K(t) = D(t) + \frac{\psi^{-1}\big(t + \psi(E(0))\big)}{\Psi\big(\psi^{-1}\big(t + \psi(E(0))\big)\big)} e^{\lambda t}, \quad \forall t \geq 0,$$
$$D(t) = \int_0^t e^{\lambda s}\,ds, \quad \forall t \geq 0,$$
$$T_0 = D^{-1}\Big(\frac{E(0)}{\Psi(E(0))}\Big), \quad \tau_0 = \begin{cases} 0, & \text{if } t > T_0, \\ 1, & \text{if } t \in [0, T_0]. \end{cases}$$

**Remark 2.2.** If $\lambda = 0$ (that is $E$ is non increasing), then we have

$$E(t) \leq \psi^{-1}\Big(h(t) + \psi(E(0))\Big), \quad \forall t \geq 0 \tag{2.6}$$

where $\psi(t) = \int_t^1 \frac{1}{\Psi(s)}\,ds$ for $t > 0$, $h(t) = 0$ for $0 \leq t \leq \frac{E(0)}{\Psi(E(0))}$ and

$$h^{-1}(t) = t + \frac{\psi^{-1}\Big(t + \psi(E(0))\Big)}{\Psi\Big(\psi^{-1}\Big(t + \psi(E(0))\Big)\Big)}, \quad t > 0.$$

This particular result generalizes the one obtained by Martinez [29] in the particular case $\Psi(t) = dt^{p+1}$ with $p \geq 0$ and $d > 0$, and improves the one obtained by Eller, Lagnese and Nicaise [15].

*Proof of Lemma 2.2.* Because $E'(t) \le \lambda E(t)$ implies $E(t) \le e^{\lambda(t-t_0)}E(t_0)$ for all $t \ge t_0 \ge 0$, then, if $E(t_0) = 0$ for some $t_0 \ge 0$, then $E(t) = 0$ for all $t \ge t_0$, and then there is nothing to prove in this case. So we assume that $E(t) > 0$ for all $t \ge 0$ without lose of generality. Let

$$L(s) = \int_s^{+\infty} \Psi(E(t))\,dt, \quad \forall s \ge 0.$$

We have $L(s) \le E(s)$, for all $s \ge 0$. The function $L$ is positive, decreasing and of class $C^1(\mathbb{R}_+)$ satisfying

$$-L'(s) = \Psi(E(s)) \ge \Psi(L(s)), \quad \forall s \ge 0.$$

The function $\psi$ is decreasing, then

$$\Big(\psi(L(s))\Big)' = \frac{-L'(s)}{\Psi(L(s))} \ge 1, \quad \forall s \ge 0.$$

Integration on $[0, t]$, we obtain

$$\psi(L(t)) \ge t + \psi(E(0)), \quad \forall t \ge 0. \tag{2.7}$$

Since $\Psi$ is convex and $\Psi(0) = 0$, we have

$$\Psi(s) \le \Psi(1)s,\ \forall s \in [0,1] \quad \text{and} \quad \Psi(s) \ge \Psi(1)s,\ \forall s \ge 1,$$

then $\lim_{t\to 0} \psi(t) = +\infty$ and $[\psi(E(0)), +\infty[ \subset \operatorname{Image}(\psi)$. Then (2.7) implies that

$$L(t) \le \psi^{-1}\Big(t + \psi(E(0))\Big), \quad \forall t \ge 0. \tag{2.8}$$

Now, for $s \ge 0$, let

$$f_s(t) = e^{-\lambda t}\int_s^t e^{\lambda\tau}\,d\tau, \quad \forall t \ge s.$$

The function $f_s$ is increasing on $[s, +\infty[$ and strictly positive on $]s, +\infty[$ such that

$$f_s(s) = 0 \quad \text{and} \quad f_s'(t) + \lambda f_s(t) = 1, \quad \forall t \ge s \ge 0,$$

and the function $d$ is well defined, positive and increasing such that

$$d(t) \le \Psi(t) \quad \text{and} \quad \lambda t d'(t) = \lambda\Psi(t), \quad \forall t \ge 0,$$

then

$$\begin{aligned}
\partial_\tau\Big(f_s(\tau)d(E(\tau))\Big) &= f_s'(\tau)d(E(\tau)) + f_s(\tau)E'(\tau)d'(E(\tau)) \\
&\le \Big(1 - \lambda f_s(\tau)\Big)\Psi(E(\tau)) + \lambda f_s(\tau)\Psi(E(\tau)) \\
&= \Psi(E(\tau)), \quad \forall \tau \ge s \ge 0.
\end{aligned}$$

Integrating on $[s, t]$, we obtain

$$L(s) \ge \int_s^t \Psi(E(\tau))\,d\tau \ge f_s(t)d(E(t)), \quad \forall t \ge s \ge 0. \tag{2.9}$$

Since $\lim_{t\to+\infty} d(s) = +\infty$, $d(0) = 0$ and $d$ is increasing, then (2.8) and (2.9) imply

$$E(t) \le d^{-1}\Big(\inf_{s\in[0,t[} \frac{\psi^{-1}\Big(s + \psi(E(0))\Big)}{f_s(t)}\Big), \quad \forall t > 0. \tag{2.10}$$

Now, let $t > T_0$ and

$$J(s) = \frac{\psi^{-1}\Big(s + \psi(E(0))\Big)}{f_s(t)}, \quad \forall s \in [0, t[.$$

The function $J$ is differentiable and we have

$$J'(s) = f_s^{-2}(t)\Big[e^{-\lambda(t-s)}\psi^{-1}\Big(s + \psi(E(0))\Big) - f_s(t)\Psi\Big(\psi^{-1}\Big(s + \psi(E(0))\Big)\Big)\Big].$$

Then

$$J'(s) = 0 \;\Leftrightarrow\; K(s) = D(t) \quad \text{and} \quad J'(s) < 0 \;\Leftrightarrow\; K(s) < D(T).$$

Since $K(0) = \frac{E(0)}{\Psi(E(0))}$, $D(0) = 0$ and $K$ and $D$ are increasing (because $\psi^{-1}$ is decreasing and $s \mapsto \frac{s}{\Psi(s)}$, $s > 0$, is non increasing thanks to the fact that $\Psi$ is convex). Then, for $t > T_0$,

$$\inf_{s \in [0,t[} J(s) = J\Big(K^{-1}(D(t))\Big) = J(h(t)).$$

Since $h$ satisfies $J'(h(t)) = 0$, we conclude from (2.10) our desired estimate for $t > T_0$.

For $t \in [0, T_0]$, we have just to note that $E'(t) \le \lambda E(t)$ and the fact that $d \le \Psi$ imply

$$E(t) \le e^{\lambda t}E(0) \le e^{\lambda T_0}E(0) \le e^{\lambda T_0}\Psi^{-1}\Big(e^{\lambda t}\Psi(E(0))\Big) \le e^{\lambda T_0}d^{-1}\Big(e^{\lambda t}\Psi(E(0))\Big).$$

□

**Remark 2.3.** Under the hypotheses of Lemma 2.2, we have $\lim_{t\to+\infty} E(t) = 0$. Indeed, we have just to choose $s = \frac{1}{2}t$ in (2.10) instead of $h(t)$ and note that $d^{-1}(0) = 0$, $\lim_{t\to+\infty} \psi^{-1}(t) = 0$ and $\lim_{t\to+\infty} f_{\frac{1}{2}t}(t) > 0$.

Before stating the global existence theorem, we will give some notions of the theory of Orlitz spaces (see [2] and [27]) which is suitable for a large class of quasilinear equations.

**Definition 2.1.** A function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is called an N-function if it is continuous, convex, strictly increasing and such that

$$\lim_{s\to 0} \frac{\Phi(s)}{s} = 0 \text{ and } \lim_{s\to 0} \frac{\Phi(s)}{s} = +\infty.$$

The N-function complementary to $\Phi$ is defined by $\tilde{\Phi}(s) = \max_{\sigma\ge 0}(s\sigma - \Phi(\sigma))$. The Simonenko indices $p(\Phi)$ and $q(\Phi)$ are defined by

$$p(\Phi) = \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)}, \quad q(\Phi) = \sup_{t>0} \frac{t\Phi'(t)}{\Phi(t)}.$$

Clearly, $1 \le p(\Phi) \le q(\Phi) \le \infty$, and if $q(\Phi) < \infty$, then

$$p(\Phi)\frac{\Phi(t)}{t} \le \Phi'(t) \le q(\Phi)\frac{\Phi(t)}{t} \text{ for all } t > 0. \tag{2.11}$$

Integrating these inequalities, one sees that (2.11) is equivalent to

$$\frac{\Phi(t)}{t^{p(\Phi)}} \text{ is increasing, and } \frac{\Phi(t)}{t^{q(\Phi)}} \text{ is decreasing for all } t > 0. \tag{2.12}$$

We say that two N-function $\Phi$ and $\Phi_1$ are equivalent if there exists two constants $C_1$ and $C_2$ such that

$$C_1\Phi_1(t) \le \Phi(t) \le C_2\Phi_1(t) \quad \text{for all } t \ge 0.$$

We denote by $i(\Phi)$ and $I(\phi)$ the reciprocal Boyd indices of $\Phi$. Sometimes, $i(\Phi)$ is called the lower index and $I(\Phi)$ the upper index of $\Phi$. We have the following characterisations:

$$i(\Phi) = \sup_{\Phi_1 \sim \Phi} p(\Phi_1) \quad \text{and} \quad I(\Phi) = \inf_{\Phi_1 \sim \Phi} q(\Phi_1).$$

Let $\Phi$ be an N-function satisfying $I(\Phi) < \infty$. The Orlitz space $L^\Phi = L^\Phi(\Omega)$ is the space of all measurable functions $f$ defined on $\Omega$ such that $\int_\Omega \Phi(|f|)\,dx < +\infty$. It is endowed with the norm

$$\|f\|_\Phi = \inf\big\{\lambda > 0; \int_\Omega \Phi\big(\frac{|f|}{\lambda}\big)\,dx \le 1\big\}$$

For every $f \in L^\Phi$ and every $g \in L^{\tilde\Phi}$ the following Hölder type inequality holds:

$$\int_\Omega |fg|\,dx \le 2\|f\|_\Phi \|g\|_{\tilde\Phi}.$$

Let us denote by $\mathcal{W}^{1,\Phi} = \mathcal{W}^{1,\Phi}(\Omega)$ the space of all functions in $L^\Phi$ such that the distributional partial derivatives belong to $L^\Phi$, and by $\mathcal{W}_0^{1,\Phi}(\Omega)$ the closure of the test functions in this space. Such spaces are well known in the literature as Orlitz-Sobolev spaces (see [2]). We have Poincaré's inequality for Orlitz-Sobolev spaces

$$\|u\|_\Phi \le C\|\nabla_x u\|_\Phi, \quad u \in \mathcal{W}_0^{1,\Phi}(\Omega), \tag{2.13}$$

so that $\|\nabla_x u\|_\Phi$ defines an equivalent norm in $\mathcal{W}_0^{1,\Phi}(\Omega)$. By $\mathcal{W}_0^{-1,\tilde\Phi} = \mathcal{W}_0^{-1,\tilde\Phi}(\Omega)$ we denote the dual space of $\mathcal{W}_0^{1,\Phi}(\Omega)$.

The classical Sobolev embedding theorem has been extended into Orlitz setting. In the following we only need that if $\Phi$ is an N-function such that for $n' = \frac{n}{n-1}$ $(n > 1)$

$$\int_1^{+\infty} \frac{\tilde\Phi(s)}{s^{n'+1}}\,ds = +\infty, \tag{2.14}$$

then it is possible to define an optimal N-function $\Phi^*$ such that the embedding

$$\mathcal{W}_0^{1,\Phi}(\Omega) \hookrightarrow L^{\Phi^*} \tag{2.15}$$

holds; optimality means that $L^{\Phi^*}$ is the smallest Orlitz space for which (2.15) holds. If the integral in (2.14) is finite or $n = 1$, then

$$\mathcal{W}_0^{1,\Phi}(\Omega) \hookrightarrow L^\infty(\Omega). \tag{2.16}$$

We assume that $\Phi$ is N-function satisfying

$$i(\Phi) \in \Big]\frac{2n}{n+2}, +\infty\Big[ \cap ]1, +\infty[ \quad \text{and} \quad I(\Phi) < +\infty. \tag{2.17}$$

and where $\Phi^*$ is the N-function from (2.15), and identifying $L^2$ with its dual, we have

$$W_0^{1,\Phi} \hookrightarrow L^2 \hookrightarrow W^{-1,\tilde\Phi}.$$

These dense inclusions also hold by the Sobolev embedding (2.16) in the case that the integral in (2.14) is finite. Set

$$\Phi(s) = \frac{1}{2}\int_0^{s^2} \phi(t)\,dt.$$

**Theorem 2.1.** *Assume that* $(u_0, u_1) \in \mathcal{W}_0^{1,\Phi}(\Omega) \times L^2(\Omega)$. *Then problem* (1.1) *admits a unique strong solution on* $\Omega \times [0,\infty[$ *in the class*

$$C([0,\infty[, \mathcal{W}_0^{1,\Phi}(\Omega)) \cap C^1([0,\infty[, L^2(\Omega))$$

*Proof.* The theory of maximal monotone operators associated with subdifferentials (see [21], [22], [10] and [4]) imply that, for every $(u_0, u_1) \in \mathcal{W}_0^{1,\phi}(\Omega) \times L^2(\Omega)$, the problem (1.1) admits a unique global strong solution. $\square$

Our main result on stabilization is the following.

**Theorem 2.2.** *Assume that* (H1)-(H3) *hold. Let* $\tilde{\sigma}(t) = \int_0^t \sigma(\tau)d\tau$. *Then there exist* $\omega$, $\epsilon_0 > 0$ *such that the energy* $E$ *satisfies*

$$E(t) \leq \varphi_1\Big(\psi^{-1}\Big(h(\tilde{\sigma}(t)) + \psi({\varphi_1}^{-1}(E(0)))\Big)\Big), \quad \forall t \geq 0 \tag{2.18}$$

*where*

$$\psi(t) = \int_t^1 \frac{1}{\omega\varphi(\tau)}d\tau \quad \textit{for } t > 0; \quad h(t) = 0 \quad \textit{for } 0 \leq t \leq \frac{E(0)}{\omega\varphi(E(0))};$$

$$h^{-1}(t) = t + \frac{\psi^{-1}\Big(t + \psi(E(0))\Big)}{\omega\varphi\Big(\psi^{-1}\Big(t + \psi(E(0))\Big)\Big)}, \quad \textit{for } t > 0;$$

$$\varphi(s) = \begin{cases} \tilde{\phi}(s) & \textit{if } r = 1 \textit{ and } G \textit{ is linear on } [0, \epsilon_1], \\ \frac{(\tilde{\phi}(s))^{1+\frac{1}{r}}}{s^{1/r}} & \textit{if } r \neq 1 \textit{ and } G \textit{ is linear on } [0, \epsilon_1], \\ \frac{2\epsilon_0 s^2}{\tilde{\phi}^{-1}(s)} G'\left(\frac{\epsilon_0^2 s^2}{\tilde{\phi}^{-1}(s)}\right) & \textit{if } G'(0) = 0; \end{cases}$$

$$\varphi_1(s) = \begin{cases} \tilde{\phi}(s) & \textit{if } G \textit{ is linear on } [0, \epsilon_1], \\ s & \textit{if } G'(0) = 0. \end{cases}$$

**Remark 2.4.** 1. Under the hypotheses of Theorem 2.2 and thanks to Remark 2.3, we have strong stability of (1.1); that is,

$$\lim_{t \to +\infty} E(t) = 0. \tag{2.19}$$

2. Thanks to (H2) and (H3), the function $\varphi$ (defined in Theorem 2.2) is of class $C^1(\mathbb{R}_+)$ and satisfies the same hypotheses as the function $\Psi$ in Lemma 2.2. Then we can apply Lemma 2.2 for $\Psi = \omega\varphi$.

3. We obtain same results for the problem

$$u'' - \Delta_\phi u - \sigma(t)\,\mathrm{div}(\psi(|\nabla_x u'|^2)\nabla_x u') = 0$$

such that

$$c_1 \leq \psi(s^2) \leq c_2 \quad \text{if } |s| \geq \epsilon_1, \tag{2.20}$$

$$|s|^2 + \psi^2(s) \leq G^{-1}(sg(s)) \quad \text{if } |s| \leq \epsilon_1. \tag{2.21}$$

Using (2.18), we give several significant examples of growth at the origine of $g$ and $\phi$, and the corresponding decay estimates. Some of these examples were given (in less general form) by Liu and Zuazua [28] and Alabau-Boussouira [5] for the wave equation, and Eller, Lagnese and Nicaise [15] for Maxwell system.

**Polynomial or logarithmic growth for $\phi$ and polynomial growth for $g$.** If $\phi(t) = ct^m\Big(\ln(t+1)\Big)^q$ (degeneracy of finite order) and $G(t) = c't^{\frac{p+1}{2}}$ for $c$, $c' > 0$ (that is $c'_1|s|^{\frac{l(p+1)-2}{2}} \leq |g(s)| \leq c'_2|s|^{\frac{1}{p}}$ on $[-\epsilon_1, \epsilon_1]$ for some $c'_1$, $c'_2 > 0$), $m \geq 0$, $q \geq -m$ and $p \geq 1$ (note that $c_3 s^{m+q+1} \leq \tilde{\phi}(s) \leq c'_3 s^{m+q+1}$ for $c_3$, $c'_3 > 0$ when $s$ is near 0), then there exists $\alpha > 0$ such that for all $t \geq 0$

$$E(t) \leq \begin{cases} \alpha e^{-\omega\tilde{\sigma}(t)} & \text{if } (m+q, p) = (0,1), \\ \alpha\Big(\tilde{\sigma}(t)+1\Big)^{-\frac{r(m+q+1)}{(r+1)(m+q)}} & \text{if } m+q > 0,\ p = 1,\ r > 1, \\ \alpha\Big(\tilde{\sigma}(t)+1\Big)^{-\frac{2(m+q+1)}{2p(m+q)+p-1}} & \text{otherwise.} \end{cases}$$

Moreover, we can obtain more precise rate of decay, in the case $\phi(s) = s^m$ with $m \geq 0$ and

$$c_1|s|^p \leq |g(s)| \leq c_2|s|^\theta \quad \text{if } |s| \leq \epsilon_1$$

where $\frac{1}{p} \leq \theta \leq p$. We have the following estimates: If $l \geq p+1$, then for all $t \geq 0$,

$$E(t) \leq \begin{cases} \alpha e^{-\omega\tilde{\sigma}(t)} & \text{if } 2m+1 \leq \theta, \\ \alpha(\tilde{\sigma}(t)+1)^{-\frac{2\theta(m+1)}{2m+1-\theta}} & \text{if } 2m+1 > \theta. \end{cases}$$

If $l < p+1$, then for all $t \geq 0$,

$$E(t) \leq \begin{cases} \alpha(\tilde{\sigma}(t)+1)^{-\frac{2\theta(m+1)}{2m+1-\theta}} & \text{if } l \geq \frac{2\theta(m+1)(p+1)}{(\theta+1)(2m+1)}, \\ \alpha(\tilde{\sigma}(t)+1)^{-\frac{l}{p+1-l}} & \text{if } l < \frac{2\theta(m+1)(p+1)}{(\theta+1)(2m+1)}. \end{cases}$$

**Polynomial or logarithmic growth for $\phi$ and exponential growth for $g$.** If $\phi(t) = ct^m\Big(\ln(t+1)\Big)^q$ (degeneracy of finite order) and $H(|s|) \leq |g(s)| \leq H^{-1}(|s|)$ on $[-\epsilon_1, \epsilon_1]$ where $H(s) = \frac{1}{s}e^{-s^{-\gamma}}$, $m \geq 0$, $q \geq -m$ and $c, \gamma > 0$ (note that $c_3 s^{m+q+1} \leq \tilde{\phi}(s) \leq c'_3 s^{m+q+1}$ and $G(s) = e^{-2^{\frac{\gamma}{2}} s^{-\frac{\gamma}{2}}}$ for $c_3$, $c'_3 > 0$ when $s$ is near 0, and $\psi(s) \leq c'_1 e^{c'_2 s^{-\frac{\gamma}{2}}}$ on $]0,1]$ for $c'_1$, $c'_2 > 0$), then there exist $\alpha, \beta > 0$ such that

$$E(t) \leq \beta\Big(\ln\Big(\alpha h(\tilde{\sigma}(t)) + 2\Big)\Big)^{\frac{-2(m+q+1)}{\gamma(2(m+q)+1)}}, \quad \forall t \geq 0.$$

**Polynomial or logarithmic growth for $\phi$ and faster than exponential growth for $g$.** If $\phi(t) = ct^m\Big(\ln(t+1)\Big)^q$ (degeneracy of finite order) and $H(|s|) \leq |g(s)| \leq H^{-1}(|s|)$ on $[-\epsilon_1, \epsilon_1]$ where $H(s) = \frac{1}{s}H_n(s)$, $m \geq 0$, $q \geq -m, c$, $\gamma > 0$ and

$$H_1(s) = e^{-s^{-\gamma}} \quad \text{and} \quad H_n(s) = e^{-\frac{1}{H_{n-1}(s)}},\ n = 2, 3, \cdots,$$

then (as in the example 2) there exist $\alpha, \beta, \delta > 0$ such that

$$E(t) \leq \beta\Big(\bar{H}_n(h(\tilde{\sigma}(t)))\Big)^{\frac{-2(m+q+1)}{\gamma(2(m+q)+1)}}, \quad \forall t \geq 0.$$

where

$$\bar{H}_1(t) = \ln(\alpha t + \delta) \quad \text{and} \quad \bar{H}_n(t) = \ln(\bar{H}_{n-1}(t)),\ n = 2, 3, \cdots.$$

**Polynomial or logarithmic growth for $\phi$ and between polynomial and exponential growth for $g$.** If $\phi(t) = ct^m\Big(\ln(t+1)\Big)^q$ (degeneracy of finite order) and $H(|s|) \le |g(s)| \le H^{-1}(|s|)$ on $[-\epsilon_1, \epsilon_1]$ where $H(s) = \frac{1}{s}e^{-(H_n(s))^\gamma}$, $\gamma > 1$, $m \ge 0$, $q \ge -m, c > 0$ and

$$H_1(s) = -\ln s \quad \text{and} \quad H_n(s) = \ln(H_{n-1}(s)),\ n = 2,\, 3, \cdots$$

(then $G(s) = e^{-(\frac{-1}{2}\ln\frac{s}{2})^\gamma}$ when $s$ is near 0), then there exist $\alpha,\, \beta, \delta > 0$ such that

$$E(t) \le \beta e^{-\frac{2(m+q+1)}{2(m+q)+1}\bar{H}_n(h(\tilde{\sigma}(t)))}, \quad \forall t \ge 0.$$

where

$$\bar{H}_1(t) = \ln(\alpha t + \delta)^{\frac{1}{\gamma}} \quad \text{and} \quad \bar{H}_n(t) = e^{\bar{H}_{n-1}(t)},\ n = 2,\, 3, \cdots.$$

**Exponential growth for $\phi$ (degeneracy of infinite order) and linear growth for $g$.** If $\phi(t) = e^{-t^{-\gamma}}, \gamma > 0$, (note that $c_1' t^{\gamma+1} e^{-t^{-\gamma}} \le \tilde{\phi}(t) \le c_2' t^{\gamma+1} e^{-t^{-\gamma}}$ for $c_1',\, c_2' > 0$ when $s$ is near 0) then there exist $\alpha,\, \beta > 0$ such that for all $t \ge 0$

$$E(t) \le \begin{cases} \beta(h(\tilde{\sigma}(t)))^{\frac{-r}{r+1}}\Big(\ln(\alpha h(\tilde{\sigma}(t)) + 2)\Big)^{-\frac{\gamma+1}{\gamma}}, & \text{if } r > 1, \\ \beta(h(\tilde{\sigma}(t)))^{-1}\Big(\ln(\alpha h(\tilde{\sigma}(t)) + 2)\Big)^{-\frac{\gamma+1}{\gamma}}, & \text{if } r = 1. \end{cases}$$

**Faster than exponential growth for $\phi$ (degeneracy of infinite order) and linear growth for $g$.** If $\phi(t) = e^{-e^{t^{-\gamma}}}, \gamma > 0$, (note that $c_1' s^{\gamma+1} e^{-e^{t^{-\gamma}}} e^{-t^{-\gamma}} \le \tilde{\phi}(t) \le c_2' t^{\gamma+1} e^{-e^{t^{-\gamma}}} e^{-t^{-\gamma}}$ for $c_1',\, c_2' > 0$ when $t$ is near 0) then there exist $\alpha,\, \beta > 0$ such that for all $t \ge 0$,

$$E(t) \le \begin{cases} \beta(h(\tilde{\sigma}(t)))^{\frac{-r}{r+1}}\Big(\ln(\alpha h(\tilde{\sigma}(t)) + 2)\Big)^{-1}\Big(\ln\Big(\frac{r}{r+1}\ln(\alpha h(\tilde{\sigma}(t)) + 3)\Big)\Big)^{-\frac{\gamma+1}{\gamma}}, & \text{if } r > 1, \\ \beta(h(\tilde{\sigma}(t)))^{-1}\Big(\ln(\alpha h(\tilde{\sigma}(t)) + 2)\Big)^{-1}\Big(\ln\Big(\ln(\alpha h(\tilde{\sigma}(t)) + 3)\Big)\Big)^{-\frac{\gamma+1}{\gamma}}, & \text{if } r = 1. \end{cases}$$

**Faster than polynomials, less than exponential growth for $\phi$ (degeneracy of infinite order) and linear growth for $g$.** If $\phi(t) = e^{-(-\ln t)^\gamma}, \gamma \ge 1$, (note that $c_1' t e^{-(-\ln t)^\gamma}(-\ln t)^{1-\gamma} \le \tilde{\phi}(t) \le c_2' t e^{-(-\ln t)^\gamma}(-\ln t)^{1-\gamma}$ for $c_1',\, c_2' > 0$ when $t$ is near 0) then there exist $\alpha, \beta > 0$ such for all $t \ge 0$, that

$$E(t) \le \begin{cases} \beta(h(\tilde{\sigma}(t)))^{\frac{-r}{r+1}} e^{-\left(\frac{r}{r+1}\ln(\alpha h(\tilde{\sigma}(t))+2)\right)^{\frac{1}{\gamma}}}\Big(\ln(\alpha h(\tilde{\sigma}(t)) + 2)\Big)^{-\frac{\gamma-1}{\gamma}}, & \text{if } r > 1, \\ \beta(h(\tilde{\sigma}(t)))^{-1} e^{-(\ln(\alpha h(\tilde{\sigma}(t))+2))^{\frac{1}{\gamma}}}\Big(\ln(\alpha h(\tilde{\sigma}(t)) + 2)\Big)^{-\frac{\gamma-1}{\gamma}}, & \text{if } r = 1. \end{cases}$$

**Slow than polynomials for $\phi$ (slow degeneracy) and linear growth for $g$.** If $\phi(t) = |\ln t|^{-\gamma}$ near of 0 where $\gamma > 0$, (note that $c_1' s(-\ln s)^{-\gamma} \le \tilde{\phi}(s) \le c_2' s(-\ln s)^{-\gamma}$ for $c_1',\, c_2' > 0$ when $s$ is near 0) then there exists $\alpha > 0$ such that for all $t \ge 0$,

$$E(t) \le \begin{cases} \alpha(h(\tilde{\sigma}(t)))^{-\frac{\gamma}{\gamma(1+\frac{1}{r})+1}} e^{-(h(\tilde{\sigma}(t)))^{\frac{1}{\gamma(1+\frac{1}{r})+1}}}, & \text{if } r > 1, \\ \alpha(h(\tilde{\sigma}(t)))^{-\frac{\gamma}{\gamma+1}} e^{-(h(\tilde{\sigma}(t)))^{\frac{1}{\gamma+1}}}, & \text{if } r = 1. \end{cases}$$

## 3. Proof of Theorem 2.2

For the rest of this article, we denote by $c$ various positive constants which may be different at different occurrences.

If $E(t_0) = 0$ for some $t_0 \geq 0$, then $E(t) = 0$ for all $t \geq t_0$, and then we have nothing to prove in this case. So we assume that $E(t) > 0$ for all $t \geq 0$ without loss of generality.

We multiply the first equation of (1.1) by $\sigma(t)\frac{\tilde{\varphi}(E)}{E}u$ where $\tilde{\varphi} : \mathbb{R}_+ \to \mathbb{R}_+$ is convex, increasing and of class $C^1(]0, +\infty[)$ such that $\tilde{\varphi}(0) = 0$, and we integrate by parts, we have, for all $0 \leq S \leq T$,

$$
\begin{aligned}
0 &= \int_S^T \sigma(t)\frac{\tilde{\varphi}(E)}{E}\int_\Omega u\Big((|u'|^{l-2}u')' - \Delta_\phi u + \sigma(t)g(u')\Big)\,dx\,dt \\
&= \Big[\sigma(t)\frac{\tilde{\varphi}(E)}{E}\int_\Omega uu'|u'|^{l-2}dx\Big]_S^T \\
&\quad - \int_S^T\int_\Omega u'|u'|^{l-2}\Big(\sigma'(t)\frac{\tilde{\varphi}(E)}{E}u + \sigma(t)\frac{\tilde{\varphi}(E)}{E}u' + \sigma(t)(\frac{\tilde{\varphi}(E)}{E})'u\Big)\,dx\,dt \\
&\quad + \int_S^T \sigma(t)\frac{\tilde{\varphi}(E)}{E}\int_\Omega\sum_{i=1}^n \phi(|\partial_{x_i}u|^2)|\partial_{x_i}u|^2\,dx\,dt + \int_S^T \sigma^2(t)\frac{\tilde{\varphi}(E)}{E}\int_\Omega ug(u')\,dx\,dt.
\end{aligned}
$$

Using Lemma 2.1 for $p = 2$ and $q = l$ and the definition of $E$, we have (note also that $\tilde{\phi}$ is convex and defines a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$)

$$
\begin{aligned}
\Big|\int_\Omega uu'|u'|^{l-2}dx\Big| &\leq \Big(\int_\Omega |u|^l dx\Big)^{1/l}\Big(\int_\Omega |u'|^l dx\Big)^{\frac{l-1}{l}} \\
&\leq c\Big(\int_\Omega |\nabla u|^2 dx\Big)^{1/2} E^{\frac{l-1}{l}} \\
&\leq cE^{\frac{l-1}{l}}\Big(\sum_{i=1}^n \tilde{\phi}^{-1}\Big(\int_\Omega \sum_{i=1}^n \tilde{\phi}(|\partial_{x_i}u|^2)dx\Big)\Big)^{1/2} \\
&\leq cE^{\frac{l-1}{l}}\sqrt{\tilde{\phi}^{-1}(E)}.
\end{aligned} \tag{3.1}
$$

In the other hand, we have $s\phi(s) \geq 2\tilde{\phi}(s)$, $l \geq 2$, $\tilde{\phi}^{-1}$ is non decreasing and $\tilde{\varphi}$ is convex, increasing and of class $C^1(]0, +\infty[)$ such that $\tilde{\varphi}(0) = 0$ (then $s \mapsto s^{\frac{l-1}{l}}$, $s \mapsto \tilde{\phi}^{-1}(s)$ and $s \mapsto \frac{\tilde{\varphi}(s)}{s}$ are non decreasing). Then we deduce that

$$
\begin{aligned}
&\int_S^T \sigma(t)\tilde{\varphi}(E(t))dt \\
&\leq cE^{\frac{l-1}{l}}(S)\sqrt{\tilde{\phi}^{-1}(E(S))}\frac{\tilde{\varphi}(E(S))}{E(S)} + c\int_S^T \sigma(t)\frac{\tilde{\varphi}(E)}{E}\int_\Omega(|u'|^l + |ug(u')|)\,dx\,dt.
\end{aligned} \tag{3.2}
$$

To estimate the last integral above, we distinguish three cases:

**Case 1: $r = 1$ and $G$ is linear on $[0, \epsilon_1]$:** We choose $\tilde{\varphi}(s) = s$. For all $t \geq 0$, we denote

$$
\Omega_t^+ = \{x \in \Omega : uu' \geq 0\}, \quad \Omega_t^- = \{x \in \Omega : uu' \leq 0\}.
$$

We have $C_1|s| \le |g(s)| \le C_2|s|$ for all $s \in \mathbb{R}$ (because $2 \le l \le r+1 = 2$), and then (using (2.5))

$$\int_S^T \sigma(t)\frac{\tilde{\varphi}(E)}{E}\int_\Omega |u'|^l\,dx\,dt \le c\int_S^T \sigma(t)\int_\Omega u'g(u')\,dx\,dt \le cE(S)$$

and (note that $\sigma' \le 0$)

$$\begin{aligned}
&\int_S^T \sigma(t)\frac{\tilde{\varphi}(E)}{E}\int_\Omega |ug(u')|\,dx\,dt\\
&\le c\int_S^T \sigma(t)\int_\Omega |uu'|\,dx\,dt\\
&\le c\Big[\sigma(t)\int_{\Omega_t^+} u^2dx - \sigma(t)\int_{\Omega_t^-} u^2dx\Big]_S^T + c\int_S^T \sigma'(t)\Big(-\int_{\Omega_t^+} u^2dx + \int_{\Omega_t^-} u^2dx\Big)dt\\
&\le c\tilde{\phi}^{-1}(E(S)) + c\tilde{\phi}^{-1}(E(S))\int_S^T (-\sigma'(t))dt \le c\tilde{\phi}^{-1}(E(S)).
\end{aligned}$$

Then

$$\int_S^T \sigma(t)E(t)dt \le c\Big(1 + \frac{E(S)}{\tilde{\phi}^{-1}(E(S))} + \sqrt{\frac{E(S)}{\tilde{\phi}^{-1}(E(S))}}\Big)\tilde{\phi}^{-1}(E(S)).$$

Using the fact that $\tilde{\phi}$ is convex, increasing and $\tilde{\phi}(0) = 0$ (then $s \mapsto \frac{s}{\tilde{\phi}^{-1}(s)}$ is non decreasing) we obtain from (3.2) that

$$\int_S^{+\infty} \sigma(t)E(t)dt \le c\tilde{\phi}^{-1}(E(S)).$$

Let $\tilde{E} = \tilde{\phi}^{-1} \circ E \circ \tilde{\sigma}^{-1}$ (note that $\tilde{\sigma}$ is a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$). Then, for $\omega > 0$,

$$\int_S^{+\infty} \tilde{\phi}(\tilde{E}(t))dt \le \frac{1}{\omega}\tilde{E}(S).$$

Using Lemma 2.2 for $\tilde{E}$ in the particular case $\Psi(s) = \omega\tilde{\phi}(s)$ and $\lambda = 0$, we deduce from (2.6) that

$$\tilde{E}(t) \le \psi^{-1}\left(h(t) + \psi(\tilde{\phi}^{-1}(E(0)))\right), \quad \forall t \ge 0.$$

Then, using the definition of $\tilde{E}$, we obtain (2.18) in the case where $r = 1$ and $G$ is linear on $[0, \epsilon_1]$.

**Case 2: $r > 1$ and $G$ is linear on $[0, \epsilon_1]$.** We choose $\tilde{\varphi}(s) = \frac{s^{1+\frac{1}{r}}}{(\tilde{\phi}^{-1}(s))^{1/r}}$. For all $t \ge 0$, we denote

$$\Omega_t^1 = \{x \in \Omega : |u'| \ge \epsilon_1\}, \quad \Omega_t^2 = \{x \in \Omega : |u'| \le \epsilon_1\}.$$

Using Young's and Lemma 2.1 (for $q = r+1$ and $p = 2$) and condition (2.2) we have, for all $\epsilon > 0$ (using also the fact that $s \mapsto \frac{\tilde{\varphi}(s)}{s}$ is non decreasing),

$$\begin{aligned}
&\int_S^T \sigma(t)\frac{\tilde{\varphi}(E)}{E}\int_{\Omega_t^1} (|u'|^l + |ug(u')|)\,dx\,dt\\
&\le \int_S^T \sigma(t)\frac{\tilde{\varphi}(E)}{E}\Big(\int_{\Omega_t^1} |u|^{r+1}dx\Big)^{\frac{1}{r+1}}\Big(\int_{\Omega_t^1} |g(u')|^{\frac{r+1}{r}}dx\Big)^{\frac{r}{r+1}}dt
\end{aligned}$$

$$
\begin{aligned}
&+ c\int_S^T \sigma(t)\int_{\Omega_t^1} u' g(u')\,dx\,dt\\
&\le \epsilon\int_S^T \sigma(t)\frac{\tilde\varphi^{r+1}(E)}{E^{r+1}}\int_{\Omega_t^1}|\nabla u|^2\,dx\,dt + c\int_S^T\sigma(t)\int_{\Omega_t^1}(|g(u')|^{1+\frac1r}+u'g(u'))\,dx\,dt\\
&\le \epsilon\int_S^T \sigma(t)\frac{\tilde\varphi^{r+1}(E)\tilde\phi^{-1}(E)}{E^{r+1}}dt + c\int_S^T\sigma(t)\int_{\Omega_t^1}u'g(u')\,dx\,dt\\
&\le \epsilon\int_S^T\sigma(t)\tilde\varphi(E)dt + cE(S).
\end{aligned}
$$

Choosing $\epsilon$ small enough, we obtain from (3.2) that

$$
\begin{aligned}
\int_S^T \sigma(t)\tilde\varphi(E(t))dt &\le c\Big(E(S)+E^{\frac{l-1}{l}}(S)\sqrt{\tilde\phi^{-1}(E(S))}\frac{\tilde\varphi(E(S))}{E(S)}\Big)\\
&\quad + c\int_S^T\sigma(t)\frac{\tilde\varphi(E)}{E}\int_{\Omega_t^2}(|u'|^l+|ug(u')|)\,dx\,dt.
\end{aligned}
$$

On the other hand, we have $C_1|s|^{l-1}\le|g(s)|\le C_2|s|$ for all $s\in[-\epsilon_1,\epsilon_1]$ and then (note that $s\mapsto\frac{\tilde\varphi(s)}{s}$ is non decreasing and follow the proof in the case 1)

$$
\begin{aligned}
c\int_S^T\sigma(t)\frac{\tilde\varphi(E)}{E}\int_{\Omega_t^2}(|u'|^l+|ug(u')|)\,dx\,dt &\le c\int_S^T\sigma(t)\int_\Omega(u'g(u')+|uu'|)\,dx\,dt\\
&\le c\Big(E(S)+\tilde\phi^{-1}(E(S))\Big).
\end{aligned}
$$

Then from (3.2) we deduce that

$$
\begin{aligned}
&\int_S^T\sigma(t)\tilde\varphi(E(t))dt\\
&\le c\Big(1+\frac{E(S)}{\tilde\phi^{-1}(E(S))}+E^{\frac{l-2}{2l}}(S)\frac{\tilde\varphi(E(S)))}{E(S))}\sqrt{\frac{E(S)}{\tilde\phi^{-1}(E(S))}}\Big)\tilde\phi^{-1}(E(S)).
\end{aligned}
$$

Finally (note that $s\mapsto s^{\frac{l-2}{2l}}$, $s\mapsto\frac{\tilde\varphi(s)}{s}$ and $s\mapsto\frac{s}{\tilde\phi^{-1}(s)}$ are non decreasing), we obtain

$$
\int_S^{+\infty}\sigma(t)\tilde\varphi(E(t))dt\le c\tilde\phi^{-1}(E(S)).
$$

Let $\tilde E=\tilde\phi^{-1}\circ E\circ\tilde\sigma^{-1}$. Then we deduce from this inequality that, for $\omega>0$,

$$
\int_S^{+\infty}\tilde\varphi\Big(\tilde\phi(\tilde E(t))\Big)dt\le\frac{1}{\omega}\tilde E(S).
$$

Using Lemma 2.2 for $\tilde E$ in the particular case $\Psi(s)=\omega\tilde\varphi(\tilde\phi(s))=\omega\frac{\tilde\phi(s)^{1+\frac1r}}{s^{1/r}}$ and $\lambda=0$, we deduce from (2.6) our estimate (2.18).

**Case 3:** $G'(0)=0$**.** We choose $\tilde\varphi(s)=\frac{2\epsilon_0 s^2}{\tilde\phi^{-1}(s)}G'\Big(\frac{\epsilon_0^2s^2}{\tilde\phi^{-1}(s)}\Big)$. Using the fact that $s\mapsto G'(s)$, $s\mapsto\frac{s^2}{\tilde\phi^{-1}(s)}$ and $s\mapsto\frac{\tilde\varphi^{r-1}(s)}{s^{r-1}}$ are non decreasing, we obtain (as in case 2)

$$
\int_S^T\sigma(t)\frac{\tilde\varphi(E)}{E}\int_{\Omega_t^1}(|u'|^l+|ug(u')|)\,dx\,dt\le\epsilon\int_S^T\sigma(t)\frac{\tilde\varphi^{r+1}(E)\tilde\phi^{-1}(E)}{E^{r+1}}dt+cE(S)
$$

$$
\begin{aligned}
&\le \epsilon \int_S^T \sigma(t) \frac{\tilde{\varphi}^2(E)\tilde{\phi}^{-1}(E)}{E^2} dt + cE(S) \\
&= 2\epsilon\epsilon_0 \int_S^T \sigma(t)\tilde{\varphi}(E) G'(\frac{\epsilon_0^2 E^2}{\tilde{\phi}^{-1}(E)}) dt + cE(S) \\
&\le 2\epsilon\epsilon_0 \int_S^T \sigma(t)\tilde{\varphi}(E) dt + cE(S)
\end{aligned}
$$

Choosing $\epsilon$ small enough, we obtain from (3.2) that

$$
\begin{aligned}
\int_S^T \sigma(t)\tilde{\varphi}(E(t))dt &\le c\Big(E(S) + E^{\frac{l-1}{l}}(S)\sqrt{\tilde{\phi}^{-1}(E(S))}\frac{\tilde{\varphi}(E(S))}{E(S)}\Big) \\
&\quad + c\int_S^T \sigma(t)\frac{\tilde{\varphi}(E)}{E}\int_{\Omega_t^2} (|u'|^l + |ug(u')|)\, dx\, dt.
\end{aligned}
$$

Let now $G_1(s) = G(s^2)$ (note that $G_1$ satisfies the same hypotheses as $G$) and let $G^*$ and $G_1^*$ denote the dual functions of the convex functions $G$ and $G_1$ respectively in the sense of Young (see Arnold [6, page 64], for the definition). Because $G$ is convex and $G$ is not linear near 0, then there exists $\epsilon_1' > 0$ such that $G'' > 0$ on $]0, \epsilon_1']$. Since, because $G'(0) = 0$ and $(2) - (3)$ are still satisfied for $\epsilon'' = \min\{\epsilon_1, \epsilon_1'\}$ instead of $\epsilon_1$, we can assume, without lose of generality, that $G'$ defines a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$. Then $G^*$ and $G_1^*$ are the Legendre transform of $G$ and $G_1$ respectively, which are given by (see Arnold [6, pp. 61-62])

$$
G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)], \quad G_1^*(s) = s(G_1')^{-1}(s) - G_1[(G_1')^{-1}(s)].
$$

Thanks to our choice

$$
\tilde{\varphi}(s) = \frac{2\epsilon_0 s^2}{\tilde{\phi}^{-1}(s)} G'\Big(\frac{\epsilon_0^2 s^2}{\tilde{\phi}^{-1}(s)}\Big) = \frac{s}{\sqrt{\tilde{\phi}^{-1}(s)}} G_1'\Big(\frac{\epsilon_0 s}{\sqrt{\tilde{\phi}^{-1}(s)}}\Big),
$$

we have

$$
\begin{gathered}
G^*\Big(\frac{\tilde{\varphi}(s)}{s}\Big) \le \tilde{\varphi}(s)\frac{(G')^{-1}(\frac{\tilde{\varphi}(s)}{s})}{s}, \\
G_1^*\Big(\frac{\tilde{\varphi}(s)}{s}\sqrt{\tilde{\phi}^{-1}(s)}\Big) \le \frac{\epsilon_0 s}{\sqrt{\tilde{\phi}^{-1}(s)}} G_1'\Big(\frac{\epsilon_0 s}{\sqrt{\tilde{\phi}^{-1}(s)}}\Big) = \epsilon_0\tilde{\varphi}(s).
\end{gathered}
$$

Then, by Poincaré's inequality, Young's inequality (see Arnold [6, p. 64]) and Jensen's inequality (see Rudin [35]), we deduce ($|\Omega|$ is the measure of $\Omega$ in $\mathbb{R}^n$)

$$
\begin{aligned}
&\int_S^T \sigma(t)\frac{\tilde{\varphi}(E)}{E}\int_{\Omega_t^2} (|u'|^l + |ug(u')|)\, dx\, dt \\
&\le \int_S^T \sigma(t)\frac{\tilde{\varphi}(E)}{E}\Big(\int_{\Omega_t^2} G^{-1}(u'g(u'))dx \\
&\quad + \Big(\int_\Omega |\nabla u|^2 dx\Big)^{1/2}\Big(\int_{\Omega_t^2} G^{-1}(u'g(u'))dx\Big)^{1/2}\Big)dt \\
&\le \int_S^T \sigma(t)\frac{\tilde{\varphi}(E)}{E}\Big(\sqrt{\tilde{\phi}^{-1}(E)}\sqrt{|\Omega| G^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega u'g(u')dx\Big)} \\
&\quad + |\Omega| G^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega u'g(u')dx\Big)dt\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\le c\int_S^T \sigma(t)\Big(G_1^*\Big(\frac{\tilde\varphi(E)}{E}\sqrt{\tilde\phi^{-1}(E)}\Big)+G^*\big(\frac{\tilde\varphi(E)}{E}\big)\Big)dt + c\int_S^T\sigma(t)\int_\Omega u'g(u')\,dx\,dt\\
&\le c\int_S^T\sigma(t)\Big(\epsilon_0+\frac{(G')^{-1}(\frac{\varphi(E)}{E})}{E}\Big)\tilde\varphi(E)dt + cE(S).
\end{aligned}
$$

Using the fact that $s\mapsto (G')^{-1}(s)$ and $s\mapsto \frac{s}{\tilde\phi^{-1}(s)}$ are non decreasing, we deduce that, for $0<\epsilon_0\le \frac{\tilde\phi^{-1}(E(0))}{2E(0)}$,

$$
\int_S^T\sigma(t)\frac{\tilde\varphi(E)}{E}\int_{\Omega_t^2}(|u'|^l+|ug(u')|)\,dx\,dt \le c\epsilon_0\int_S^T\sigma(t)\tilde\varphi(E)dt + cE(S).
$$

Then, choosing $\epsilon_0$ small enough, we deduce from (3.2) that

$$
\int_S^T\sigma(t)\tilde\varphi(E(t))dt \le c\Big(1+E^{\frac{l-2}{2l}}(S)\sqrt{\frac{\tilde\phi^{-1}(E(S))}{E(S)}}\frac{\tilde\varphi(E(S))}{E(S)}\Big)E(S)
$$

Finally (note that $s\mapsto s^{\frac{l-2}{2l}}$ and $s\mapsto \sqrt{\frac{\tilde\phi^{-1}(s)}{s}}\frac{\tilde\varphi(s)}{s} = 2\epsilon_0\sqrt{\frac{s}{\tilde\phi^{-1}(s)}}G'\Big(\frac{\epsilon_0^2 s^2}{\tilde\phi^{-1}(s)}\Big)$ are non decreasing), we obtain

$$
\int_S^{+\infty}\sigma(t)\tilde\varphi(E(t))dt\le cE(S).
$$

Let $\tilde E = E\circ\tilde\sigma^{-1}$. Then we deduce from this inequality that, for $\omega>0$,

$$
\int_S^{+\infty}\tilde\varphi(\tilde E(t))dt \le \frac{1}{\omega}\tilde E(S).
$$

Using Lemma 2.2 for $\tilde E$ in the particular case $\Psi(s)=\omega\tilde\varphi(s)$ and $\lambda=0$, we deduce from (2.6) our estimate (2.18). This is completes the proof.

## 4. An application to wave equations of $\phi$-Laplacian with source term

In this section we shall propose some applications of Theorem 2.2.

**Example 1.** Let us consider the Cauchy problem for the wave equation, in $\Omega\times\mathbb{R}_+$,

$$
\begin{gathered}
(|u'|^{l-2}u')' - e^{-\lambda(x)}\sum_{i=1}^n \partial_{x_i}\Big(e^{\lambda(x)}\phi(|\partial_{x_i}u|^2)\partial_{x_i}u\Big)+\sigma(t)g(u')+f(u)=0,\\
u=0 \quad \text{on } \Gamma\times\mathbb{R}_+\\
u(x,0)=u_0(x),\quad u'(x,0)=u_1(x)\quad \text{on } \Omega.
\end{gathered}
\tag{4.1}
$$

We define the energy associated to the solution as

$$
\begin{aligned}
E(t) &= \frac{l-1}{l}\int_\Omega e^{\lambda(x)}|u'|^l dx + \int_\Omega e^{\lambda(x)}\sum_{i=1}^n\tilde\phi(|\partial_{x_i}u|^2)dx + \int_\Omega e^{\lambda(x)}F(u)\,dx\\
&= \frac{l-1}{l}\int_\Omega e^{\lambda(x)}|u'|^l dx + J(u)
\end{aligned}
$$

where $F(u)=\int_0^u f(s)\,ds$. For the function $f\in C(\mathbb{R})$ we assume that there exists an N-function $\psi$ satisfying $i(\psi), I(\psi)\in ]1,+\infty[$ and

$$
|f(t)|\le \psi'(|t|)\quad \text{for every } t\in\mathbb{R}. \tag{4.2}
$$

If condition (2.14) is satisfied (so that $\Phi^*$ exists), then we assume in addition that

$$\psi(t) \le \Phi^*(Ct) \quad \text{for all large } t > 0. \tag{4.3}$$

Thus, we can verify that for all $u \in W_0^{1,\Phi}$ with norm small enough that

$$\frac{1}{C}\int_\Omega \Phi(|\nabla_x u|)\,dx \le |J(u)| \le C\int_\Omega \Phi(|\nabla_x u|)\,dx. \tag{4.4}$$

So, we obtain same results as in the theorem 2.1.

**Proof of the example 1.** We prove only the second part. The proof of the first part is a direct application of the theorem 2.2. We make an additional assumption on $g(v)$:

(H3') Suppose that there exist $c_i > 0$; $i = 1, 2, 3, 4$ such that

$$c_1|v|^p \le |g(v)| \le c_2|v|^\theta \text{if } |v| \le 1, \tag{4.5}$$

$$c_3|v|^s \le |g(v)| \le c_4|v|^r \text{for all } |v| \ge 1, \tag{4.6}$$

where $1 \le m \le r$, $\theta \le p$, $l-1 \le s \le r \le \frac{n+2}{n-2}$.

**Proof of the energy decay.** We denote by $c$ various positive constants which may be different at different occurrences. We multiply the first equation of (1.1) by $E^q\tilde\sigma' u$, where $\tilde\sigma$ is a function satisfying all the hypotheses of lemma 2.2, we obtain

$$\begin{aligned}
0 &= \int_S^T E^q\tilde\sigma'\int_\Omega u((|u'|^{l-2}u')_t - \Delta_\phi u + \sigma(t)g(u'))\,dx\,dt\\
&= \big[E^q\tilde\sigma'\int_\Omega uu'|u'|^{l-2}\,dx\big]_S^T - \int_S^T (qE'E^{q-1}\tilde\sigma' + E^q\tilde\sigma'')\int_\Omega uu'|u'|^{l-2}\,dx\,dt\\
&\quad - \int_S^T E^q\tilde\sigma'\int_\Omega |u'|^l\,dx\,dt + \int_S^T E^q\tilde\sigma'\|\nabla u\|_2^{2(\gamma+1)}\,dx\,dt\\
&\quad + \int_S^T E^q\tilde\sigma'\int_\Omega \sigma(t)ug(u')\,dx\,dt.
\end{aligned}$$

we deduce that

$$\begin{aligned}
&2(m+1)\int_S^T E^{q+1}\tilde\sigma'\,dt\\
&\le -\big[E^q\tilde\sigma'\int_\Omega uu'|u'|^{l-2}\,dx\big]_S^T + \int_S^T (qE'E^{q-1}\tilde\sigma' + E^q\tilde\sigma'')\int_\Omega uu'|u'|^{l-2}\,dx\,dt\\
&\quad + \frac{2(l-1)(m+1)+l}{l}\int_S^T E^q\tilde\sigma'\int_\Omega |u'|^l\,dx\,dt - \int_S^T E^q\tilde\sigma'\int_\Omega \sigma(t)ug(u')\,dx\,dt.
\end{aligned} \tag{4.7}$$

Since $E$ is non-increasing, $\tilde\sigma'$ is a bounded nonnegative function on $\mathbb{R}_+$ (and we denote by $\mu$ its maximum) and using Hölder inequality, we have

$$\big|E(t)^q\tilde\sigma'\int_\Omega uu'|u'|^{l-2}\,dx\,dt\big| \le c\mu E(S)^{q+\frac{l-1}{l}+\frac{1}{2(m+1)}} \qquad \forall t \ge S.$$

and

$$\begin{aligned}
&\int_S^T (qE'E^{q-1}\tilde\sigma' + E^q\tilde\sigma'')\int_\Omega uu'|u'|^{l-2}\,dx\,dt,\,dx\,dt\\
&\le c\mu\int_S^T -E'(t)E(t)^{q-\frac{1}{l}+\frac{1}{2(m+1)}}\,dt + c\int_S^T E(t)^{q+\frac{l-1}{l}+\frac{1}{2(m+1)}}(-\tilde\sigma''(t))\,dt
\end{aligned}$$

$$\leq c\mu E(S)^{q+\frac{l-1}{l}+\frac{1}{2(m+1)}}.$$

Using these estimates we conclude from the above inequality that

$$\begin{aligned}
&2(m+1)\int_S^T E(t)^{1+q}\tilde\sigma'(t)\,dt\\
&\leq cE(S)^{q+\frac{l-1}{l}+\frac{1}{2(m+1)}}+\frac{2(l-1)(m+1)+l}{l}\int_S^T E^q\tilde\sigma'\int_\Omega |u'|^l\,dx\,dt\\
&\quad-\int_S^T E^q\tilde\sigma'\int_\Omega \sigma(t)ug(u')\,dx\,dt\\
&\leq cE(S)^{q+\frac{l-1}{l}+\frac{1}{2(m+1)}}+\frac{2(l-1)(m+1)+l}{l}\int_S^T E^q\tilde\sigma'\int_\Omega |u'|^l\,dx\,dt\\
&\quad-\int_S^T E^q\tilde\sigma'\int_{|u'|\leq 1} \sigma(t)ug(u')\,dx\,dt-\int_S^T E^q\tilde\sigma'\int_{|u'|>1} \sigma(t)ug(u')\,dx\,dt.
\end{aligned}$$

Now, we estimate each terms on the right-hand side of the above inequality, to apply Lemma 2.2. Using Hölder inequality, we obtain

$$\begin{aligned}
&\int_S^T E^q\tilde\sigma'\int_\Omega |u'|^l\,dx\,dt\\
&\leq C\int_S^T E^q\tilde\sigma'\int_\Omega \frac{1}{\sigma(t)}u'\rho(t,u')\,dx\,dt+C'\int_S^T E^q\tilde\sigma'\int_\Omega\Big(\frac{1}{\sigma(t)}u'\rho(t,u')\Big)^{\frac{l}{(p+1)}}\,dx\,dt\\
&\leq C\int_S^T E^q\frac{\tilde\sigma'}{\sigma(t)}(-E')\,dt+C'(\Omega)\int_S^T E^q\frac{\tilde\sigma'}{\sigma^{\frac{l}{p+1}}(t)}(-E')^{\frac{l}{p+1}}\,dt\\
&\leq CE^{q+1}(S)+C'(\Omega)\int_S^T E^q\tilde\sigma'^{\frac{p+1-l}{p+1}}\Big(\frac{\tilde\sigma'}{\sigma(t)}\Big)^{\frac{l}{p+1}}(-E')^{\frac{l}{p+1}}\,dt.
\end{aligned}$$

Now, fix an arbitrarily small $\varepsilon>0$ (to be chosen later), by applying Young's inequality, we obtain

$$\begin{aligned}
&\int_S^T E^q\tilde\sigma'\int_\Omega |u'|^l\,dx\,dt\\
&\leq CE^{q+1}(S)+C'(\Omega)\frac{p+1-l}{p+1}\varepsilon^{\frac{(p+1)}{(p+1-l)}}\int_S^T E^{q\frac{p+1}{p+1-l}}\tilde\sigma'\,dt+C'(\Omega)\frac{l}{p+1}\frac{1}{\varepsilon^{\frac{(p+1)}{l}}}E(S).
\end{aligned}\tag{4.8}$$

If $l\geq p+1$, from (4.5) and (4.6) we obtain easily that

$$\int_S^T E^q\tilde\sigma'\int_\Omega |u'|^l\,dx\,dt\leq CE^{q+1}(S).\tag{4.9}$$

Thanks to Young's inequality,

$$\begin{aligned}
&\int_S^T E^q\tilde\sigma'\int_{|u'|\leq 1}\sigma(t)ug(u')\,dx\,dt\int_S^T E^q\tilde\sigma'\int_{|u'|\leq 1}\sigma(t)\|u\|_2\Big(\int_{|u'|\leq 1}|g(u')|^2\,dx\Big)^{\frac12}\,dt\\
&\times\int_S^T E^q\tilde\sigma'\int_{|u'|\leq 1}\sigma(t)\|\nabla_x u\|_{2m+2}\Big(\int_{|u'|\leq 1}(u'g(u'))^{\frac{2\theta}{\theta+1}}\,dx\Big)^{1/2}\,dt\\
&\leq c\int_S^T E^{q+\frac{1}{2(m+1)}}\tilde\sigma'\sigma^{\frac{1}{(\theta+1)}}(t)\Big(\int_{|u'|<1}\sigma u'g(u')\,dx\Big)^{\theta/(\theta+1)}\,dt
\end{aligned}$$

$$\leq c \int_S^T E^{q+\frac{1}{2(m+1)}} \tilde{\sigma}' \sigma^{\frac{1}{(\theta+1)}}(t)(-E')^{\frac{\theta}{\theta+1}}\, dt.$$

Applying Young's inequality, we obtain

$$\begin{aligned}\int_S^T E^q \tilde{\sigma}' \int_{|u'|\leq 1} \sigma(t) u g(u')\, dx\, dt &\leq C(\Omega)\varepsilon_2^{\theta+1} \int_S^T \Big(E^{q+\frac{1}{2(m+1)}} \tilde{\sigma}' \sigma^{\frac{1}{(\theta+1)}}(t)\Big)^{\theta+1}\, dt \\ &\quad + C(\Omega)\frac{1}{\varepsilon_2^{\frac{\theta+1}{\theta}}} \int_S^T (-E')\, dt \end{aligned} \tag{4.10}$$

and

$$\begin{aligned}&\int_S^T E^q \tilde{\sigma}' \int_{|u'|\geq 1} \sigma(t) u g(u')\, dx\, dt \\ &\leq C(\Omega)\frac{1}{(r+1)}\varepsilon_1^{(r+1)} \int_S^T E^{\left(q+\frac{1}{2(\gamma+1)}\right)(r+1)} \tilde{\sigma}' \sigma(t)^{r+1}\, dt + \frac{C(\Omega) r}{(r+1)} \frac{1}{\varepsilon_1^{\frac{r+1}{r}}} E(S), \end{aligned} \tag{4.11}$$

**The case $l \geq p+1$.** We consider two subcases

- $\theta \geq 2m+1$. Choose $q=0$ and we have $\big(\frac{1}{2(\gamma+1)}\big)(\theta+1) = 1+\alpha$, where $\alpha = \frac{\theta-(2m+1)}{2(m+1)} \geq 0$.
- $\theta < 2m+1$. Choose $q$ such that $\big(q+\frac{1}{2(m+1)}\big)(\theta+1) = q+1$. Thus, $q = \frac{2m-\theta+1}{2\theta(m+1)}$.

**The case $l < p+1$.**

- $2m+1 > \theta$ If $l \geq \frac{2\theta(m+1)(p+1)}{(\theta+1)(2m+1)}$, we choose $q$ such that $\big(q+\frac{1}{2(m+1)}\big)(\theta+1) = q+1$. Thus, $q = \frac{2m-\theta+1}{2\theta(m+1)}$ and $q\frac{p+1}{p+1-l} = q+1+\alpha$ with

  $$\alpha = \frac{l(2m+1)(\theta+1) - 2\theta(m+1)(p+1)}{2`\theta(m+1)(p+1-l)} \geq 0.$$

  If $l < \frac{2\theta(m+1)(p+1)}{(\theta+1)(2m+1)}$, we choose $q$ such that $q\frac{p+1}{p+1-l} = q+1$. Thus $q = \frac{p+1-l}{l}$ and $\big(q+\frac{1}{2(m+1)}\big)(\theta+1) = q+1+\alpha$, where

  $$\alpha = \frac{2\theta(m+1)(p+1) - l(2m+1)(\theta+1)}{2l(m+1)} > 0.$$

- $2m+1 \leq \theta$, we choose $q$ such that $q\big(\frac{p+1}{p+1-l}\big) = q+1$, thus $q = \frac{p+1-l}{l}$ and $(q+\frac{1}{2(\gamma+1)})(m+1) = q+1+\alpha$ with

  $$\alpha = m\frac{m+1-l}{l} + \frac{m-(2\gamma+1)}{2(\gamma+1)} > 0.$$

We may thus complete the proof by applying Lemma 2.2 with $\tilde{E} = E\circ\tilde{\sigma}^{-1}$ instead of $E$ and $\Psi(s) = s^q$.

## 5. Comments and open questions

1. It is interesting to study the asymptotic behavior of solutions for Klein-Gordon nonlocal equation

$$\begin{gathered}
(|u'|^{l-2}u')' - \phi_1(\|\nabla u\|_2^2, \|u\|_2^2)\Delta u + \phi_2(\|\nabla u\|_2^2, \|u\|_2^2)u + \sigma(t)g(u') = 0 \quad \text{in } \Omega \times \mathbb{R}_+ \\
u = 0 \quad \text{on } \Gamma_0 \times \mathbb{R}_+ \\
u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{on } \Omega,
\end{gathered}$$

in particular when there exists a continuous function $E(w,r,s) = \frac{l-1}{l}w + \frac{1}{2}L(r,s)$ defined for $w, r, s \geq 0$ such that for all solutions,

$$E(\|u'\|_l^l, \|\nabla u\|_2^2, \|u\|_2^2) + \int_0^t \sigma(t) \int_\Omega u'g(u')\,dx\,ds = E(\|u_1\|_l^l, \|\nabla u_0\|_2^2, \|u_0\|_2^2).$$

For example when $\phi_1(r,s) = m(r)$ and $\phi_2(r,s) = n(s)$ where $m$ and $n$ are two continuous positives functions. We can take

$$E(w,r,s) = \frac{l-1}{l}w + \frac{1}{2}\int_0^r m(\tau)\,d\tau + \frac{1}{2}\int_0^s n(\tau)\,d\tau.$$

So, $E'(t) = -\sigma(t)\int_\Omega u'g(u')\,dx\,ds$.

As another example, when $\phi_1(r,s) = \frac{r}{1+s^2}$ and $\phi_2(r,s) = -\frac{r^2 s}{(1+s^2)^2}$ we can take

$$E(w,r,s) = \frac{l-1}{l}w + \frac{r^2}{4(1+s^2)}.$$

So, $E'(t) = -\sigma(t)\int_\Omega u'g(u')\,dx$.

As another example when $\phi_1(r,s) = \frac{s}{1+r}$ and $\phi_2(r,s) = \arctan(r)$ we can take

$$E(w,r,s) = \frac{l-1}{l}w + \frac{1}{2}\arctan(r)\ s.$$

So, $E'(t) = -\sigma(t)\int_\Omega u'g(u')\,dx$.

2. An interesting problem is to study the asymptotic behavior of solutions for Kirchhoff type systems,

$$\begin{gathered}
(|v|^{l-2}v)' = \psi_1(\|v(t)\|_2^2, \|w(t)\|_2^2)v_x + \phi_1(\|v(t)\|_2^2, \|w(t)\|_2^2)w_x - \rho_1(t)g(v) \\
(|w|^{r-2}w)' = \phi_2(\|v(t)\|_2^2, \|w(t)\|_2^2)v_x + \psi_2(\|v(t)\|_2^2, \|w(t)\|_2^2)w_x - \mu_2(t)h(w)
\end{gathered}$$

where $\phi_1, \phi_2, \psi_1$ and $\psi_2$ are real and continuous functions on $\mathbb{R}_+^2$, $\phi_1\phi_2 \geq 0$, $\rho_1$ and $\mu_2$ are two positives and decreasing functions, $g, h : \mathbb{R} \to \mathbb{R}$ are non-decreasing functions of class $C(\mathbb{R})$.

If there is a $C^1$ function $L(r,s)$ defined on $\mathbb{R}_+^2$, with

$$\frac{l}{l-1}\frac{\partial L}{\partial r}\phi_1 = \frac{r}{r-1}\frac{\partial L}{\partial r}\phi_2, \quad \frac{\partial L}{\partial r} \geq 0, \quad \frac{\partial L}{\partial s} \geq 0.$$

We define the energy function $E(t) = L(\|v(t)\|_l^l, \|w(t)\|_r^r)$. So that

$$E'(t) = -\frac{l}{l-1}\rho_1\int_0^{2\pi} g(v)v\,dx - \frac{r}{r-1}\mu_2\int_0^{2\pi} h(v)v\,dx \leq 0.$$

3. Another interesting problem is to study the asymptotic behavior of solutions for Kirchhoff equation with memory,

$$(|u'|^{l-2}u')' - \phi_1(\|\nabla u\|^2)\Delta u - \int_0^t a(t-s)\phi_2(\|\nabla u\|^2)\Delta u ds = 0 \quad \text{in } \Omega \times \mathbb{R}_+$$

$$
\begin{gathered}
u=0 \quad \text{on } \Gamma_0\times\mathbb{R}_+\\
u(x,0)=u_0(x),\quad u'(x,0)=u_1(x)\quad \text{on } \Omega.
\end{gathered}
$$

In the non-degenerate case, the global existence in $H^2(\Omega)\cap H_0^1(\Omega)$ was treated by Abdelli and Benaissa [1] when $\phi_2\equiv$const and the function $a$ is a polynomial. The asymptotic behaviour of the energy play an important role to prove global existence.

In the degenerate case, when $\phi_1\geq\phi_2\geq 0$, Dix and Torrejon [13] proved a global existence of the $(-\Delta)$-analytic solution. It is an interesting question to study the decay rate of the energy (the energy is a decreasing function). It is clear that the energy decay rate depends on the order of degeneracy of $\phi_1,\phi_2$ and the form of $a$.

4. Another interesting problem is to study global existence and asymptotic behaviour for the following Kirchhoff equation with dissipation and source term with initial data less regular than as in the classical case (i.e $(u_0,u_1)\in H^2(\Omega)\cap H_0^1(\Omega)$),

$$
\begin{gathered}
(|u'|^{l-2}u')'-\phi_1(\|\nabla u\|^2)\Delta u+\sigma(t)g(u')+f(u)=0\quad \text{in } \Omega\times\mathbb{R}_+\\
u=0 \quad \text{on } \Gamma_0\times\mathbb{R}_+\\
u(x,0)=u_0(x),\quad u'(x,0)=u_1(x)\quad \text{on } \Omega.
\end{gathered}
$$

This study makes possible to consider the case when $g$ and $f$ are not Lipschitz functions (see Serrin, Todorova and Vitillaro [36] and Panizzi [34]). A convenient space is $D((-\Delta)^{\kappa/2})\cap D((-\Delta)^{\frac{\kappa-1}{2}})$ where $\kappa\geq 3/2$, in particular when $\kappa=2$, we find $D((-\Delta)^{\kappa/2})\cap D((-\Delta)^{\frac{\kappa-1}{2}})=H^2(\Omega)\cap H_0^1(\Omega)$. When $1\leq\kappa<3/2$, the problem of local existence is open for the non-degenerate Kirchhoff equation without dissipation and source term.

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